**R.Z. Zhdanov[*], I.M. Tsyfra[†], and R.O. Popovych[‡]**

# A precise definition of reduction of partial differential equations

Institute of Mathematics of NAS of Ukraine,
3 Tereshchenkivska Street, 252004 Kyiv, Ukraine

E-mail:
[*]renat@imath.kiev.ua,
[†]ivan@apmat.freenet.kiev.ua,
[‡]roman@apmat.freenet.kiev.ua

**Abstract**

We give a comprehensive analysis of interrelations between the basic concepts of the modern theory of symmetry (classical and non-classical) reductions of partial differential equations. Using the introduced definition of reduction of differential equations we establish equivalence of the non-classical (conditional symmetry) and direct (Ansatz) approaches to reduction of partial differential equations. As an illustration we give an example of non-classical reduction of the nonlinear wave equation in $1+3$ dimensions. The conditional symmetry approach when applied to the equation in question yields a number of non-Lie reductions which are far-reaching generalization of the well-known symmetry reductions of the nonlinear wave equations.

Running title:

**A precise definition of reduction**

Corresponding author:

Prof. Renat Z. Zhdanov
Institute of Mathematics of NAS of Ukraine,
3 Tereshchenkivska Street, 252004 Kyiv, Ukraine
E-mail: renat@imath.kiev.ua

# 1 Introduction

The notion of a non-classical symmetry was introduced by Bluman and Cole as early as in 1969 [1]. However, non-trivial examples of non-classical symmetries for nonlinear partial differential equations (PDEs) appeared much later in the papers by Olver & Rosenau [2, 3] and by Fushchych & Tsyfra [4]. These papers together with the ones by Fushchych & Zhdanov, [5], Clarkson & Kruskal [6], Levi & Winternitz [7] gave a start to an intensive search for non-classical symmetries of a wide range of nonlinear differential equations. Following the suggestion by Fushchych [8]–[10] we call this kind of non-Lie symmetries conditional symmetries (CSs).

The vast majority of the papers devoted to constructing CSs of nonlinear PDEs consider equations having two independent variables only. This is explained by the fact that the determining PDEs for CSs are nonlinear and have the dimension which is equal to the sum of the number of dependent and independent variables of the PDE under study. That is why, there is no systematic general procedure for obtaining CSs of multi-dimensional nonlinear PDEs. Constructing CS for a specific multi-dimensional differential equation requires preliminary guesswork enabling one to reduce the dimension of the system of determining PDEs. In the papers [5], [11]–[18] devoted to studying CSs of multi-dimensional nonlinear equations of quantum field theory (wave, Dirac, Lèvi-Leblond and $SU(2)$ Yang-Mills equations) we developed an efficient approach based on fixing a special Ansatz for a conditionally-invariant solution to be found. The underlying idea for choosing such an Ansatz was a proper use of Lie symmetry properties of the equation under consideration (a complete account of the results obtained in this way can also be found in the monograph [19]).

There exists a number of different approaches to utilizing CSs of PDEs in order to reduce these to equations with fewer number of independent variables. However, with all the differences between these methods they can be classified into two major groups. The first one is composed by the methods that are close to the traditional Lie approach and can be regarded as the 'infinitesimal methods for finding CS' [1]–[5], [7], [20]–[25]. The central role is played by infinitesimal CSs within the class of first-order differential operators. Given an operator of conditional symmetry, we can construct an Ansatz reducing the dimension of PDE under study. The second group of the methods are the 'direct' ones [6], [11]–[18], [26]–[28] (see also [29] and the references therein) that goes up, probably, to the papers by Fourier and Euler devoted to finding particular solutions of the two-dimensional heat equation with the help of a substitution of a special (separated) form. Namely, the methods in question are based on fixing a special Ansatz for a solution to be found. As a rule, these Ansätze contain arbitrary functions which are to be so chosen that some reduction requirements must be met. One of the principal motivations for writing the present article is studying interrelations between these approaches. A necessary ingredient of such study is a precise mathematical definition of reduction of PDEs. We attempt to give this definition which is the core result of the paper. Based on this definition is our proof of equivalence of the above two approaches to reduction of PDEs provided some reasonable restrictions are met (see, also [30]). The present paper is a natural continuation of our earlier papers [20, 31], where some ideas presented below were indicated. We present these ideas in a rigorous mathematical form which, as we believe, should give new insights into the theory of conditional symmetries of PDEs.

## 2 Ansätze and involutive sets of operators

Consider a family of first-order differential operators in the variables $x=(x_1,\ldots,x_n)$, $u$

$$Q_a=\sum_{i=1}^{n}\xi_{ai}(x,u)\frac{\partial}{\partial x_i}+\eta_a(x,u)\frac{\partial}{\partial u},\quad a=1,\ldots,m, \tag{1}$$

where $\xi_{ai},\eta_a$ are some continuously differentiable functions in an open domain in $\mathbb{R}^{n+1}$, $m<n$. The variable $u$ is regarded as dependent, i.e., it corresponds to the function $u=u(x)$. In a sequel, we suppose that the conditions of the theorem about implicit function are fulfilled, wherever applicable.

**Definition 1.** Family of first-order differential operators (1) is called involutive if there exist smooth functions $\mu_{ab}^c(x,u)$, $a,b,c=1,\ldots,m$, such that

$$[Q_a,\,Q_b]=\sum_{c=1}^{m}\mu_{ab}^cQ_c,\quad a,b=1,\ldots,m. \tag{2}$$

The simplest example of an involutive family of operators is given by first-order differential operators forming a Lie algebra. In this case $\mu_{ab}^c=\text{const}$, $a,b,c=1,\ldots,m$ are called structure constants of the Lie algebra. This implication has far-reaching consequences in the modern theory of non-Lie reductions of PDEs. This is explained by the fact that involutive families of operators of the form (1) play the same role in the theory of non-classical symmetry reductions of PDEs having non-trivial conditional symmetries as that played by finite-dimensional Lie algebras in the theory of symmetry reductions of invariant PDEs.

In a sequel we consider involutive families of operators (1) satisfying an additional constraint

$$\mathrm{rank}\,\|\xi_{ai}(x,u)\|_{a=1}^{m}{}_{i=1}^{n} = \mathrm{rank}\,\|\xi_{ai}(x,u),\eta(x,u)\|_{a=1}^{m}{}_{i=1}^{n} = m. \tag{3}$$

By direct computation we check that, if operators (1) form an involutive family, then the family of differential operators

$$Q'_a = \sum_{b=1}^{m} \lambda_{ab}(x,u)Q_b, \quad \det\|\lambda_{ab}(x,u)\|_{a,b=1}^{m} \neq 0 \tag{4}$$

is also involutive (see, also [20]). Furthermore the involutive family (4) is easily seen to obey the condition (3).

Provided the relation of the form (4) holds, two involutive families of operators $\{Q_a\}$ and $\{Q'_a\}$ are called equivalent. This equivalence relation splits the set of involutive families of $m$ operators into equivalence classes forming the quotient set. We denote this set as $\mathcal{I}$.

It is a common knowledge that conditions (2) are sufficient for the system of PDEs

$$Y_a(x,u,\underset{1}{u}) = \sum_{i=1}^{n} \xi_{ai}(x,u)\frac{\partial u}{\partial x_i} - \eta_a(x,u) = 0, \quad a = 1,\ldots,m \tag{5}$$

to be compatible (the Frobenius theorem [33]). Its general solution can be locally represented in the form

$$F(W_1,\,\ldots,W_{n+1-m}) = 0, \tag{6}$$

where $F$ is an arbitrary smooth function of the variables $W_j$, $W_j = W_j(x,u)$, $j = 1,\ldots,n+1-m$ are functionally-independent first integrals of system of PDEs (5).

Due to constraint (3) there exists a first integral $W_k(x,u)$ such that the condition $\partial W_k/\partial u \neq 0$ holds locally, since otherwise integrals $W_1$, $W_2$, …, $W_{n+1-m}$ would be functionally-dependent.

Changing, if necessary, enumeration, we can put $k = 1$. Solving (6) with respect to $W_1$ and introducing the notations

$$\omega(x,u) = W_1(x,u), \quad \omega_j(x,u) = W_{j+1}(x,u), \quad j = 1,\ldots,n-m$$

we get the following expression:

$$\omega(x,u) = \varphi(\omega_1(x,u),\ldots,\omega_{n-m}(x,u)), \tag{7}$$

where $\varphi$ is an arbitrary smooth function of the variables $\omega_1, \ldots, \omega_{n-m}$.

**Definition 2.** We call an expression of the form (7), where $\varphi$ is an arbitrary smooth function, $\omega(x,u)$, $\omega_1(x,u)$, $\ldots$, $\omega_{n-m}(x,u)$ are functionally-independent and $\partial\omega/\partial u \neq 0$, an Ansatz for the field $u = u(x)$.

**Lemma 1** *There is one-to-one correspondence between the set of Ansätze for the field $u = u(x)$ and the elements of the space $\mathcal{I}$.*

Proof. While constructing the general solution of system (5) we have shown that each involutive family obeying (3) gives rise to the Ansatz of the form (7). Furthermore, by construction equivalent involutive families of operators have the same set of functionally-independent first integrals. Hence we conclude that each element of $\mathcal{I}$ corresponds to one and only one Ansatz (7).

Let us prove the inverse, namely, that each Ansatz (7) corresponds to one and only one element of the space $\mathcal{I}$. Choose the functions $\theta_a(x,u), a = 1,\ldots,m$ so that the expressions

$$\theta_a(x,u), \quad \omega(x,u), \quad \omega_j(x,u), \quad a = 1,\ldots,m,\ j = 1,\ldots,n-m \tag{8}$$

are functionally-independent. Then the functions (8) form the new coordinate system in the space of variables $x, u$

$$\begin{aligned} &y_a = \theta_a(x,u), \quad z_j = \omega_j(x,u), \quad v = \omega(x,u), \\ &a = 1,\ldots,m, \quad j = 1,\ldots,n-m. \end{aligned} \tag{9}$$

Rewriting (7) in the new variables $y, z, v$ we arrive at the following expression:

$$v = \varphi(z_1,\ldots,z_{n-m}). \tag{10}$$

Evidently, the formula (10) give the general solution of the system of PDEs $\partial v/\partial y_a = 0$, $a = 1,\ldots,m$. The operators $Q_a = \partial/\partial y_a, a = 1,\ldots,m$ form an involutive family (since they commute) and fulfill the condition (3). These properties are preserved after rewriting the operators $Q_a$ in the initial variables $x, u$.

Thus we have constructed an involutive family of operators which corresponds to a given Ansatz for the field $u = u(x)$. However, this correspondence is not one-to-one, since the choice of the functions $\theta_a(x,u)$ is ambiguous. Let us show that choosing another set of functions $\chi_1(x,u),\ldots,\chi_m(x,u)$ will lead to an involutive family which is equivalent to the above obtained involutive family.

Indeed, consider the transformation of variables

$$y'_i = \chi_a(x,u), \quad z'_j = \omega_j(x,u), \quad v' = \omega(x,u), \qquad (11)$$
$$a = 1,\ldots,m, \quad j = 1,\ldots,n-m$$

which reduces the initial Ansatz to become (10). Comparing (9) and (11) we conclude that the map relating the coordinate systems $y, z, v$ and $y', z', v'$ is of the form

$$y'_a = F_a(y,z,v), \quad z'_j = z_j, \quad v' = v$$

with $a = 1,\ldots,m$, $j = 1,\ldots,n-m$. Consequently, the operators $\partial/\partial y_a$ after being rewritten in the new variables $y', z', v'$ read

$$\frac{\partial}{\partial y_a} = \sum_{b=1}^{m} \frac{\partial y'_b}{\partial y_a}\frac{\partial}{\partial y'_b} = \sum_{b=1}^{m} \frac{\partial F_b}{\partial y_a}\frac{\partial}{\partial y'_b} = \sum_{b=1}^{m} \frac{\partial F_b}{\partial y_a} Q'_b, \quad a = 1,\ldots,m.$$

Since $\det \|\partial F_b/\partial y_a\|_{a,b=1}^{m} \neq 0$, hence it follows that the involutive families $Q_a = \partial/\partial y_a$ and $Q'_a = \partial/\partial y'_a$ are equivalent.

Thus each involutive family that corresponds to a fixed Ansatz for the field $u = u(x)$ belongs to the same equivalence class. This is the same as what was to be proved. $\rhd$

# 3 Conditional symmetry of PDEs

Consider PDE of the form

$$L(x, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \qquad (12)$$

where $x = (x_1, x_2, \ldots, x_n)$, $u = u(x)$ is a sufficiently smooth function and the symbol $\underset{s}{u}$ stands for the set of partial derivatives of the function $u(x)$ of the order $s$, i.e.,

$$\underset{s}{u} = \left\{ u_{i_1\ldots i_s} \stackrel{\text{def}}{=} \frac{\partial^s u}{\partial x_{i_1}\ldots\partial x_{i_s}},\ 1 \leq i_1 \leq n,\ \ldots,\ 1 \leq i_s \leq n \right\}.$$

Within the local approach (used throughout the paper) PDE (12) is treated as an algebraic equation in the jet space $J^{(r)}$ of the order $r$. Then $L$ is a smooth function from $\mathcal{D}$ into $\mathbb{R}$, where $\mathcal{D}$ is an open domain in $J^{(r)}$.

Denote the manifold defined by the equation $L = 0$ in $J^{(r)}$ by $\mathcal{L}$, the set of all differential consequences of the system of PDEs (5) of the order not higher than $r-1$ (we remind that $r$ is the order of the initial equation (12)) by the symbol $M$ and the corresponding manifold in $J^{(r)}$ by $\mathcal{M}$.

The most widely used definition of conditional invariance is the following one.

**Definition 3.** PDE (12) is conditionally-invariant with respect to involutive family of operators (1) if the relation

$$\underset{(r)}{Q_a} L \Big|_{\mathcal{L}\cap\mathcal{M}} = 0 \tag{13}$$

holds $\forall a = 1, \ldots, m$. Here the symbol $\underset{(r)}{Q}$ stands for the $r$th prolongation of the operator $Q_a$.

This definition is very useful when computing CSs for a specific PDE. However, for theoretical considerations it is preferable to utilize the alternative definition of conditional invariance given below.

**Definition 4.** PDE (12) is conditionally-invariant with respect to involutive family of operators (1) if the relation

$$\underset{(r)}{Q_a} \Lambda \Big|_{\mathcal{L}\cap\mathcal{M}} = 0, \quad \text{where} \quad \Lambda = L \Big|_{\mathcal{M}}, \tag{14}$$

holds $\forall a = 1, \ldots, m$. It will be shown below that Definitions 3 and 4 are equivalent. Note that there are some other ways to define CS [1, 2, 7, 34], however Definition 4 is the most convenient for the purposes of this paper.

**Note.** Definitions 3, 4 make sense provided $\mathcal{L}\cap\mathcal{M} \neq \emptyset$. If this is not the case, namely, if there exists an involutive family such that $\mathcal{L}\cap\mathcal{M} = \emptyset$, then we suppose by definition that PDE (12) is conditionally invariant with respect to this family.

**Lemma 2** *Let system of PDEs (12) be conditionally-invariant under involutive family of differential operators (1). Then, it is conditionally-invariant under involutive family (4) with arbitrary smooth functions* $\lambda_{ab}$.

**Proof.** To prove the lemma we use the special representation for the coefficients of the $s$th prolongation of a first-order operator $Q$ given in [32]:

$$\underset{(s)}{Q} = Q + \sum_{k=1}^{s} \sum_{i_1,\ldots,i_k=1}^{n} \eta_{i_1\ldots i_k} \frac{\partial}{\partial u_{i_1\ldots i_k}} \qquad \text{if} \qquad Q = \sum_{i=1}^{n} \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u},$$

where

$$\eta_{i_1\ldots i_k} = D_{i_1}\ldots D_{i_k}\Big(\eta - \sum_{i=1}^{n}\xi_i u_i\Big) + \xi_i u_{i_1\ldots i_k i},$$
$$i_1, \ldots, i_k = 1, \ldots, n, \quad k = 1, \ldots, s$$

and

$$D_i = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + \sum_{p=1}^{\infty} \sum_{i_1,\ldots,i_p=1}^{n} u_{i_1\ldots i_p i} \frac{\partial}{\partial u_{i_1\ldots i_k}}, \quad i = 1, \ldots, n$$

is a total differentiation operator with respect to the variable $x_i$.

Using the above identity yields the chain of equations that correspond to condition (13) for the operators (4):

$$\left. \underset{(r)}{Q'_a} L \right|_{\mathcal{L}\cap\mathcal{M}} = \left. \left( \sum_{b=1}^{m} \lambda_{ab}(x,u) \underset{(r)}{Q_b} L \right) \right|_{\mathcal{L}\cap\mathcal{M}} = \sum_{b=1}^{m} \lambda_{ab}(x,u) \left( \left. \underset{(r)}{Q_b} L \right|_{\mathcal{L}\cap\mathcal{M}} \right) = 0. \quad (15)$$

Evidently, the same arguments apply if we use Definition 4. The chain of equations analogous to the above equations (15) is obtained, where one should replace $L$ by $\Lambda$. The lemma is proved. $\triangleright$

One of the important consequences of the above lemma is that while studying conditional symmetry of PDEs we can restrict our considerations to elements of the quotient space $\mathcal{I}$. This enables choosing the most simple representative of each equivalence class in the way described below.

Let (1) be an involutive family of differential operators satisfying condition (3). Then it is possible to choose the functions $\lambda_{ab}(x,u)$ and, if it is necessary, to change enumeration of the variables $x_1, \ldots, x_n$ in such a way that operators (4) take the form

$$Q'_a = \sum_{b=1}^{m} \lambda_{ab} \left( \sum_{i=1}^{n} \xi_{bi} \frac{\partial}{\partial x_i} + \eta_b \frac{\partial}{\partial u} \right) = \frac{\partial}{\partial x_a} + \sum_{j=m+1}^{n} \xi'_{aj} \frac{\partial}{\partial x_j} + \eta'_a \frac{\partial}{\partial u},$$
$$a = 1, \ldots, m.$$

Since the family of operators $Q'_a,\ a = 1, \ldots, m$ is also involutive, there exist functions $\tilde{\mu}^c_{ab}(x,u)$ such that

$$[Q'_a,\, Q'_b] = \sum_{c=1}^{m} \tilde{\mu}^c_{ab} Q'_c, \quad a, b = 1, \ldots, m.$$

Computing commutators on the left-hand sides of the above equalities and equating coefficients of the linearly independent differential operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$ we have $\tilde{\mu}^c_{ab} = 0,\ a, b, c = 1, \ldots, m$. Consequently, operators $Q'_a$ form a commutative Lie algebra. Hence, we conclude that there is a local coordinate system (9) such that the operators $Q'_a$ after being rewritten in the variables $y, z, v$ read

$$Q'_a = \frac{\partial}{\partial y_a}, \quad a = 1, \ldots, m. \quad (16)$$

Consequently, without loss of generality we may consider commuting families of operators. This fact simplify calculations, since the latter can always be represented in the form (16).

**Lemma 3** *Relation (14) holds true if and only if relation (13) holds true.*

**Proof.** It suffices to consider the case $\mathcal{L}\cap\mathcal{M}\neq\emptyset$. Let us fix an arbitrary point $\mathbf{j}^0=(x^0,u^0,\underset{1}{u}^0,\ldots,\underset{r}{u}^0)\in J^{(r)}:\ \mathbf{j}^0\in\mathcal{L}\cap\mathcal{M}$.

As established above we can suppose without loss of generality that the operators $Q_a$ commute. Choosing an appropriate coordinate transformation (9) in a neighborhood of $(x^0,u^0)$ we reduce them to become $Q_a=\partial/\partial y_a$. Now the manifold $\mathcal{M}$ is determined by the following set of $N$ algebraic equations in the space of variables $y,z,\underset{1}{v},\ldots,\underset{r}{v}$:

$$\mathcal{M}=\left\{(y,z,\underset{1}{v},\ldots,\underset{r}{v})\ \middle|\ \begin{array}{c}\forall s=1,\ldots,r,\ \forall i_1,\ldots,i_s=1,\ldots,n\\ (\exists k=1,\ldots,s:\ i_k\le m)\end{array}:\ v_{i_1\ldots i_s}=0\right\},$$

where the variable $v_{i_1\ldots i_s}$ of the jet space $J^{(r)}$ correspond to the derivative $\partial^s v/(\partial t_{i_1}\ldots\partial t_{i_s})$, $t_a=y_a$, $a=1,\ldots,m$ and $t_j=z_{j-m}$, $j=m+1,\ldots,n$.

Taking into account the fact that the relation

$$L(\mathbf{j}_1)=\Lambda(\mathbf{j}_1)$$

holds for any point $\mathbf{j}_1\in\mathcal{L}\cap\mathcal{M}$ and using the definition of the partial derivative yield the equality

$$\frac{\partial L}{\partial y_a}(\mathbf{j}^0)=\frac{\partial\Lambda}{\partial y_a}(\mathbf{j}^0).$$

Since $\mathbf{j}^0$ is an arbitrary point of $\mathcal{L}\cap\mathcal{M}$, the equation

$$\left.\frac{\partial L}{\partial y_a}\right|_{\mathcal{L}\cap\mathcal{M}}=\left.\frac{\partial\Lambda}{\partial y_a}\right|_{\mathcal{L}\cap\mathcal{M}}\tag{17}$$

holds.

Now taking into account the fact that an arbitrary order prolongation of the operator $\partial/\partial y_a$ is equal to $\partial/\partial y_a$ we see that the left-hand side of (17) coincides with the left-hand side of (13) and the right-hand side of (17) coincides with the left-hand side of (14). Hence it follows the validity of the assertion of the lemma. $\triangleright$

# 4 Reduction of PDEs

We say that Ansatz (7) reduces PDE (12) if the substitution of formulae (7) into (12) gives rise to an equation which is *equivalent* to PDE containing "new" independent $\omega_1$, …, $\omega_{n-m}$ and dependent $\varphi$ variables only. To give a formal definition let us insert Ansatz (7) into the initial equation (12). As a result, we get some $p$th ($p \leq r$) order PDE

$$W(x, u, \varphi, \underset{1}{\varphi}, \ldots, \underset{p}{\varphi}) = 0,$$

where the symbol $\underset{k}{\varphi}$ stands for the set of $k$th order derivatives of the function $\varphi$ with respect to the variables $\omega_1, \ldots, \omega_{n-m}$. Eliminating the variables $x, u$ with the help of formulae (8) yields

$$W'(\theta_1, \ldots, \theta_m, \omega_1, \ldots, \omega_{n-m}, \varphi, \underset{1}{\varphi}, \ldots, \underset{p}{\varphi}) = 0.$$

**Definition 4.** Ansatz (7) reduces PDE (12) if the relation

$$\begin{aligned} W' = {} & H(\theta_1, \ldots, \theta_m, \omega_1, \ldots, \omega_{n-m}, \varphi, \underset{1}{\varphi}, \ldots, \underset{p}{\varphi}) \\ & \times \widetilde{L}(\omega_1, \ldots, \omega_{n-m}, \varphi, \underset{1}{\varphi}, \ldots, \underset{p}{\varphi}) \end{aligned} \tag{18}$$

holds with some function $H$ that does not vanish in $\mathcal{D} \cap \mathcal{M}$. The equation $\tilde{L} = 0$ is called the reduced differential equation.

**Remark** The reduced differential equation is determined up to a non-vanishing multiplier depending on $\omega_1, \ldots, \omega_{n-m}$, $\varphi, \underset{1}{\varphi}, \ldots, \underset{p}{\varphi}$.

As mentioned in Introduction there exist two different approaches to reduction of PDEs that are based on their conditional symmetry. The first one is solving the determining equations (13) in order to obtain an involutive family of operators $Q_a$ such that the equation under study is conditionally-invariant with respect to these operators. According to [20] an Ansatz corresponding to thus obtained involutive family reduces the PDE under study in the sense of Definition 4. Alternatively, one can try to construct an Ansatz (7) reducing the PDE under study without solving an intermediate problem of finding involutive families of operators obeying (13). The first approach is usually addressed to as the non-classical or conditional symmetry reduction method. The second one is called the Ansatz or direct reduction method. Note that within the framework of the direct reduction method one always supposes an explicit dependence of an Ansatz on $u$, thus restricting the choice of Ansätze to the following particular form:

$$u = f\Big(x, \varphi(\omega_1(x), \ldots, \omega_{n-m}(x))\Big). \tag{19}$$

This assumption simplify substantially calculation involved but, on the other hand, it may result in loosing some classes of Ansätze which have implicit dependence on

$u$. This is indeed the case for the relativistic eikonal equation where some invariant Ansätze cannot be represented in the form (19) [29].

Now we are going to prove that the conditional symmetry reduction and Ansatz approaches are equivalent.

**Theorem 1** *Let system of PDEs (12) be conditionally-invariant under the involutive family of differential operators (1) satisfying condition (3) and let the function* $\Lambda = L\,|_{\mathcal{M}}$ *have the maximal rank on* $\mathcal{L}\cap\mathcal{M}$ *or be identically equal to 0. Then, the Ansatz (7) corresponding to (1) reduces system of PDEs (12). Inversely, let Ansatz (7) reduce PDE (12). Then there is an involutive family of operators (1) obeying (3) and corresponding to Ansatz (7) such that PDE (12) is conditionally-invariant with respect to this involutive family.*

**Proof.** *Conditional symmetry* $\Rightarrow$ *reduction.* Let PDE (12) be conditionally-invariant with respect to an involutive family of differential operators (1) obeying the relation (3) and let the function $\Lambda$ have the maximal rank 1 on $\mathcal{L}\cap\mathcal{M}$. If $\mathcal{L}\cap\mathcal{M}=\emptyset$, then we can choose $H = W'$, $\widetilde{L} = 1$ in (18), which means that PDE (12) is reduced to the incompatible equation 1=0.

Suppose now that $\mathcal{L}\cap\mathcal{M}\neq\emptyset$. Using the arguments analogous to those applied to prove Lemma 3, we rewrite (14) as follows

$$\frac{\partial\Lambda}{\partial y_a}\bigg|_{\mathcal{L}\cap\mathcal{M}} = 0.$$

Making use of the Hadamard lemma, we represent the above relation in the equivalent form:

$$\Lambda_{y_a} = F_a\Lambda, \quad a = 1,\ldots,m. \tag{20}$$

Consequently, the function $\Lambda$ is a solution of the over-determined system of PDEs (20), where $F_a$ are smooth functions of the variables $y, z, v, \underset{1}{v}^{(z)},\ldots,\underset{r}{v}^{(z)}$. Here the symbol $\underset{s}{v}^{(z)}$ corresponds to the set of the $s$-order derivatives of the function $v$ with respect to the variables $z$ only. The necessary and sufficient compatibility conditions of (20) read

$$\left(\frac{\partial F_a}{\partial y_b} - \frac{\partial F_b}{\partial y_a}\right)\Lambda = 0, \quad a,b = 1,\ldots,m. \tag{21}$$

As the function $\Lambda$ has the maximal rank on $\mathcal{L}\cap\mathcal{M} = \{M = 0,\ \Lambda = 0\}$, in an arbitrary neighborhood of any point $\mathbf{j}\in\mathcal{L}\cap\mathcal{M}$ in $\mathcal{M}$ there exists a point $\mathbf{j}'$ such that $\Lambda(\mathbf{j}')\neq 0$. In view of this we get from (21) the following system of PDEs:

$$\frac{\partial F_a}{\partial y_b} = \frac{\partial F_b}{\partial y_a}, \quad a,b = 1,\ldots,m.$$

Consequently, there is a function $F$ of the variables $y, z, v, \underset{1}{v}^{(z)}, \ldots, \underset{r}{v}^{(z)}$ such that $F_a = \partial F/\partial y_a$, $\forall a$. Hence we get the general solution of (20)

$$\Lambda = \exp\{F\}\widetilde{L}, \tag{22}$$

where $\widetilde{L}$ is an arbitrary function of $z, v, \underset{1}{v}, \ldots, \underset{r}{v}$.

Inserting the Ansatz $v = \varphi(z_1, \ldots, z_{n-m})$ invariant under the family of operators $\partial/\partial y_1, \ldots, \partial/\partial y_m$ into $L$ yields

$$L\Big|_{v=\varphi(z_1,\ldots,z_{n-m})} = \Lambda\Big|_{v=\varphi(z_1,\ldots,z_{n-m})} = \exp\{F\}\widetilde{L}\Big|_{v=\varphi(z_1,\ldots,z_{n-m})}.$$

For the case $\Lambda \equiv 0$ the proof is obvious (for example, we can choose $H \equiv 1$ and $L \equiv 0$).

*Reduction $\Rightarrow$ conditional symmetry.* Let the Ansatz (7) reduce PDE (12). Let us make the change of variables (9) in order to represent (7) in the form (10). Then a corresponding involutive family of differential operators can be chosen as follows, $Q_a = \partial/\partial y_a$. Since the function $\varphi$ is arbitrary, to insert the Ansatz (10) into PDE (12) written in the new variables $y, z, v(y, z)$ is the same as to consider the intersection of the manifold $\mathcal{L}$ with the manifold $\mathcal{M}$ with a subsequent identifying $\varphi$ with $v$.

By assumption of the theorem a relation of the form

$$L\Big|_{v=\varphi(z_1,\ldots,z_{n-m})} = H(y, z, \varphi, \underset{1}{\varphi}, \ldots, \underset{p}{\varphi})\, \widetilde{L}(z_1, \ldots, z_{n-m}, \varphi, \underset{1}{\varphi}, \ldots, \underset{p}{\varphi})$$

holds with some non-vanishing $H$. Consequently,

$$\Lambda = L\Big|_{\mathcal{M}} = H(y, z, v, \underset{1}{v}^{(z)}, \ldots, \underset{p}{v}^{(z)})\, \widetilde{L}(z_1, \ldots, z_{n-m}, v, \underset{1}{v}^{(z)}, \ldots, \underset{p}{v}^{(z)}).$$

As the $r$th prolongation of the operator $Q_a = \partial/\partial y_a$ is equal to $\partial/\partial y_a$, $\forall a$, we have

$$\underset{(r)}{Q_a}\Lambda = \frac{\partial}{\partial y_a}\Lambda = \frac{\partial H(y, z, v, \underset{1}{v}^{(z)}, \ldots, \underset{p}{v}^{(z)})}{\partial y_a}\, \widetilde{L}(z_1, \ldots, z_{n-m}, v, \underset{1}{v}^{(z)}, \ldots, \underset{p}{v}^{(z)}).$$

Next, as the function $H$ does not vanish in $\mathcal{D}\cap\mathcal{M}$, the set of solutions of the equation $\Lambda = 0$ coincides with the set of solutions of the equation $\widetilde{L} = 0$. Consequently, the relation

$$\left(\underset{(r)}{Q_a}\Lambda\Big|_{\Lambda=0}\right)\Big|_{\mathcal{M}} = \underset{(r)}{Q_a}\Lambda\Big|_{\mathcal{L}\cap\mathcal{M}} = 0,$$

holds $\forall a = 1, \ldots, m$. Hence, we conclude that the initial PDE (12) written in the variables $y, z, v(y,z)$ is conditionally-invariant with respect to the involutive family $\partial/\partial y_a$. Rewriting (12) and the involutive family $\partial/\partial y_a$ in the initial variables $x, u$ completes the proof of the theorem. $\triangleright$

# 5 Application: the nonlinear wave equation

Now we are going to consider a specific example enlightening the peculiarities of the Ansatz (direct) and conditional (non-classical) symmetry approaches to the problem of dimensional reduction of PDEs. As a basic model we take the nonlinear (1+3)-dimensional wave equation

$$\Box u = F(u). \tag{23}$$

Here $\Box = \partial^2/\partial x_0^2 - \Delta$ is the d'Alembertian, $u = u(x)$ is a real-valued function of four real variables $x_0, x_1, x_2, x_3$ and $F$ is an arbitrary continuous function.

First, we apply the Ansatz approach to reduction of PDE (23). To this end we utilize an idea suggested in [5] and make use of the Lie symmetry properties of the equation under study for the sake of elucidating of a possible structure of the Ansatz for the $u(x)$.

As is well-known the maximal in Lie's sense symmetry group admitted by Eq.(23) with an arbitrary $F$ is the ten-parameter Poincaré group $P(1,3)$ having the generators

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad \mu, \nu = 0, 1, 2, 3, \ \mu < \nu. \tag{24}$$

Hereafter raising and lowering the indices is performed with the help of the metric tensor of the Minkowski space $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$ and the summation convention is used. For example,

$$x^\mu = g_{\mu\nu} x_\nu = \begin{cases} x_0, & \mu = 0, \\ -x_a, & \mu = a = 1, 2, 3. \end{cases}$$

Symmetry reduction of Eq.(23) by subgroups of the Poincaré group has been performed in [27, 45]. An analysis of thus obtained invariant Ansätze for the scalar field $u(x)$ shows that they have the same structure

$$u(x) = \varphi(\omega(x)). \tag{25}$$

The form of a real-valued function $\omega(x)$ is determined by the choice of a specific subgroup of the group $P(1,3)$.

Thus as a first step of our approach we fix the Ansatz for solutions of Eq.(23) to be of the form (25). However, we do not impose *a priori* restrictions on the choice of unknown function $\omega(x)$. The only requirement to be met is that inserting the expression (25) into Eq.(23) should yield an ordinary differential equation (ODE) for the function $\varphi(\omega)$. This requirement gives rise to a compatible over-determined system of nonlinear partial differential equations for the functions $\omega(x)$. Any solution of the latter after being inserted into formula (25) yields an Ansatz for the scalar field $u(x)$ reducing Eq.(23) to ODE.

Inserting (25) into the nonlinear wave equation (23) gives

$$\left(\frac{\partial\omega}{\partial x_\mu}\frac{\partial\omega}{\partial x^\mu}\right)\frac{d^2\varphi}{d\omega^2}+\Box\omega\,\frac{d\varphi}{d\omega}=F(\varphi). \tag{26}$$

As the above equation has to be equivalent to ODE for the function $\varphi(\omega)$ under arbitrary $F$, the coefficients of $d^2\varphi/d\omega^2$, $d\varphi/d\omega$ have to be some functions of $\omega$. This requirement yields that there exist real-valued functions $f_1(\omega), f_2(\omega)$ such that

$$\frac{\partial\omega}{\partial x_\mu}\frac{\partial\omega}{\partial x^\mu}=f_1(\omega),\quad \Box\omega=f_2(\omega). \tag{27}$$

System of nonlinear PDEs (27) is the necessary and sufficient condition for the Ansatz (25) to reduce the nonlinear wave equation (23) to an ordinary differential equation. And what is more, the equation for the function $\varphi(\omega)$ reads

$$f_1(\omega)\,\frac{d^2\varphi}{d\omega^2}+f_2(\omega)\,\frac{d\varphi}{d\omega}=F(\varphi). \tag{28}$$

Summing up we conclude that any solution of over-determined system of nonlinear PDEs (27) gives rise to an Ansatz for the field $u(x)$ reducing Eq.(23) to ODE of the form (28). In particular, any Ansatz corresponding to the Lie symmetry of the nonlinear wave equation (23) can be obtained in this way. However, the Lie Ansätze do not exhaust the set of all possible substitutions of the form (25) reducing Eq. (23) to ODEs. This is explained by an existence of wide classes of Ansätze (25) that correspond to conditional symmetry of the nonlinear wave equation and cannot be, in principle, obtained within the framework of the Lie symmetry approach.

Now we utilize the conditional symmetry approach for obtaining Ansatz (25) that reduces PDE in four dimensions (23) to ODE. Consider conditional symmetry of the nonlinear wave equation within the class of first-order differential operators

$$Q=\xi_\mu(x)\frac{\partial}{\partial x_\mu}. \tag{29}$$

As we are looking for conditional symmetries that enable reduction of (23) to ODE, it is necessary to consider an involutive family of three differential operators of the form (29), namely, $Q_a = \xi_{a\mu}(x)\frac{\partial}{\partial x_\mu}, \quad a = 1, 2, 3$. And what is more, we require that the restriction (3) is respected, which means that

$$\mathrm{rank}\, \|\xi_{a\mu}(x)\|_{a=1}^{3}{}_{\mu=0}^{3} = 3$$

Taking into account the above relation, Lemma 2 and also making use of the Poincaré invariance of the equation under study we can always transform the operators $Q_1, Q_2, Q_3$ to become

$$Q_a = \frac{\partial}{\partial x_a} - f_a(x)\frac{\partial}{\partial x_0}, \quad a = 1, 2, 3. \tag{30}$$

It is straightforward to check that the family of operators (30) is involutive if and only if $Q_1, Q_2, Q_3$ commute each with another. Hence we get the system of three PDEs for the functions $f_1(x), f_2(x), f_3(x)$

$$\frac{\partial f_a}{\partial x_b} - f_b\frac{\partial f_a}{\partial x_0} = \frac{\partial f_b}{\partial x_a} - f_a\frac{\partial f_b}{\partial x_0},$$

where $a, b = 1, 2, 3,\ a < b$. Its general solution can be represented in the form (see, e.g., [25])

$$f_a = \frac{\partial \omega}{\partial x_a}\left(\frac{\partial \omega}{\partial x_0}\right)^{-1}, \quad a = 1, 2, 3, \tag{31}$$

where $\omega = \omega(x)$ is an arbitrary twice continuously differentiable function, $\partial\omega/\partial x_0 \neq 0$.

The condition (13) of invariance of the nonlinear wave equation (23) with respect to operators (30) after some involved straightforward algebraic manipulations reduces to over-determined system of six PDEs for the functions $f_1, f_2, f_3$

$$\Box f_a - 2\frac{\partial f_a}{\partial x_b}\frac{\partial f_b}{\partial x_0} = 0, \quad \frac{\partial f_a}{\partial x_0} - f_b\frac{\partial f_a}{\partial x_b} = 0, \quad a = 1, 2, 3. \tag{32}$$

Inserting the expressions for $f_a$ (31) into (32) and rearranging the obtained PDEs for the function $\omega(x)$ yields

$$\left(\frac{\partial \omega}{\partial x_0}\frac{\partial}{\partial x_a} - \frac{\partial \omega}{\partial x_a}\frac{\partial}{\partial x_0}\right)\frac{\partial \omega}{\partial x_\mu}\frac{\partial \omega}{\partial x^\mu} = 0, \quad \left(\frac{\partial \omega}{\partial x_0}\frac{\partial}{\partial x_a} - \frac{\partial \omega}{\partial x_a}\frac{\partial}{\partial x_0}\right)\Box\omega = 0$$

with $a = 1, 2, 3$. Hence we conclude that there are smooth functions $f_1(\omega), f_2(\omega)$ such that the relations $\frac{\partial\omega}{\partial x_\mu}\frac{\partial\omega}{\partial x^\mu} = f_1(\omega),\ \Box\omega = f_2(\omega)$ hold and we arrive at system of PDEs (27). Consequently, the involutive family (30) takes necessarily the form

$$Q_a = \frac{\partial}{\partial x_a} - \frac{\partial \omega}{\partial x_a}\left(\frac{\partial \omega}{\partial x_0}\right)^{-1}\frac{\partial}{\partial x_0}, \quad a = 1, 2, 3, \tag{33}$$

where the function $\omega = \omega(x)$ is a solution of system (27).

As the function $\omega = \omega(x)$ is the first integral of the system of PDEs $Q_a f(x) = 0,\ a = 1, 2, 3$, the Ansatz for the field $u(x)$ corresponding to the family $Q_1, Q_2, Q_3$ is given by (25).

Thus both Ansatz (direct) and conditional symmetry (non-classical) approaches to reduction of the nonlinear wave equation (23) to ODEs lead to the same reduction conditions, namely, to the system of differential equations (23) consisting of the nonlinear wave and relativistic Hamilton-Jacobi equations. Following [12] we call this system the d'Alembert-Hamilton system.

The d'Alembert-Hamilton system in three dimensions was studied by Jacobi [35], Smirnov & Sobolev [36, 37] and later on by Collins [38]. Collins constructed the general solution of system of nonlinear PDEs (26) for a complex-valued function of three complex variables. Some exact solutions of the d'Alembert-Hamilton system in four dimensions have been constructed by Cartan [39], Bateman [40] and Erugin [41]. Recently, we have constructed the general solution of system (26) for the complex-valued function of four complex variables [42, 43].

As established in [19], system of PDEs (26) for the real-valued function $\omega(x)$ is compatible if and only if it is locally equivalent to the system

$$\Box\omega = \epsilon N \omega^{-1}, \quad (\partial_\mu \omega)(\partial^\mu \omega) = \epsilon, \quad \epsilon = \pm 1, 0, \tag{34}$$

where $N = 0, 1, 2, 3$.

The real form of the general solution of the system of PDEs (34) is given by one of the formulae below [15, 19]

I. $\epsilon = -1$

1) $N = 0$

$$\omega = A_\mu(\tau) x^\mu + R_1(\tau), \tag{35}$$

where $\tau = \tau(x)$ is determined in implicit way

$$B_\mu(\tau) x^\mu + R_2(\tau) = 0$$

and $A_\mu(\tau),\ B_\mu(\tau),\ R_1(\tau),\ R_2(\tau)$ are arbitrary smooth real-valued functions satisfying the conditions

$$A_\mu(\tau) A^\mu(\tau) = -1, \quad A_\mu(\tau) B^\mu(\tau) = 0, \quad \dot{A}_\mu(\tau) B^\mu(\tau) = 0, \quad B_\mu(\tau) B^\mu(\tau) = 0;$$

2) $N = 1$

$$\omega^2 = (d_\mu x^\mu + g_2)^2 - (a_\mu x^\mu + g_1)^2, \tag{36}$$

$$\omega^2 = (b_\mu x^\mu + C_1)^2 + (c_\mu x^\mu + C_2)^2, \tag{37}$$

where $g_i = g_i(a_\mu x^\mu + d_\mu x^\mu) \in C^2(\mathbf{R}^1, \mathbf{R}^1)$ are arbitrary functions;

3) $N = 2$

a)
$$\omega^2 = -\Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big) - \Big\{B_\mu(\tau)\Big(x^\mu + A^\mu(\tau)\Big)\Big\}^2, \qquad (38)$$

where $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + A_\mu(\tau)\Big)\dot{B}^\mu(\tau) = 0,$$

$A_\mu(\tau),\ B_\mu(\tau)$ are arbitrary smooth real-valued functions satisfying the conditions

$$B_\mu(\tau)B^\mu(\tau) = -1, \quad \dot{B}_\mu(\tau)\dot{B}^\mu(\tau) = 0, \quad \dot{A}_\mu(\tau) = R(\tau)\dot{B}_\mu(\tau)$$

with an arbitrary $R(\tau) \in C^1(\mathbf{R}^1,\ \mathbf{R}^1)$;

b)
$$\omega^2 = -\Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big) - \Big\{b_\mu\Big(x^\mu + A^\mu(\tau)\Big)\Big\}^2, \qquad (39)$$

where $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + A_\mu(\tau)\Big)\Big(\dot{A}^\mu(\tau) + b^\mu b_\nu \dot{A}^\nu(\tau)\Big) = 0,$$

$A_\mu(\tau)$ are arbitrary smooth real-valued functions satisfying the condition

$$\dot{A}_\mu(\tau)\dot{A}^\mu(\tau) + \Big(b_\mu \dot{A}^\mu(\tau)\Big)^2 = 0;$$

c)
$$\omega^2 = (b_\mu x^\mu + C_1)^2 + (c_\mu x^\mu + C_2)^2 + (d_\mu x^\mu + C_3)^2; \qquad (40)$$

4) $N = 3$
$$\omega^2 = -\Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big), \qquad (41)$$

where $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + A_\mu(\tau)\Big)B^\mu(\tau) = 0,$$

$A_\mu(\tau),\ B_\mu(\tau)$ are arbitrary smooth real-valued functions satisfying the conditions

$$\dot{A}_\mu(\tau)B^\mu(\tau) = 0, \quad B_\mu(\tau)B^\mu(\tau) = 0. \qquad (42)$$

II. $\epsilon = 1$

1) $N = 0$

$$\omega = a_\mu x^\mu + C_1; \tag{43}$$

2) $N = 1$

$$\omega^2 = (a_\mu x^\mu + C_1)^2 - (d_\mu x^\mu + C_2)^2; \tag{44}$$

3) $N = 2$

$$\omega^2 = (a_\mu x^\mu + C_1)^2 - (c_\mu x^\mu + C_2)^2 - (d_\mu x^\mu + C_3)^2; \tag{45}$$

4) $N = 3$

$$\omega^2 = (x_\mu + C_\mu)(x^\mu + C^\mu). \tag{46}$$

III. $\epsilon = 0$, $N = 0$

$$A_\mu(\omega)x^\mu + B(\omega) = 0,$$

where $A_\mu, B$ are arbitrary smooth real-valued functions such that $A_\mu A^\mu = 0$.

In the above formulae $C_0, \ldots, C_3$ are arbitrary real constants and $a_\mu$, $b_\mu$, $c_\mu$, $d_\mu$ are arbitrary real constants satisfying the conditions

$$\begin{aligned} &a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1, \\ &a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0. \end{aligned}$$

Let us emphasize that all the functions $\omega(x)$ defined by formulae (35), (36), (38), (39), (41) give rise to conditionally-invariant Ansätze for the field $u(x)$ of the form (25). Using these one can construct broad families of new (non-Lie) exact solutions even for such a well studied model as the nonlinear wave equation (see, also [46]). Consider, for example, the confomally-invariant nonlinear wave equation

$$\Box u = \lambda u^3. \tag{47}$$

The Ansatz

$$u = \varphi\left(-\Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big)^{1/2}\right),$$

where $\tau = \tau(x)$ is defined in (41) and $A_\mu(\tau), B_\mu(\tau)$ are arbitrary smooth functions satisfying (42), reduces (47) to ODE for $\varphi = \varphi(\omega)$

$$\frac{d^2\varphi}{d\omega^2} + 3\omega^{-1}\frac{d\varphi}{d\omega} = -\lambda\varphi^3.$$

Two particular solutions of the latter $\varphi = \lambda^{-1/2}\omega^{-1}$ and $\varphi = a(\omega^2 + \lambda a/8)^{-1}$, $a$ =const give rise to two families of new exact solutions of the cubic wave equation (47)

$$\begin{aligned} u(x) &= \lambda^{-1/2}\left[-\Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big)\right]^{-1/2}, \\ u(x) &= a\left[\frac{\lambda a^2}{8} - \Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big)\right]^{-1}. \end{aligned}$$

Choosing arbitrary functions $A_\mu(\tau)$ to be constant yields the well-known exact solutions of (47) obtained in [45] within the symmetry reduction routine. However, if $A_\mu(\tau)$ are not constants, the constructed solutions are new and cannot be found using the symmetry reduction procedure.

# 6 Concluding Remarks

Thus introducing a rigorous definition of reduction of PDEs enables a systematic treatment of the problem of studying interrelations between the Ansatz (direct) and non-classical (conditional symmetry) approaches to dimensional reductions of multi-dimensional PDEs. We have proved that the direct approach, taken in a full generality, is equivalent to the non-classical approach provided some natural restrictions are met (see Theorem 1). When we say 'in a full generality' we mean that the most general form of the similarity Ansatz should be taken. For example, the Ansatz (25) is a particular case of the general similarity Ansatz for PDE (23)

$$U(x, u) = \varphi(\omega(x, u)). \tag{48}$$

Imposing the restrictions $U(x, u) = u, \omega(x, u) = \omega(x)$ results in loosing some reductions. On the other hand, with this choice of the form of the Ansatz we were able to get a full solution of the problem of constructing the corresponding conditionally-invariant Ansätze of the form (25) (see, formulae (35)–(46)), as the system of nonlinear determining equations for $\omega(x)$ proves to be integrable. Integrating it yields broad classes of principally new reductions and exact solutions for nonlinear wave equations containing several arbitrary functions of one argument.

So both direct and nonclassical approaches can be used on equal footing and the choice of one of them is, in fact, a matter of taste. Nevertheless, the direct approach has an evident benefit of being comparatively simple, since only some basics of the standard university course on partial differential equations are required for understanding and implementing it. Another merit of the direct approach is its flexibility. A similarity Ansatz can be easily modified in order to yield, for example, 'nonlinear separation of variables' in the spirit of [47] (see, also [21, 48]). However, if we wish to take into consideration the case of implicit Ansätze (say, of the form (48)), then the nonclassical approach is preferable. These points are illustrated by the considerations of Section 5, where both approaches are applied to the nonlinear wave equation and the direct method provides a shorter way to obtain conditional symmetries of the equation under study.

The fact that we restrict our considerations to scalar PDEs, namely, to PDEs with one dependent variable, is explained by the major difficulties arising when handling systems of PDEs. The first problem is the fact that different equations of system may have different orders. Next, if the number of equations is greater than the number of dependent variables, there arises a natural question of compatibility of this system. However, the implication *conditional invariance* $\Rightarrow$ *reduction* can be proved in almost the same way as it is done for the case of a single PDE in Theorem 1 [20]. The problem is how to modify the proof in order to establish a validity of an assertion *reduction* $\Rightarrow$ *conditional invariance.* We postpone the investigation of this problem to our future publications.

# Acknowledgments

This work is partially supported by the Ukrainian DFFD Foundation under the project 1.4/356.

# References

[1] G. Bluman and J.D. Cole, The general similarity solution of the heat equation, *J. Math. Mech.* **18**, N 11 (1969), 1025–1042.

[2] P.J. Olver and P. Rosenau, The construction of special solutions to partial differential equations, *Phys. Lett.* **114A**, N 3 (1986), 107–112.

[3] P.J. Olver and P. Rosenau, Group-invariant solutions of differential equations, *SIAM J. Appl. Math.* **47**, N 2 (1987), 263–278.

[4] W.I. Fushchych and I.M. Tsyfra, On a reduction and solutions of nonlinear wave equations with broken symmetry, *J. Phys. A: Math. Gen.* **20**, N 2 (1987), L45–L48.

[5] W.I. Fushchych and R.Z. Zhdanov, Symmetry and exact solutions of nonlinear spinor equations, *Phys. Repts.* **172**, N 4 (1989), 123–174.

[6] P. Clarkson and M.D. Kruskal, New similarity solutions of the Boussinesq equation, *J. Math. Phys.* **30**, N 10 (1989), 2201–2213.

[7] D. Levi and P. Winternitz, Non-classical symmetry reduction: example of the Boussinesq equation, *J. Phys. A: Math. Gen.* **22**, N 15 (1989), 2915–2924.

[8] W.I. Fushchych, How to extend symmetry of differential equations?, *in* "Symmetry and Solutions of Nonlinear Mathematical Physics Equations", Inst. of Math. Acad. of Sci. Ukraine, Kiev, 1987, 4–16.

[9] W.I. Fushchych, On symmetry and exact solutions of multi-dimensional nonlinear wave equations, *Ukrain. Math. J.* **39**, N 1 (1987), 116–123.

[10] W.I. Fushchych and A.G. Nikitin, "Symmetries of Maxwell's Equations", Reidel, Dordrecht, 1987.

[11] W.I. Fushchych and R.Z. Zhdanov, Symmetry and exact solutions of the nonlinear Dirac equation, *Fizika Elementar. Chastits i Atom. Yadra*, **19**, N 5 (1988), 1154–1196.

[12] W.I. Fushchych and R.Z. Zhdanov, On some new exact solutions of the nonlinear d'Alembert-Hamilton system, *Phys. Lett.* **141A**, N 3-4 (1989), 113–115.

[13] W.I. Fushchych and R.Z. Zhdanov, Non-Lie Ansätze and exact solutions of the nonlinear spinor equation, *Ukrain. Math. J.*, **42**, N 7 (1990), 958–962.

[14] W.I. Fushchych and R.Z. Zhdanov, On non-Lie reduction of the nonlinear Dirac equation, *J. Math. Phys.* **32**, N 12, 3488-3490 (1991).

[15] W.I. Fushchych, R.Z. Zhdanov and I.A. Yehorchenko, On the reduction of the nonlinear multi-dimensional wave equations and compatibility of the d'Alembert-Hamilton system, *J. Math. Anal. Appl.* **161**, N 2 (1991), 352–360.

[16] R.Z. Zhdanov and W.I. Fushchych, On non-Lie ansatzes and new exact solutions of the classical Yang-Mills equations, *J. Non. Math. Phys.* **2**, N 2 (1995), 172-181.

[17] R.Z. Zhdanov and W.I. Fushchych, Conditional symmetry and new classical solutions of the Yang-Mills equations, *J. Phys. A: Math. Gen.* **28**, N 21 (1995), 6253-6263.

[18] R.Z. Zhdanov, On integration of nonlinear d'Alembert-eikonal system and conditional symmetry of nonlinear wave equations, *J. Non. Math. Phys.* **4**, N 1–2 (1997), 49–62.

[19] W.I. Fushchych and R.Z. Zhdanov, "Symmetries and Exact Solutions of Nonlinear Dirac Equations", Naukova Ukraina Publishing, Kyiv, 1997.

[20] W.I. Fushchych and R.Z. Zhdanov, Conditional symmetry and reduction of partial differential equations, *Ukr. Math. J.* **44**, N 7 (1992), 970–982.

[21] R.Z. Zhdanov, Conditional Lie-Bäcklund symmetry and reduction of evolution equations, *J. Phys. A: Math. Gen.* **28**, N 13 (1995), 3841–3850.

[22] P.J. Olver and E.M. Vorob'ev, Nonclassical and conditional symmetries, *in* "Handbook of Lie group analysis", Boca Raton , FL: Chemical Rubber Company, 1994.

[23] R.O. Popovych, On reduction and $Q$-conditional symmetry, *in* Proceedings of the Second International Conference "Symmetry in Nonlinear Mathematical Physics" (Kyiv, July 7-13, 1997), Inst. of Math. Acad. of Sci. Ukraine, Kyiv, 1997, 437–443.

[24] R.O. Popovych and I.P. Korneva, On $Q$-conditional symmetry of the linear $n$-dimensional heat equation, *in* "Symmetry and Analytic Methods in Mathematical Physics", Inst. of Math. Acad. of Sci. Ukraine, Kyiv, 1998, 200–211.

[25] R.O. Popovych, On a class of $Q$-conditional symmetries and solutions of evolutionary equations, *in* "Symmetry and Analytic Methods in Mathematical Physics", Inst. of Math. Acad. of Sci. Ukraine, Kyiv, 1998, 194–199.

[26] W.I. Fushchych, On symmetry and particular solutions of some multi-dimensional mathematical physics equations, *in* "Algebraic-Theoretical Methods in Mathematical Physics Problems", Inst. of Math. Acad. of Sci. Ukraine, Kiev, 1983, 4–23.

[27] W.I. Fushchych and N.I. Serov, The symmetry and some exact solutions of nonlinear many-dimensional Liouville, d'Alembert and eikonal equations, *J. Phys. A: Math. Gen.* **16**, N 15 (1983), 3645–3656.

[28] P.J. Olver, Direct reduction and differential constraints, *Proc. R. Soc. Lond. A*, **444** (1994), 509–523.

[29] W.I. Fushchych, W.M. Shtelen, and Serov M.I., "Symmetry Analysis and Exact Solutions of Equations of Non-Linear Mathematical Physics", Kluwer Academic Publisher, Dordrecht, 1992.

[30] P.G. Estevez and P.R. Gordoa, Nonclassical symmetries and the singular manifold method - theory and 6 examples, *Studies in Applied Mathematics*, **95**, N 5 (1995) 517-557.

[31] R.Z. Zhdanov and I.M. Tsyfra, Reduction of differential equations and conditional symmetry, *Ukr. Math. J.* **48**, N 5 (1996), 595–602.

[32] P.J. Olver, "Applications of Lie Groups to Differential Equations", Springer-Verlag, New-York, 1989.

[33] B.F. Schutz, "Geometrical methods of mathematical physics", Cambridge University Press, London, 1982.

[34] E.M. Vorob'ev, Partial symmetries of system of differential equations, *Soviet Math. Dokl.* **33**, N 4 (1986), 408–411.

[35] H. Bateman, "Partial Differential Equations of Mathematical Physics", Univ. Press, Cambridge, 1922.

[36] V.I. Smirnov and S.L. Sobolev, New method of solution of the problem of elastic plane vibrations, *Proc. of Seismological Inst. Acad. Sci. USSR*, **20** (1932), 37–40.

[37] V.I. Smirnov and S.L. Sobolev, On application of the new method to study of elastic vibrations in the space with axial symmetry, *Proc. of Seismological Inst. Acad. Sci. USSR*, **29** (1933), 43–51.

[38] C.B. Collins, Complex potential equations. I. A technique for solution, *Proc. Cambr. Phyl. Soc.* **80**, N 1 (1976), 165–171.

[39] E. Cartan, *in* "Oeuvres completes", Gauthier-Villars, Paris, Part 3, Vol. 2, 1955, 1431–1445.

[40] H. Bateman, "Mathematical Analysis of Electrical and Optical Wave-Motion", New York, Dover, 1955.

[41] N.P. Erugin, On functionally-invariant solutions, *Proc. Acad. of Sci. USSR*, **5** (1944), 385–386.

[42] W.I. Fushchych, R.Z. Zhdanov and I.V. Revenko, "Compatibility and solutions of the nonlinear d'Alembert and Hamilton equations", *Preprint*, Inst. of Math. Acad. of Sci. Ukraine, N 90-39, Kiev, 1990.

[43] W.I. Fushchych, R.Z. Zhdanov and I.V. Revenko, General solutions of the nonlinear wave and eikonal equations, *Ukrain. Math. J.* **43**, N 11 (1991), 1471–1486.

[44] J. Patera, R.T. Sharp, P. Winternitz and H. Zassenhaus, Subgroups of the Poincaré group and their invariants, *J. Math. Phys.* **17**, N 6 (1976), 977–985.

[45] A. Grundland, J. Harnad and P. Winternitz, Symmetry reduction for nonlinear relativistically invariant equations, *J. Math. Phys.* **25**, N 4 (1984), 791–806.

[46] R.Z. Zhdanov, O.A. Panchak, New conditional symmetries and exact solutions of the nonlinear wave equation *J. Phys. A: Math. Gen.* **31** N 43 (1998), 8727–8734.

[47] V.A. Galaktionov, On new exact blow-up solutions for nonlinear heat conduction equations with source and applications, *Diff. and Int. Equations*, **3**, N 5 (1990), 863–874.

[48] M.C. Nucci, Iterating the nonclassical symmetries method, *Physica D* **78**, N 3 (1994), 124–134.