# Separation of variables in the Kramers equation

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**Abstract**

We consider the problem of separation of variables in the Kramers equation admitting a non-trivial symmetry group. Provided the external potential $V(x)$ is at most quadratic, a complete solution of the problem of separation of variables is obtained. Furthermore, we construct solutions of the Kramers equation with separated variables in explicit form.

## 1 Introduction

A lot of phenomena in physics and, especially, in chemical physics may be modeled as the Brownian motion of particles in an external potential $V(x)$, the appropriate transport equation being (1+2)-dimensional Fokker-Plank equation of a special form

$$u_t = \nu u_{yy} - y u_x + (\nu y + V'(x))u_y + \nu u, \tag{1}$$

where $u = u(t,x,y)$ is a sufficiently smooth real-valued function, $\nu$ is a real parameter.

The first relevant result on studying partial differential equation (PDE) (1) has been obtained by Kramers [1]. He got solution of the escape problem

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of a classical particle subjected to a Gaussian white noise out of a deep potential well. That is why the equation is question is called the Kramers equation (KE) (see, for more details [2]–[4]).

As KE is a partial differential equation with variable coefficients, we cannot apply the Fourier transform in order to solve it. In fact the only possibilities to get exact solutions of KE are either to utilize its Lie symmetry or to apply the method of separation of variables. The first possibility has been exploited recently in [5, 6], where symmetry classification of the class of PDEs (1) has been carried out. The principal result of the above papers is that KE has a symmetry group that is wider than a trivial one-parameter group of time translations if and only if $V''(x) = 0$.

The principal aim of the present paper is applying the direct approach to variable separation in partial differential equations suggested in [7]–[9] to solving KE. As is well-known, separability of PDE is intimately connected to its symmetry within the class of second-order differential operators [10]. That is why, we will concentrate on the case $V(x) = kx,\ k = \text{const}$, namely, we consider KE having nontrivial Lie symmetry

$$u_t = \nu u_{yy} - yu_x + (\nu y + kx)u_y + \nu u. \tag{2}$$

In a classical setting the method of separation of variables (say, in the Cartesian coordinate system) is based on a special representation of a solution to be found in a factorized form:

$$u(t, x, y) = \varphi_0(t)\varphi_1(x)\varphi_2(y),$$

where $\varphi_i, i = 0, 1, 2$ are solutions of some ordinary differential equations (ODEs). However, one can try to separate variables in the equation under study in another coordinate system, for example, in the polar one and look for a solution of the form

$$u(t, x, y) = \varphi_0(t)\varphi_1\left(\sqrt{x^2 + y^2}\right)\varphi_2\left(\arctan\frac{y}{x}\right).$$

So, if we are given some coordinate system, then it is clear what to do in order to get exact solutions with separated variables. However, the classical approach gives no general routine for finding all possible coordinate systems providing separability of the equation under study. Our approach to the problem of separation of variables in the evolution type equations (to be specific, we take the case of an equation having three independent variables $t, x, y$) is based on the following observations:

- All solutions with separated variables known to us can be represented in the form

$$u(t,x,y) = Q(t,x,y)\varphi_0(t)\varphi_1(\omega_1(t,x,y))\varphi_2(\omega_2(t,x,y)), \qquad (3)$$

  where $Q, \omega_1, \omega_2$ are sufficiently smooth functions and $\varphi_i, i = 0,1,2$ satisfy some ODEs.

- The functions $\varphi_i, i = 0,1,2$ depend on two arbitrary parameters $\lambda_1, \lambda_2$ called spectral parameters or separation constants. Furthermore, the functions $Q, \omega_1, \omega_2$ are independent of $\lambda_1, \lambda_2$.

By properly postulating the above features we have formulated an efficient approach to the problem of variable separation in linear partial differential equations [9]. Applying it to the Kramers equation (2) we are looking for its particular solutions of the form (0.2), where functions $Q, \omega_1, \omega_2$ are chosen in such a way that inserting (3) into KE yields three ODEs for functions $\varphi_0(t), \varphi_1(\omega_1), \varphi_2(\omega_2)$

$$\begin{aligned} &U_0(t, \varphi_0, \dot\varphi_0; \lambda_1, \lambda_2) = 0, \\ &U_i(\omega_i, \varphi_i, \dot\varphi_i, \ddot\varphi_i; \lambda_1, \lambda_2) = 0, \quad i = 1, 2. \end{aligned} \qquad (4)$$

Here $U_0, U_1, U_2$ are some smooth functions of the indicated variables, $\lambda_1, \lambda_2$ are real parameters and, what is more,

$$\operatorname{rank} \left\| \begin{matrix} \frac{\partial U_0}{\partial \lambda_1} & \frac{\partial U_0}{\partial \lambda_2} \\ \frac{\partial U_1}{\partial \lambda_1} & \frac{\partial U_1}{\partial \lambda_2} \\ \frac{\partial U_2}{\partial \lambda_1} & \frac{\partial U_2}{\partial \lambda_2} \end{matrix} \right\| = 2. \qquad (5)$$

Note that the functions $Q, \omega_1, \omega_2$ are independent of $\lambda_1, \lambda_2$.

Provided the above requirements are met, we say that KE is separable in the coordinate system $t, \omega_1(t,x,y), \omega_2(t,x,y)$.

Due to the fact that the equation under study is linear, the reduced equations prove to be linear as well. Furthermore, we have to consider two distinct cases.

Case 1. System of equations (4) has the form

$$\begin{aligned}\dot{\varphi}_0 &= A_0(t;\lambda_1,\lambda_2)\varphi_0,\\ \dot{\varphi}_1 &= A_1(\omega_1;\lambda_1,\lambda_2)\varphi_1,\\ \ddot{\varphi}_2 &= A_2(\omega_2;\lambda_1,\lambda_2)\dot{\varphi}_2+A_3(\omega_2;\lambda_1,\lambda_2)\varphi_2.\end{aligned} \tag{6}$$

Case 2. System of equations (4) has the form

$$\begin{aligned}\dot{\varphi}_0 &= A_0(t;\lambda_1,\lambda_2)\varphi_0,\\ \dot{\varphi}_1 &= A_1(\omega_1;\lambda_1,\lambda_2)\varphi_1,\\ \dot{\varphi}_2 &= A_2(\omega_2;\lambda_1,\lambda_2)\varphi_2.\end{aligned} \tag{7}$$

In the above formulae $A_0,\ldots,A_3$ are some smooth real-valued functions of the indicated variables.

Consequently, there are two different possibilities to separate variables in KE, either to reduce it to two first-order and one second-order ODEs or to three first-order ODEs. It is impossible to reduce KE to two or three second-order ODEs because it contains a second-order derivative with respect to one variable only.

Provided the system of reduced ODEs has the form (6), separation of variables in (2) is performed in the following way:

1. We insert the Ansatz (3) into KE and express the derivatives $\dot{\varphi}_0$, $\dot{\varphi}_1$, $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ in terms of functions $\varphi_0,\varphi_1,\varphi_2,\dot{\varphi}_2$ using equations (6) and their differential consequences (where necessary).

2. The equality obtained is split by $\varphi_0,\varphi_1,\varphi_2,\dot{\varphi}_2,\lambda_1,\lambda_2$ which are regarded as independent variables. This yields an over-determined system of nonlinear PDEs for unknown functions $Q,\omega_1,\omega_2$.

3. After solving the above system we get an exhaustive description of coordinate systems providing separability of KE.

Clearly, if we adopt a more general definition of separation of variables, then additional coordinate systems providing separability of KE may appear. However, all solutions with separated variables of the Schrödinger and heat conductivity equations known to us can be obtained within the above described approach.

The case when the system of reduced ODEs is of the form (7) is handled in a similar way.

Next, we introduce an equivalence relation on the set of all coordinate systems providing separability of KE. We say that two coordinate systems $t,\ \omega_1,\ \omega_2$ and $t',\ \omega_1',\ \omega_2'$ are equivalent if the corresponding solutions with separated variables are transformed one into another by

- the group transformations from the Lie transformation group admitted by KE,
- the transformations of the form

$$t \to t' = f_0(t), \quad \omega_i \to \omega_i' = f_i(\omega_i), \tag{8}$$

$$Q \to Q' = Q h_0(t) h_1(\omega_1) h_2(\omega_2), \tag{9}$$

  where $f_0, f_i, h_0, h_i$ are some smooth functions.

It can be proved that formulae (8), (9) define the most general transformation preserving the class of Ansätze (3). The above equivalence relation split the set of all possible coordinate systems into equivalence classes. In a sequel, when presenting the lists of coordinate systems enabling us to separate variables in KE we will give only one representative for each equivalence class.

# 2 Principal results

In this section we give a complete account of our results on separation of variables in KE obtained within the framework of the approach described in Introduction. We write down explicit forms of the functions $Q(t,x,y)$, $\omega_1(t,x,y)$, $\omega_2(t,x,y)$ and the corresponding reduced ODEs for functions $\varphi_0(t)$, $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$.

**Theorem 1** *Equation (2) admits separation of variables into two first-order and one second-order ODEs if and only if $k$ takes one of the three values $0,\ 3\nu^2/16,\ -3\nu^2/4$. Furthermore, equations separates into three first-order ODEs with an arbitrary $k$.*

The above theorem gives a general description of separable KEs. The solution of problem of separation of variables in corresponding KEs is provided by Theorems 2–6 below.

**Theorem 2** *The set of inequivalent coordinate systems providing separability of KE with $k = \nu^2/4$ is exhausted by the following ones:*

$$
\begin{aligned}
&\omega_i = \frac{f_i y - \dot{f}_i x}{\dot{f}_2 f_1 - \dot{f}_1 f_2}, \quad i = 1, 2, \\
&Q = \exp\left\{ \left( -\frac{1}{4\nu} \frac{\ddot{f}_2 f_1 - \ddot{f}_1 f_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} - \frac{1}{4} \right) y^2 + \frac{1}{2\nu} \left( \frac{\ddot{f}_2 \dot{f}_1 - \ddot{f}_1 \dot{f}_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} - k \right) x \right. \\
&\quad \left. + \left( \frac{1}{4\nu} \frac{\dddot{f}_2\, \dot{f}_1 - \dddot{f}_1\, \dot{f}_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} - \frac{k}{4} \right) x^2 - \frac{1}{2} \ln |\dot{f}_2 f_1 - \dot{f}_1 f_2| + \frac{\nu}{2} t \right\}; \\
&\dot{\varphi}_0 = \nu \left( \frac{f_1 \lambda_1 + f_2 \lambda_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} \right)^2 \varphi_0, \quad \dot{\varphi}_1 = \lambda_1 \varphi_1, \quad \dot{\varphi}_2 = \lambda_2 \varphi_2,
\end{aligned}
\tag{10}
$$

*where*

$$
\begin{aligned}
f_1 &= t \left( A_1 \sinh \frac{\nu}{2} t + A_2 \cosh \frac{\nu}{2} t \right) + A_3 \sinh \frac{\nu}{2} t + A_4 \cosh \frac{\nu}{2} t, \\
f_2 &= t \left( B_1 \sinh \frac{\nu}{2} t + B_2 \cosh \frac{\nu}{2} t \right) + B_3 \sinh \frac{\nu}{2} t + B_4 \cosh \frac{\nu}{2} t,
\end{aligned}
$$

*and $A_1, \ldots, B_4$ are arbitrary real constants satisfying the condition $2C_{12} - \nu(C_{13} - C_{24}) = 0$. Hereafter we use the notations*

$$
C_{ij} = B_i A_j - A_i B_j, \quad i, j = 1, \ldots, 4.
$$

**Theorem 3** *The set of inequivalent coordinate systems providing separability of KE with $k > \nu^2/4$ is exhausted by those given in (10) with*

$$
f_1 = \sin bt (A_1 \sinh at + A_2 \cosh at) + \cos bt (A_3 \sinh at + A_4 \cosh at),
$$

$$
f_2 = \sin bt (B_1 \sinh at + B_2 \cosh at) + \cos bt (B_3 \sinh at + B_4 \cosh at),
$$

*where $a = \frac{\nu}{2}, b = \sqrt{k - \frac{\nu^2}{4}}$ and $A_1, \ldots, B_4$ are constants fulfilling the condition $(C_{12} + C_{34})b + (C_{13} - C_{24})a = 0$. The explicit form of the function $Q$ and the reduced ODEs are also obtained from the formulae (10) with $f_1, f_2$ given above.*

**Theorem 4** *The set of inequivalent coordinate systems providing separability of KE with $k < \nu^2/4$ and $k \neq 0,\ 3\nu^2/16,\ -3\nu^2/4$ is exhausted by those given in (10) with*

$$f_1 = \sinh bt(A_1 \sinh at + A_2 \cosh at) + \cosh bt(A_3 \sinh at + A_4 \cosh at),$$

$$f_2 = \sinh bt(B_1 \sinh at + B_2 \cosh at) + \cosh bt(B_3 \sinh at + B_4 \cosh at),$$

*where $a = \frac{\nu}{2}, b = \sqrt{\frac{\nu^2}{4} - k}$ and $A_1, \ldots, B_4$ are constants fulfilling the condition $(C_{12} - C_{34})b + (C_{13} - C_{24})a = 0$. The explicit form of the function $Q$ and reduced ODEs are also obtained from the formulae (10) with $f_1, f_2$ given above.*

**Theorem 5** *The set of inequivalent coordinate systems providing separability of KE with $k = 0$ is exhausted by*

1) *those given in (10) with*

$$f_1 = A_1 \sinh \nu t + A_2 \cosh \nu t + A_3 t + A_4,$$

$$f_2 = B_1 \sinh \nu t + B_2 \cosh \nu t + B_3 t + B_4,$$

*where $A_1, \ldots, B_4$ are constants fulfilling the equation $\nu C_{12} - C_{34} = 0$. The explicit form of the function $Q$ and reduced ODEs are also obtained from the formulae (10) with $f_1, f_2$ given above;*

2) *the following coordinate systems:*

$$\omega_1 = x, \quad \omega_2 = y, \quad Q = \exp\left(-\frac{y^2}{4}\right);$$
$$\dot{\varphi}_0 = \nu\lambda_1\varphi_0, \quad \dot{\varphi}_1 = \nu\lambda_2\varphi_1, \quad \ddot{\varphi}_2 = \left(\frac{y^2}{4} + \lambda_2 y + \lambda_1 - \frac{1}{2}\right)\varphi_2.$$

**Theorem 6** *The set of inequivalent coordinate systems providing separability of KE with $k = 3\nu^2/16$ or $k = -3\nu^2/4$ is exhausted by*

1) *those given in Theorem 4 under $k = 3\nu^2/16$ or $k = -3\nu^2/4$;*

*2) the following coordinate systems:*

$$\begin{aligned}
&\omega_1 = R^3 x, \quad \omega_2 = Ry + 3\dot{R}x,\\
&Q = \exp\left\{ \left( \frac{\dot{R}}{\nu R} - \frac{1}{4} \right) y^2 + \frac{1}{2\nu}\left( 3\frac{\ddot{R}}{R} - k \right) xy + \left( -\frac{3\,\dddot{R}}{4\nu R} + \frac{15\dot{R}\ddot{R}}{4\nu R^2} \right.\right.\\
&\quad \left.\left. - \frac{k}{4} \right) x^2 + \frac{\nu}{2} t + 2\ln R \right\};\\
&\dot{\varphi}_0 = \nu\lambda_1 R^2 \varphi_0, \quad \dot{\varphi}_1 = \nu\lambda_2\varphi_1, \quad \ddot{\varphi}_2 = (\lambda_2\omega_2 + \lambda_1)\varphi_2,
\end{aligned}$$

*where*

$$R(t) = \begin{cases} \dfrac{1}{\cosh at}, \\ \dfrac{1}{\sinh at}, \\ \exp\{\pm at\} \end{cases} \qquad \text{with} \quad a = \begin{cases} \dfrac{\nu}{4}, & \textit{under } k = \dfrac{3\nu^2}{16}, \\ \dfrac{\nu}{2}, & \textit{under } k = -\dfrac{3\nu^2}{4}. \end{cases}$$

# 3 Proof of Theorems 1–6

In order to prove the assertions of the previous section one should apply to equation (2) the algorithm of variable separation described in Introduction.

We give a detailed proof for the case when system of reduced ODEs is of the form (6). Inserting Ansatz (3) into KE (2) and expressing the derivatives $\dot{\varphi}_0, \dot{\varphi}_1, \ddot{\varphi}_1, \ddot{\varphi}_2$ in terms of functions $\varphi_0, \varphi_1, \varphi_2, \dot{\varphi}_2$ with the use of equations (6) and their differential consequences yield the system of two nonlinear PDEs

$$\begin{aligned}
&Q\omega_{2t} + yQ\omega_{2x} = \nu(yQ\omega_{2y} + 2Q_y\omega_{2y} + 2QA_1\omega_{1y}\omega_{2y}\\
&\quad + QA_2\omega_{2y}^2 + Q\omega_{2yy}) + kxQ\omega_{2y}, \qquad (11)\\
&Q_t + QA_0 + QA_1\omega_{1t} + yQ_x + yQA_1\omega_{1x} = \nu(Q + yQ_y\\
&\quad + yQA_1\omega_{1y} + Q_{yy} + 2Q_yA_1\omega_{1y} + Q(A_1^2 + A_{1\omega_1})\omega_{1y}^2\\
&\quad + QA_1\omega_{1yy} + QA_3\omega_{2y}^2) + kx(Q_y + QA_1\omega_{1y}). \qquad (12)
\end{aligned}$$

The above system is to be split with respect to variables $\lambda_1, \lambda_2$ (we remind that the functions $\omega_1, \omega_2$ are independent of $\lambda_1, \lambda_2$). To this end we differentiate (11) with respect to $\lambda_i$ and get the relation

$$(2A_{1\lambda_i}\omega_{1y} + A_{2\lambda_i}\omega_{2y})\omega_{2y} = 0, \quad i = 1, 2$$

Due to the fact that $\omega_{2y}$ does not vanish identically (since otherwise it follows from (11) that $\omega_2$ =const), the equation

$$2A_{1\lambda_i}\omega_{1y} + A_{2\lambda_i}\omega_{2y} = 0, \quad i = 1, 2. \tag{13}$$

holds.

Let us show first that we can without loss of generality put $\omega_{1y} = 0$. Suppose the inverse, namely, that the inequality $\omega_{1y} \neq 0$ holds true. It follows from the second equation of system (6) that $A^2_{1\lambda_1} + A^2_{1\lambda_2} \neq 0$. Let the function $A_{1\lambda_1}$ be non-vanishing, then by force of (13) $A_{2\lambda_1} \neq 0$. Denoting

$$A_{1\lambda_1} = g(\omega_1, \lambda_1, \lambda_2), \quad -2A_{2\lambda_1} = f(\omega_2, \lambda_1, \lambda_2)$$

we rewrite (13) as follows

$$\frac{\omega_{1y}}{\omega_{2y}} = \frac{f(\omega_2, \lambda_1, \lambda_2)}{g(\omega_1, \lambda_1, \lambda_2)}. \tag{14}$$

Differentiating (14) with respect to $\lambda_1$ yields

$$\frac{f_{\lambda_1}}{f} = \frac{g_{\lambda_1}}{g}.$$

Hence we conclude that there is a function $k = k(\lambda_1, \lambda_2)$ such that

$$\frac{f_{\lambda_1}}{f} = \frac{g_{\lambda_1}}{g} = k(\lambda_1, \lambda_2).$$

Integrating the above equations we get

$$f = k_1(\lambda_1, \lambda_2) f_1(\omega_2, \lambda_2), \quad g = k_1(\lambda_1, \lambda_2) g_1(\omega_1, \lambda_1),$$

so that (14) reduces to the relation

$$\frac{\omega_{1y}}{\omega_{2y}} = \frac{f(\omega_2, \lambda_2)}{g(\omega_1, \lambda_2)}.$$

In a similar way we establish that the last relation is equivalent to the following one

$$\frac{\omega_{1y}}{\omega_{2y}} = \frac{f_2(\omega_2)}{g_2(\omega_1)},$$

whence

$$g_2(\omega_1)\omega_{1y} = f_2(\omega_2)\omega_{2y}.$$

Taking into account the equivalence relation (8) we can put $g_2 = 1, f_2 = 1$ in the above equality thus getting $\omega_{1y} = \omega_{2y}$. Integrating this PDE yields

$$\omega_1 = \omega_2 + h(t, x)$$

with an arbitrary smooth function $h$. By force of the above equation relation (13) takes the form

$$2A_{1\lambda_i} + A_{2\lambda_i} = 0, \quad i = 1, 2.$$

Hence we conclude that there exists a function $\Lambda(\lambda_1, \lambda_2)$ such that

$$A_1 = \Lambda(\lambda_1, \lambda_2) + \tilde{A}_1(\omega_1), \quad A_2 = -2\Lambda(\lambda_1, \lambda_2) + \tilde{A}_2(\omega_2). \tag{15}$$

Within the equivalence transformation (9) with properly chosen functions $h_1, h_2$ we can put $\tilde{A}_1(\omega_1) = 0, \tilde{A}_2(\omega_2) = 0$. Furthermore, defining new separation constants as

$$\lambda_1' = \Lambda(\lambda_1, \lambda_2), \quad \lambda_2' = \lambda_2$$

and omitting the primes we represent (15) in the form

$$A_1 = \lambda_1, \quad A_2 = -2\lambda_1.$$

Consequently, system (6) takes the form

$$\begin{aligned} \dot{\varphi}_0(t) &= A_0(t)\varphi_0(t), \\ \dot{\varphi}_1(\omega_1) &= \lambda_1\varphi_1(\omega_1), \\ \ddot{\varphi}_2(\omega_2) &= -2\lambda_1\dot{\varphi}_2(\omega_2) + A_3(\omega_1, \lambda_1, \lambda_2)\varphi_2(\omega_2). \end{aligned} \tag{16}$$

Making the change of variables $\varphi_2 = \phi\exp\{-\lambda_1\omega_2\}$ reduces the third equation of system (16) to become

$$\ddot{\phi} = (\lambda_1^2 + A_3)\phi.$$

Let $\phi = \phi(\omega_2, \lambda_1, \lambda_2)$ be a solution of this equation. Then the corresponding solution with separated variables takes the form

$$u = Q(t, x, y)\phi(\omega_2, \lambda_1, \lambda_2)\exp\left\{\int A_0(t)dt + \lambda_1(\omega_1 - \omega_2)\right\}.$$

The structure of thus obtained solution with separated variables is such that dependence of $\omega_1$ on $y$ is inessential. Indeed, the function $\omega_1$ enters into the above solution only as a combination $\omega_1 - \omega_2$ and the latter is equal to $h(t,x)$. Consequently, we have proved that without loss of generality we may choose $\omega_{1y} = 0$.

Given the condition $\omega_{1y} = 0$, equation (13) reduces to the relations $A_{2\lambda_i} = 0,\ i = 1,2$, whence we get $A_2 = A_2(\omega_2)$. Choosing appropriately the function $h_2$ in (9) we can put $A_2 = 0$. Next, differentiating (12) with respect to $\lambda_i$ we arrive at the equations

$$A_{0\lambda_i} + A_{1\lambda_i}(\omega_{1t} + y\omega_{1x}) = \nu A_{3\lambda_i}\omega_{2y}^2, \quad i = 1,2. \tag{17}$$

Differentiating twice the above equations with respect to $y$ yields

$$A_{3\omega_2\omega_2\lambda_i}\omega_{2y}^4 + 5A_{3\omega_2\lambda_i}\omega_{2y}^2\omega_{2yy} + 2A_{3\lambda_i}(\omega_{2yy}^2 + \omega_{2y}\omega_{2yyy}) = 0, \tag{18}$$

where $i = 1,2$.

Note that due to (17) the inequality $A_{3\lambda_i} \neq 0$ holds. Dividing (18) into $A_{3\lambda_i}$ and differentiating the equality obtained by $\lambda_j, j = 1,2$ we get

$$\left(\frac{A_{3\omega_2\omega_2\lambda_i}}{A_{\lambda_i}}\right)_{\lambda_j}\omega_{2y}^2 + 5\left(\frac{A_{3\omega_2\lambda_i}}{A_{\lambda_i}}\right)_{\lambda_j}\omega_{2yy} = 0, \quad i,j = 1,2. \tag{19}$$

**Case 1.** At least one of the four expressions

$$\left(\frac{A_{3\omega_2\lambda_i}}{A_{\lambda_i}}\right)_{\lambda_j}$$

does not vanish. Then it is easy to become convinced of the fact that the relation

$$\frac{\omega_{2yy}}{\omega_{2y}^2} = f(\omega_2)$$

holds true. Integration of the above relation yields

$$\omega_{2y} = g_1(t,x)\exp\left\{\int f(\omega_2)d\omega_2\right\},$$

where $g_1(t,x)$ is an arbitrary smooth function.

Next using the equivalence relation (8) we reduce the equation obtained to the form

$$\omega_{2y} = g(t, x),$$

whence

$$\omega_2 = y g_1(t, x) + g_2(t, x), \tag{20}$$

$g_2(t, x)$ being an arbitrary smooth function.

In view of this result, (18) takes the form $A_{3\omega_2\omega_2\lambda_i} = 0, i = 1, 2$, whence

$$A_3 = \Lambda_1(\lambda_1, \lambda_2)\omega_2 + \Lambda_2(\lambda_1, \lambda_2) + F(\omega_2), \tag{21}$$

where $\Lambda_1, \Lambda_2, F$ are arbitrary smooth functions of the indicated variables. Furthermore, it is not difficult to prove that $\Lambda_1, \Lambda_2$ are functionally independent (since otherwise the condition (5) would be broken) and, consequently, after redefining $\lambda_1, \lambda_2$ we can represent (21) in the form

$$A_3 = \lambda_1\omega_2 + \lambda_2 + F(\omega_2). \tag{22}$$

**Case 2.** Suppose now that

$$\left(\frac{A_{3\omega_2\lambda_i}}{A_{\lambda_i}}\right)_{\lambda_j} = 0, \quad i, j = 1, 2.$$

Integrating the above system of partial differential equations gives the following form of $A_{3\lambda_i}$

$$A_{3\lambda_i} = B_i(\omega_2)L_i(\lambda_1, \lambda_2), \quad i = 1, 2, \tag{23}$$

where $B_i, L_i$ are arbitrary smooth functions and, what is more, $B_1^2 + B_2^2 \neq 0$.

As a compatibility condition of system (23) we get

$$B_1 L_{1\lambda_2} = B_2 L_{2\lambda_1}.$$

**Subcase 2.1.** $L_{1\lambda_2} \neq 0, \quad L_{2\lambda_1} \neq 0$. Given this restrictions the above compatibility condition is transformed to become

$$\frac{B_1(\omega_1)}{B_2(\omega_2)} = \frac{L_{1\lambda_2}}{L_{2\lambda_1}} = \text{const}. \tag{24}$$

Integrating system (23) with account of (24) yields

$$A_3 = \Lambda(\lambda_1, \lambda_2)F_1(\omega_2) + F_2(\omega_2),$$

where $\Lambda, F$ are arbitrary smooth functions of indicated variables. After redefining separation parameters $\lambda_1, \lambda_2$ we represent the above relation as follows

$$A_3 = \lambda_1 F_1(\omega_2) + F_2(\omega_2). \tag{25}$$

**Subcase 2.2** $L_{1\lambda_2} = 0, \quad L_{2\lambda_1} = 0$. Integrating system (23) and redefining the separation parameters $\lambda_1, \lambda_2$ yield

$$A_3 = \lambda_1 S_1(\omega_2) + \lambda_2 S_2(\omega_2) + S_0(\omega_2), \tag{26}$$

where $S_1, S_2, S_0$ are arbitrary smooth functions. An analysis of formulae (22), (25), (26) shows that the first two of them are particular cases of formula (26). Thus the most general form of the function $A_3$ is given by (26).

Inserting (26) into (17) and differentiating the equality obtained with respect to $x$ and $\lambda_j$ give $A_{1\lambda_i\lambda_j} = 0, i, j = 1, 2$. Hence we get the form of $A_1$

$$A_1 = \lambda_1 L_1(\omega_2) + \lambda_2 L_2(\omega_2) + L_0(\omega_2), \tag{27}$$

where $L_1, L_2, L_0$ are arbitrary smooth functions.

Next, inserting (26), (27) into (17) and differentiating the equation obtained with respect to $\lambda_j$ we get $A_{0\lambda_i\lambda_j} = 0, i, j = 1, 2$, whence

$$A_0 = \lambda_1 R_1(\omega_2) + \lambda_2 R_2(\omega_2) + R_0(\omega_2), \tag{28}$$

where $R_1, R_2, R_0$ are arbitrary smooth functions.

With these results in hand we can split equations (11), (12) by $\lambda_1, \lambda_2$ thus getting the system of four nonlinear PDEs for three functions $\omega_1, \omega_2, Q$

$$Q\omega_{2t} + yQ\omega_{2x} = (\nu y + kx)Q\omega_{2y} + 2\nu Q_y\omega_{2y} + \nu Q\omega_{2yy}, \tag{29}$$

$$\begin{aligned} &Q_t + QR_0 + QL_0(\omega 1t + y\omega_{1x}) + yQ_x = \nu Q + (\nu y + kx)Q_y \\ &\quad + \nu Q_{yy} + \nu Q S_0 \omega_{2y}^2, \end{aligned} \tag{30}$$

$$R_1 + L_1(\omega_{1t} + y\omega_{1x}) = \nu S_1 \omega_{2y}^2, \tag{31}$$

$$R_2 + L_2(\omega_{1t} + y\omega_{1x}) = \nu S_2 \omega_{2y}^2. \tag{32}$$

Making an equivalence transformation (9) with appropriately chosen functions we can put $L_0 = 0, R_0 = 0$. Next, due to the requirement (5) $S_1 S_2 \neq 0$.

There are two inequivalent cases $L_2 = 0$ and $L_2 \neq 0$. Since they are handled in a similar way, we consider in detail the case $L_2 = 0$ only. In view of (5) $L_1$ does not vanish. Choosing appropriately the functions $f_1, f_2$ in (8) we can put $L_1 = 1, S_2 = \pm 1$ in formulae (29)–(32). Integrating (32) with account of (31) yields the form of $\omega_2$

$$\omega_2 = R(t)y + F(t,x), \quad R(t) \neq 0, \tag{33}$$

where $R, F$ are arbitrary smooth functions and, what is more, $R_2 = \pm\nu R^2$.

Differentiating (31) twice with respect to $y$ and taking into account (33) we arrive at the equation $S_{1\omega_2\omega_2} = 0$, whence

$$S_1 = C_1\omega_2 + C_2,$$

where $C_1 \neq 0, C_2$ are arbitrary constants. Next, integrating (31) we obtain the forms of $\omega_1, F(t,x)$

$$\begin{aligned} \omega_1 &= \nu C_1(R^3x + P(t)) - \textstyle\int R_1(t)dt, \\ F(t,x) &= 3\dot{R} + R^{-2}\dot{P}(t) - C_1^{-1}C_2. \end{aligned} \tag{34}$$

Here $P(t)$ is an arbitrary smooth function.

Hence we conclude that the corresponding solution with separated variables reads as

$$\begin{aligned} u &= Q(t,x,y)\exp\{\lambda_1 \int R_1(t)dt + \lambda_2 \int R_2(t)dt\}\exp\{\lambda_1\omega_1\}\varphi_2(\omega_2) \\ &= Q(t,x,y)\exp\{\lambda_2 \int R_2(t)dt\}\exp\{\lambda_1(\nu C_1(R^3x + P(t)))\}\varphi_2(\omega_2). \end{aligned}$$

Thus the function $R_1(t)$ does not enter into the solution with separated variables and, therefore, we can put $R_1 = 0$ in (34). Furthermore, within an equivalence transformation (8) we can choose $C_1 = \nu^{-1}, C_2 = 0$, thus getting

$$\omega_1 = R(t)^3x + P(t), \tag{35}$$

$$\omega_2 = R(t)y + 3\dot{R}(t)x + \dot{P}R(t)^{-2}. \tag{36}$$

Provided $L_2 \neq 0$, the forms of the functions $\omega_1, \omega_2$ is the same as those given in (35), (36).

Inserting (35), (36) into (29) and integrating by $y$ we get the form of the factor $Q(t,x,y)$

$$
\begin{aligned}
Q &= \exp\left\{\left(\frac{4\dot{R}-\nu R}{4\nu R}\right)y^2 + 3\left(\frac{\ddot{R}-R\dot{R}}{2\nu R}\right)xy + \frac{y}{2\nu R}\frac{d}{dt}\left(\frac{\dot{P}}{R^2}\right)\right. \\
&\quad \left. +M(t,x)\right\}.
\end{aligned} \tag{37}
$$

Substituting (37) into (30) we come to the following relation

$$
\begin{aligned}
&\frac{1}{\nu}\frac{d}{dt}\left(\frac{\dot{R}}{R}\right)y^2 + \frac{3}{2\nu}\frac{d}{dt}\left(\frac{\ddot{R}}{R}\right)xy + \frac{1}{2\nu}\dot{Z}y + M_t + \frac{1}{2\nu}\left(3\frac{\ddot{R}}{R}-k\right)y^2 + yM_x \\
&= \frac{\nu}{2} + 2\frac{\dot{R}}{R} + (\nu y + kx)\left(\left(\frac{2\dot{R}}{\nu R}-\frac{1}{2}\right)y + \frac{1}{2\nu}\left(3\frac{\ddot{R}}{R}-k\right)x + \frac{1}{2\nu}Z\right) \\
&\quad +\nu\left(\left(\frac{2\dot{R}}{\nu R}-\frac{1}{2}\right)y + \frac{1}{2\nu}\left(3\frac{\ddot{R}}{R}-k\right)x + \frac{1}{2\nu}Z\right)^2 + \nu S_0 R^2,
\end{aligned} \tag{38}
$$

where we use the notation

$$
Z(t) = R^{-1}\frac{d}{dt}\left(\frac{\dot{P}}{R^2}\right).
$$

Differentiating (38) three times with respect to $y$ yields $S_{0\omega_2\omega_2\omega_2} = 0$, whence

$$
S_0 = C_1\omega_2^2 + C_2\omega_2 + C_3,
$$

where $C_1, C_2, C_3$ are arbitrary constants. Next, differentiating (38) with respect to $y$ twice and with respect to $x$ once we get $M_{xxx} = 0$, or

$$
M = M_1(t)x^2 + M_2(t)x + M_3(t),
$$

where $M_1, M_2, M_3$ are arbitrary smooth functions.

At last, inserting the obtained expressions for $S_0, M$ into (38) and splitting by the variables $x, y$ we come to the following system of ODEs:

$$
\frac{\ddot{R}}{R} = 2\frac{\dot{R}^2}{R^2} + \frac{2\nu^2}{5}C_1R^4 - \frac{\nu^2}{10} + \frac{k}{5}, \tag{39}
$$

$$
M_1 = -\frac{3}{4\nu}\frac{\dddot{R}}{R} + \frac{15}{4\nu} + 3\nu C_1\dot{R}R^3 - \frac{k}{4}, \tag{40}
$$

$$\dot{M}_1 = \frac{9\ddot{R}^2}{4\nu R^2} + 9\nu C_1 R^2 \dot{R}^2 - \frac{k^2}{4\nu}, \tag{41}$$

$$M_2 = -\frac{1}{2\nu}\dot{Z} + \frac{2\dot{R}}{\nu R} Z + \nu C_2 R^3 + 2\nu R \dot{P} C_1, \tag{42}$$

$$\dot{M}_2 = \frac{3\ddot{R}}{2\nu R} Z + 3\nu R^2 \dot{R} C_2 + 6\nu \dot{R} \dot{P} C_1, \tag{43}$$

$$\dot{M}_3 = \frac{\nu}{2} + 2\frac{\dot{R}}{R} + \frac{1}{4\nu} Z^2 + \nu C_1 \frac{\dot{P}^2}{\dot{R}^2} + \nu C_2 \dot{P} + \nu C_3 R^2. \tag{44}$$

Differentiating (40) with respect to $t$ and subtracting the resulting equation from (41) yield the fourth-order ODE for the functions $R, P$

$$-\frac{R^{(IV)}}{R} + 6\frac{\dot{R}\,\dddot{R}}{R^3} + 2\frac{\ddot{R}^2}{R^2} - 10\frac{\dot{R}^2\ddot{R}}{R^3} + 4\nu^2 C_1 \ddot{R} R^3 + \frac{k^3}{3} = 0.$$

Reducing the order of the above ODE with the help of equation (39) and its first- and second-order differential consequences we arrive at the following relation:

$$\frac{4\nu^2}{25} C_1^2 R^8 = \frac{\nu^4}{100} + \frac{k^2}{25} - \frac{\nu^2 k}{25} - \frac{k^2}{9}. \tag{45}$$

If in (45) $C_1 \neq 0$, then in view of (39) $k = 0$. Provided, $C_1 = 0$, $k$ is a root of the quadratic equation

$$64k^2 + 36\nu^2 k - 9\nu^2 = 0,$$

whence $k = 3\nu^2/16$ or $k = -3\nu^2/4$.

Thus system of ODEs (39)–(44) is consistent only if the parameter $k$ takes one of three values $0,\ 3\nu^2/16,\ -3\nu^2/4$. Consequently, KE (2) has solutions with separated variables in the case considered (i.e., provided the system (4) takes the form (6)) only for the values of the parameter $k$ given above. This provides the proof of the first part of Theorem 1.

We examine the three possible cases $0,\ 3\nu^2/16,\ -3\nu^2/4$ separately.

**Case 1.** $k = 0$. Then the equality $R(t) = \pm 2^{-1/2} S_1^{-1/4} = \text{const}$ holds. We denote this constant as $r$. Next, it follows from (43) that $M_2 = m = \text{const}$. In view of these facts we get from (42) ODE for $P(t)$

$$-\dddot{P} + \nu^2 \dot{P} + 2\nu r^3(\nu S_2 r^3 - m) = 0$$

which general solutions reads

$$P(t) = C_4 \mathrm{e}^{\nu t} + C_5 \mathrm{e}^{-\nu t} + 2r^3(m\nu^{-1} - S_2 r^3) + C_6, \tag{46}$$

where $C_4, C_5, C_6$ are arbitrary constants.

A direct check shows that applying finite transformations from the symmetry group admitted by KE under $k = 0$ to the obtained solution with separated variables (3), (35), (36), (46) we can cancel $P(t)$.

Scaling when necessary $\omega_1, \omega_2$ in (35), (36) we can choose $r = 1$. Hence we get the equality $C_1 = \frac{1}{4}$. Summing up we conclude that the following relations hold:

$$\begin{aligned}
&Q = \exp\left(-\frac{y^2}{4} + \nu C_2 x + \nu\left(C_3 + \frac{1}{2}\right)t\right), \quad \omega_1 = x, \quad \omega_2 = y;\\
&\dot{\varphi}_0 = \nu\left(\lambda_1 - \left(C_3 + \frac{1}{2}\right)\right)\varphi_0,\\
&\dot{\varphi}_1 = \nu(\lambda_2 - C_2)\varphi_1, \quad \ddot{\varphi}_2 = \left(\frac{\omega_2^2}{4} + \lambda_2\omega_2 + \lambda_1 - \frac{1}{2}\right)\varphi_2.
\end{aligned}$$

Then the corresponding solution with separated variables reads as

$$u = \varphi_2 \exp\left(-\frac{y^2}{4} + \nu(\lambda_1 t + \lambda_2 x)\right).$$

Consequently, the constants $C_2$ and $C_3 + \frac{1}{2}$ do not enter the final form of the solution with separated variables. This means that we can put $C_2 = 0$ and $C_3 = -\frac{1}{2}$.

Thus we have proved the validity of the first part of Theorem 5.

**Cases 2,3.** $k = \frac{3\nu^2}{16}$ or $k = -\frac{3\nu^2}{4}$. In these cases we get from (39)

$$\frac{\ddot{R}}{R} - 2\left(\frac{\dot{R}}{R}\right)^2 = -a^2,$$

where

$$a = \begin{cases} \dfrac{\nu}{4}, & \text{under } k = \dfrac{3\nu^2}{16},\\ \dfrac{\nu}{2}, & \text{under } k = -\dfrac{3\nu^2}{4}. \end{cases}$$

Integrating the above ODEs yields

$$R(t) = (C_1 \sinh at + C_2 \cosh at)^{-1},$$

where $C_1, C_2$ are arbitrary constants.

Using shifts with respect to $t$ and the equivalence transformation (8) we get the four inequivalent forms of the function $R(t)$

$$R(t) = \frac{1}{\cosh at}, \quad R(t) = \frac{1}{\sinh at}, \quad R(t) = \exp\{\pm at\}.$$

Comparing (42) and the first-order differential consequence of (43) yields the second-order ODE for $Z(t) = \frac{d}{dt}\frac{\dot{P}}{R^2}$

$$-\ddot{Z} + 4\frac{\dot{R}}{R}\dot{Z} + \left(\frac{\ddot{R}}{R} - 4\frac{\dot{R}}{R}\right) Z = 0. \tag{47}$$

The general solution of this equation has the following structure:

$$Z(t) = C_1 Z_1(t) + C_2 Z_2(t),$$

where $C_1, C_2$ are integration constants. Hence we conclude that the function $P(t)$ is of the form

$$P(t) = C_1 P_1(t) + C_2 P_2(t) + C_3 P_3(t) + C_4 P_4(t), \tag{48}$$

where $C_3, C_4$ are integration constants.

On the other hand, if we apply to the solution with separated variables (3), (35), (36) with $P(t) = 0$ finite transformations from the symmetry group of KE under $k = \frac{3\nu^2}{16}$ or $k = -\frac{3\nu^2}{4}$, then we get an equivalent solution with separated variables such that $P(t)$ is of the form

$$P(t) = C_1' P_1'(t) + C_2' P_2'(t) + C_3' P_3'(t) + C_4' P_4'(t). \tag{49}$$

Here $C_1', \ldots, C_4'$ are arbitrary constants and the functions $P_1'(t), \ldots, P_4'(t)$ are linearly independent. Hence we conclude that due to the theorem on existence and uniqueness of the Cauchy problem for a fourth-order ODE (47) (considered as an equation for the function $P(t)$) the expressions on the right-hand sides of (48) and (49) coincides within the choice of constants

$C_i, C_i', i = 1, \ldots, 4$. Consequently, without loss of generality we can put $P(t) = 0$ in formulae (35), (36).

Using the reasonings analogous to those of Case 1 we can put $r = 1, C_2 = 0, C_3 = 0$. The second part of Theorem 6 is proved.

A similar analysis of the separability of KE into three ODEs (7) yields the proofs of the remaining assertions from Section 2.

# 4 Exact solutions

Remarkably, for the equation under study it is possible to give a complete account of solutions with separated variables. For the case when KE separates into three first-order ODEs (7), we get the following family of its exact solutions:

$$
\begin{aligned}
u \ = \ & \exp\Bigg\{\nu \int \left(\frac{f_1\lambda_1 + f_2\lambda_2}{\dot f_2 f_1 - \dot f_1 f_2}\right)^2 dt + \lambda_1 \frac{f_1 y - \dot f_1 x}{\dot f_2 f_1 - \dot f_1 f_2} + \lambda_2 \frac{f_2 y - \dot f_2 x}{\dot f_2 f_1 - \dot f_1 f_2} + \\
& + \left(-\frac{1}{4\nu}\frac{\ddot f_2 f_1 - \ddot f_1 f_2}{\dot f_2 f_1 - \dot f_1 f_2} - \frac{1}{4}\right) y^2 + \frac{1}{2\nu}\left(\frac{\ddot f_2 \dot f_1 - \ddot f_1 \dot f_2}{\dot f_2 f_1 - \dot f_1 f_2} - k\right) xy + \\
& + \left(\frac{1}{4\nu}\frac{\dddot f_2\, \dot f_1 - \dddot f_1\, \dot f_2}{\dot f_2 f_1 - \dot f_1 f_2} - \frac{k}{4}\right) x^2 - \frac{1}{2}\ln|\dot f_2 f_1 - \dot f_1 f_2| + \frac{\nu}{2} t\Bigg\},
\end{aligned}
$$

where $k, f_1(t), f_2(t)$ are given by the corresponding formulae from Theorems 2–5.

Next, for the case when KE separates into three ODEs of the form (6) we obtain the following families of its exact solutions:

1. $k = 0$ (this case has been considered in Theorem 5)

$$
\begin{aligned}
u \ = \ & (y + 2\lambda_2)^{-\frac{1}{2}} \exp\left(-\frac{y^2}{4} + \nu(\lambda_1 t + \lambda_2 x)\right) \\
& \times Y\left(\frac{\lambda_2^2}{2} - \frac{\lambda_1}{2} + \frac{1}{4}, \frac{1}{4}, \frac{1}{2}(y + 2\lambda_2)^2\right),
\end{aligned}
$$

   where $Y$ is the Whittaker function.

2. $k = 3\nu^2/16$ or $k = -3\nu^2/4$

$$\begin{aligned} u &= \exp\Bigg\{\nu\lambda_1 \int R^2 dt + \nu\lambda_2 R^3 x \left(\frac{\dot R}{\nu R} - \frac{1}{4}\right) y^2 + \frac{1}{2\nu}\left(3\frac{\ddot R}{R} - k\right) xy \\ &+ \left(-\frac{3}{4\nu}\frac{\ddot R}{R} + \frac{15\dot R \ddot R}{4\nu R^2} - \frac{k}{4}\right) x^2 + \frac{\nu}{2} t + 2\ln R\Bigg\}\{\lambda_2(Ry + 3\dot R x) \\ &+ \lambda_1\}^{\frac{1}{2}} Z_{\frac{1}{3}}\left(\frac{2}{3\lambda_2}(\lambda_2(Ry + 3\dot R x) + \lambda_1)^{\frac{3}{2}}\right), \end{aligned}$$

where $R$ is given by the corresponding formula from Theorem 6 and $Z_{\frac{1}{3}}$ is the cylindric function.

Note that the above obtained families of exact solutions of KE contain two continuous parameters $\lambda_1, \lambda_2$. These parameters have the meaning of eigenvalues of two commuting symmetry operators of KE, while the corresponding solution with separated variables is the eigenfunction of these operators. Provided some appropriate boundary and initial conditions are imposed, the parameters become discrete and thus we get a basis for expanding sufficiently smooth solutions of KE into series.

# 5 Conclusions

It is a remarkable feature of the Kramers equation (2) that a classical problem of variable separation can be solved in a full generality. The results obtained on this way are in good correspondence with the ones on symmetry classification of KEs of the form (2). As it follows from the papers [5, 6], the cases $k = 3\nu^2/16, k = -3\nu^2/4$ are distinguished by the fact that the corresponding KEs (2) admit the most extensive symmetry groups. For these choices of $k$ KE (2) is invariant with respect to eight-parameter Lie transformation groups, while for all other value of the parameter $k$ the maximal group is six-parameter.

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