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COMPATIBILITY CRITERIA AND GENERAL SOLUTION OF THE NONLINEAR D'ALEMBERT-EIKONAL SYSTEM

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We study integrability of a system of nonlinear partial differential equations consisting of the nonlinear d'Alembert equation $\Box u = F(u)$ and nonlinear eikonal equation $u_{x_\mu} u_{x^\mu} = G(u)$ in the complex Minkowski space $R(1,3)$. A method suggested makes it possible to establish necessary and sufficient compatibility conditions and construct general solution of the d'Alembert-eikonal system for all cases when it is compatible. The results obtained can be applied, in particular, to construct principally new (non-Lie, non-similarity) solutions of the nonlinear d'Alembert and Dirac equations.

Speaking about the d'Alembert-eikonal system we mean the system of two nonlinear partial differential equations (PDEs) which consists of the nonlinear d'Alembert and eikonal equations taken together

$$\begin{cases} \Box_n u \equiv \partial_\mu \partial^\mu u = F(u)\,, \\ (\partial_\mu u)(\partial^\mu u) = G(u)\,. \end{cases} \tag{1}$$

Here $u = u(x) \in C^2(C^n, C^1)$; $\partial_\mu = \partial/\partial x_\mu$, $\mu = \overline{0, n-1}$; $F(u)$, $G(u)$ are some smooth functions. Hereafter summation over the repeated indices in the pseudo-Euclidean space with the metric tensor $g_{\mu\nu} = \delta_{\mu\nu} \times (1, -1, -1, \ldots, -1)$ is understood.

The three-dimensional elliptic analogue of system (1) with $F = G = 0$

$$\begin{cases} u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} = 0\,, \\ u_{x_1}^2 + u_{x_2}^2 + u_{x_3}^2 = 0 \end{cases} \tag{2}$$

was studied by Jacobi, who had constructed a wide class of its exact solutions. Later on, the general solution of (2) was constructed by Smirnov and Sobolev. Some important results on exact solutions of system of PDEs (1) were obtained by Bateman, Cartan,

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Erugin and Collins. In particular, Collins [1] using differential geometry technique has obtained the general solution of the three-dimensional d'Alembert-eikonal system.

We have developed a technique making it possible to study compatibility of the overdetermined system of PDEs (1) and to construct its general solution provided $n = 4$. Here we present only principal results omitting the proofs (they can be found in [2–5]).

Evidently, provided $G(u) \neq 0$ system (1) is reduced to the form

$$\begin{cases} \Box_n u' = \tilde{F}(u')\,, \\ (\partial_\mu u')(\partial^\mu u') = 1\,. \end{cases}$$

by the change of the dependent variable

$$u \to u' = \int^{u} \Big(G(\tau)\Big)^{-1/2} d\tau\,.$$

Consequently, instead of (1) we can study system of PDEs of the form

$$\begin{cases} \Box_n u = F(u)\,, \\ (\partial_\mu u)(\partial^\mu u) = \lambda\,, \end{cases} \tag{3}$$

where $\lambda = 0,\ 1$.

THEOREM 1. (necessary compatibility condition for the system (3))
Let the system of PDEs (3) be compatible. Then the function $F(u)$ is given by the formula

$$F(u) = \lambda \frac{d}{du} \ln\, P(u) \tag{4}$$

and what is more

$$\frac{d^{n+1}P}{du^{n+1}} = 0\,.$$

It occurs that in the case $\lambda = 0$ the formula (4) gives necessary and sufficient compatibility condition for the system (3).

THEOREM 2. *System (3) with $\lambda = 0$ is compatible if and only if $F(u) = 0$.*

But in the case $\lambda = 1$ the condition (4) is not sufficient for the d'Alembert-eikonal system to be compatible under arbitrary $n \in \mathbb{N}$. We have succeeded in obtaining compatibility criteria for system (3) in the four-dimensional case ($n = 4$).

THEOREM 3. *The system of PDEs*

$$\begin{cases} u_{x_0x_0} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = F(u)\,, \\ u_{x_0}^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 = \lambda \end{cases} \tag{5}$$

is compatible if and only if the function $F(u)$ has the form

$$F(u) = \frac{\lambda N}{(u + C)}\,, \tag{6}$$

where C is an arbitrary complex constant, $N = 0, 1, 2, 3$.

Note: We guess that the formula (6) with $N = \overline{0, n-1}$ gives compatibility criteria for system (3) under arbitrary $n \in \mathbb{N}$ but so far we cannot prove such assertion.

Thus, a compatible system of PDEs (1) with $n = 4$ is equivalent to the following one:

$$\begin{cases} u_{x_0x_0} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = \dfrac{\lambda N}{u}\,, \\ u_{x_0}^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 = \lambda\,, \end{cases} \tag{7}$$

where $\lambda = 0$ or $\lambda = 1$.

The general solution of the system of nonlinear PDEs (7) is given by the following assertions.

THEOREM 4. *General solution of the system of nonlinear PDEs*

$$\begin{cases} u_{x_0x_0} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0\,, \\ u_{x_0}^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 = 0 \end{cases}$$

is given by the formula

$$A_\mu(u,\,\tau)x^\mu + A(u,\,\tau) = 0\,,$$

where $\tau = \tau(x, u)$ is determined in implicit way

$$B_\mu(u,\tau)x^\mu + B(u,\,\tau) = 0$$

and $A_\mu(u,\,\tau)$, $B_\mu(u,\,\tau)$, $A(u,\,\tau)$, $B(u,\,\tau)$ are arbitrary complex-valued functions satisfying the conditions

$$A_\mu A^\mu = A_\mu B^\mu = B_\mu B^\mu = 0\,, \qquad B_\mu \frac{\partial A^\mu}{\partial \tau} = 0\,.$$

THEOREM 5. *The general solution of system of nonlinear PDEs*

$$\begin{cases} u_{x_0x_0} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = \dfrac{3}{u}\,, \\ u_{x_0}^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 = 1 \end{cases}$$

is given by the formula

$$u^2 = \Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big),$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + A_\mu(\tau)\Big)B^\mu(\tau) = 0$$

and the functions $A_\mu(t)$, $B_\mu(\tau)$ satisfy the relations

$$\dot{A}_\mu B^\mu = 0\,, \qquad B_\mu B^\mu = 0\,.$$

Hereafter an overdot denotes differentiation with respect to the corresponding argument.

THEOREM 6. *General solution of system of nonlinear PDEs*

$$\begin{cases} u_{x_0x_0} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = \dfrac{2}{u}\,, \\ u^2_{x_0} - u^2_{x_1} - u^2_{x_2} - u^2_{x_3} = 1 \end{cases}$$

is given by the following formulae:

1. $$u^2 = \Big(x_\mu + R_\mu(\tau)\Big)\Big(x^\mu + R^\mu(\tau)\Big) + \Big[B_\mu(\tau)\Big(x^\mu + R^\mu(\tau)\Big)\Big]^2,$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + R_\mu(\tau)\Big)\dot{B}^\mu(\tau) = 0$$

and R_μ, B_μ are arbitrary smooth functions satisfying the relations

$$\dot{R}_\mu = T(\tau)B_\mu\,, \qquad B_\mu B^\mu = -1\,, \qquad \dot{B}_\mu\dot{B}^\mu = 0$$

with arbitrary $T(\tau)$;

2. $$u^2 = \Big(x_\mu + R_\mu(\tau)\Big)\Big(x^\mu + R^\mu(\tau)\Big) + \Big[d_\mu\Big(x^\mu + R^\mu(\tau)\Big)\Big]^2,$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + R_\mu(\tau)\Big)\dot{R}^\mu(\tau) + \Big(x_\mu + R_\mu(\tau)\Big)d^\mu d_\nu \dot{R}^\nu(\tau) = 0\,,$$

$d_\mu = const$, $d_\mu d^\mu = -1$; R_μ are arbitrary functions satisfying the relation

$$\dot{R}_\mu\dot{R}^\mu + (d_\mu\dot{R}^\mu)^2 = 0\,.$$

THEOREM 7. *The general solution of system of nonlinear PDEs*

$$\begin{cases} u_{x_0x_0} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = \dfrac{1}{u}\,, \\ u^2_{x_0} - u^2_{x_1} - u^2_{x_2} - u^2_{x_3} = 1 \end{cases}$$

is given by the formula

$$u^2 = \Big(a_\mu x_\mu + h_1(\theta_\mu x^\mu)\Big)^2 + \Big(b_\mu x^\mu + h_2(\theta_\mu x^\mu)\Big)^2,$$

where h_1, h_2 are arbitrary smooth functions; a_μ, b_μ, θ_μ are arbitrary complex constants satisfying the relations

$$a_\mu a^\mu = 1\,, \qquad b_\mu b^\mu = -1\,, \qquad a_\mu b^\mu = a_\mu\theta^\mu = b_\mu\theta^\mu = \theta_\mu\theta^\mu = 0\,.$$

THEOREM 8. *The general solution of the system of nonlinear PDEs*

$$\begin{cases} u_{x_0x_0} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = 0\,, \\ u^2_{x_0} - u^2_{x_1} - u^2_{x_2} - u^2_{x_3} = 1 \end{cases}$$

is given by the formula

$$u = A_\mu(\tau)x^\mu + R_1(\tau)\,,$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$B_\mu(\tau)x^\mu + R_2(\tau) = 0$$

and $A_\mu(\tau)$, $B_\mu(\tau)$, $R_1(\tau)$, $R_2(\tau)$ are arbitrary smooth functions satisfying the relations

$$A_\mu A^\mu = 1\,, \qquad \dot{A}_\mu B^\mu = 0\,, \qquad A_\mu B^\mu = 0\,, \qquad B_\mu B^\mu = 0\,.$$

The principal idea of the proof of the above theorems is a linearization of the eikonal equation $u^2_{x_0} - u^2_{x_1} - u^2_{x_2} - u^2_{x_3} = 1$ by a suitable contact transformation

$$x'_\mu = f_\mu(x,\,u,\,\overset{1}{u})\,, \qquad u' = g(x,\,u,\,\overset{1}{u})\,, \qquad u'_{x_\mu} = f_\mu(x,\,u,\,\overset{1}{u})\,, \tag{8}$$

where $\overset{1}{u} = \{\partial u/\partial x_\mu,\ \mu = \overline{0,3}\}$.

Then, integrating the linear PDE and substituting the result obtained into the nonlinear d'Alembert equation we get an overdetermined system of PDEs having three independent variables. This system is transformed to integrable form by a sequence of changes of variables (8).

The d'Alembert-eikonal system has various applications in mathematical physics. We will demonstrate how its general solution can be used to construct broad classes of exact solutions of the nonlinear d'Alembert

$$U_{x_0x_0} - U_{x_1x_1} - U_{x_2x_2} - U_{x_3x_3} = R(U) \tag{9}$$

and Dirac equations

$$i\gamma_\mu\psi_{x_\mu} - R(\bar{\psi}\psi)\psi = 0\,. \tag{10}$$

Let $u = u(x)$ be arbitrary solution of (5). Then, substitution of the Ansatz

$$U(x) = \varphi(u) \tag{11}$$

into Eq. (9) reduces it to ordinary differential equation for the function $\varphi(u)$

$$\lambda\frac{d^2\varphi}{du^2} + F(u)\frac{d\varphi}{du} = R(\varphi)\,.$$

Solving the above equation and substituting the result obtained into the Ansatz (11) we get a class of exact solutions of the nonlinear d'Alembert equation containing arbitrary functions.

Similarly, substitution of the Ansatz [2]

$$\psi(x) = \Big(f(u) + i\gamma_\mu u_{x_\mu} g(u)\Big)\chi\,, \tag{12}$$

where χ is an arbitrary four-component constant column and $u = u(x)$ is an arbitrary solution of (5), into Eq. (10) reduces it to the system of two ordinary differential equations

$$\frac{df}{du} = R(f^2 + \lambda g^2)\,, \qquad \lambda\frac{dg}{du} + F(u)g = -R(f^2 + \lambda g^2)\,.$$

Each particular solution of the above system gives rise to a class of exact solution of the nonlinear Dirac equation (10) containing arbitrary functions.

Let us emphasize that the above presented approach to the reduction of the nonlinear d'Alembert equation is more general than the symmetry reduction procedure [6]. In the framework of the said procedure a solution of Eq. (9) is looked for in the form (11), where $u(x)$ is an invariant of a three-parameter subgroup of the symmetry group of the equation under study. Since maximal symmetry group of the nonlinear d'Alembert equation with arbitrary $R(U)$ is the ten-parameter Poincaré group $P(1,3)$, $u(x)$ should be an invariant of some three-dimensional subgroup of the group $P(1,3)$. By easy computation one can check that all invariants $u(x)$ of three-parameter subgroups of $P(1,3)$ (which were obtained in [7]) satisfy the d'Alembert-eikonal system (1) with $n = 4$. But the inverse assertion is not true. Not every solution of the system of PDEs (5) is an invariant of some three-parameter subgroup of the Poincaré group (for more detail, see [5]).

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