Conditional symmetry of a porous medium equation

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Abstract

We obtain an exhaustive description of conditional symmetries admitted by a nonlinear porous medium equation in one spatial dimension. It is proved that solving the determining equations for the conditional symmetries either yields classical (Lie symmetries) or is equivalent to solving the porous medium equation itself. The second possibility is shown to be typical for an arbitrary evolution-type partial differential equation in one spatial variable. We show that this very property is responsible for the well-known phenomenon when using non-classical symmetries yield exact solutions which prove to be invariant ones.

1 Introduction

The nonlinear equation describing nonstationary flows of soil water with free surface

$$u_t = \frac{k}{m}(uu_x)_x \tag{1}$$

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has appeared first in the paper by Boussinesq [1]. Here k is the filtration coefficient, m is the porosity of soil and $u = u(t,x)$ is the pressure of the ground water at the time t and at the cross-section measured by the x-axis. This equation is usually addressed to as the porous medium equation (PME) and we will use this term throughout the paper.

The porous medium equation (1) can be derived from some general principles of the theory of soil water filtration under two assumptions: a) deviations of the free surface from the plane $z = const$ are small, b) horizontal velocities are independent of z (for more details see, e.g. [2]). Its first exact solution has been obtained by Boussinesq itself [1]. He was looking for a solution in a separated form

$$u(t,x) = X(x)T(t)$$

with the following conditions:

$$u(t,0) = 0, \quad u_x|_{x=L} = 0.$$

Thus constructed solution reads as

$$u = \frac{H_0 F(\xi)}{1 + \frac{3b^2 k H_0}{2mL^2} t}, \tag{2}$$

where H_0 is a constant, $\xi = x/L$ and the function $F = F(\xi)$ is defined implicitly

$$\xi = \frac{1}{b} \int\limits_0^F \frac{\lambda d\lambda}{\sqrt{1-\lambda^3}}, \quad b = \int\limits_0^1 \frac{\lambda d\lambda}{\sqrt{1-\lambda^3}} = \frac{1}{3} B\left(\frac{2}{3}, \frac{1}{2}\right).$$

The above solution characterizes the so-called Boussinesq ordered regime. It corresponds to an infinite number of drains on the impermeable soil.

The next exact solution of the porous medium equation (1) was found much later by Barenblatt (the instant source solution) [3]

$$u = \frac{m}{6kt} \left[\left(\frac{9kt}{m} \right)^{\frac{2}{3}} - x^2 \right], \quad 0 \le x \le \left(\frac{9kt}{m} \right)^{\frac{1}{3}} = l). \tag{3}$$

Note that the paper [3] contained a misprint which had been corrected by Sokolov in [4], where the instant source solution had been presented in the form (3). The above solution describes spreading of the soil water knoll over the impermeable subsoil [2].

It is easy to become convinced of the fact that both solutions (2), (3) correspond to the Lie symmetry of the porous medium equation (1). Indeed, the solution (2) is a particular case of the Ansatz

$$u = (1 + \alpha t)^{-1}\varphi(x), \quad \alpha = \frac{3b^2 k H_0}{2mL^2},$$

that is invariant with respect to the one-parameter Lie transformation group having the generator

$$Q_1 = (1 + \alpha t)\partial_t - u\partial_u, \quad \left(\partial_t = \frac{\partial}{\partial t}, \quad \partial_u = \frac{\partial}{\partial u}\right).$$

Furthermore, the Barenblatt's solution (3) is invariant with respect to the one-parameter transformation group generated by the operator

$$Q_2 = 3t\partial_t + x\partial_x - u\partial_u, \quad \left(\partial_x = \frac{\partial}{\partial x}\right).$$

The operators Q_1, Q_2 belong to the maximal invariance algebra admitted by PME [5]

$$T = \partial_t, \quad P = \partial_x, \quad D_1 = t\partial_t - u\partial_u, \quad D_2 = x\partial_x + 2u\partial_u. \tag{4}$$

Later on a systematic search for invariant solutions within the framework of the symmetry reduction technique has been undertaken (an account of the results obtained in this way and the comprehensive list of relevant references can be found in [6, 7]). As a choice of exact invariant solutions of PME is very restricted, there were several attempts to utilize its conditional (non-classical) symmetries for the sake of constructing new analytical solutions. The notion of non-classical symmetry was firstly introduced by Bluman and Cole in the paper [8], where the determining equations for non-classical symmetries admitted by the linear one-dimensional heat equation had been obtained. An active study of conditional symmetries of nonlinear partial differential equations (PDEs) has been initiated by papers [9]–[13] which yielded a lot of principally new (non-Lie) exact solutions for a number of nonlinear PDEs (see, e.g. [7, 14] and references cited there). Note that an overwhelming majority of the papers devoted to studying conditional symmetries of PDEs consider equations in two dimensions. Some results on study of conditional symmetries of multi-dimensional PDEs are given in [15], where we present wide classes of conditional symmetries admitted by the nonlinear Dirac, wave and $SU(2)$ Yang-Mills equations. Let us also mention the general geometric approach to studying local and higher order symmetries and conservation laws of nonlinear PDEs suggested in [16, 17].

To the best of our knowledge the first papers devoted to studying conditional symmetries of PME appeared in 1988-89 [19, 20], then other papers followed (see, e.g. [7] and the references therein). However, no new solutions were found in this way. One obtained new conditional symmetries that after performing the symmetry reduction routine gave rise to invariant solutions corresponding to the Lie symmetry of PME. One of the aims of the present paper is to provide an explanations to this "experimental data". We prove that integrating the determining equations for conditional symmetries of PME either yields classical Lie symmetries or is equivalent to solving equation (1). Furthermore, we will prove that a similar assertion holds in part for an *arbitrary* evolution type PDE in one dependent and two independent variables.

2 Conditional symmetries of PME

Denoting the coefficient $\frac{m}{k}$ as λ we represent PME (2) in the form

$$u_t = \lambda(uu_{xx} + u_x^2). \tag{5}$$

Following the usual procedure for finding conditional symmetries (see, e.g., [7, 18]) we are looking for a conditional symmetry operator of the general form

$$Q = T(t,x,u)\partial_t + X(t,x,u)\partial_x + U(t,x,u)\partial_u, \tag{6}$$

where T, X, U are some sufficiently smooth functions. Taking into account the fact that, provided Q is a conditional symmetry operator of PME, an operator $f(t,x,u)\,Q$ with an arbitrary function f is a conditional symmetry operator of PME as well [15], we can simplify substantially the structure of an operator to be found. Namely, it is sufficient to consider the two particular cases of the general formula (6), namely

1. $T = 1$,

2. $T = 0, \quad X = 1$.

Case 1. The system of determining equations for the coefficients of operator $Q = \partial_t + X(t,x,u)\partial_x + U(t,x,u)\partial_u$ reads

$$\begin{aligned}
&U_t = \lambda u U_{xx} + u^{-1}U^2 - 2UX_x, \quad uX_{uu} - X_u = 0,\\
&X_t - u^{-1}XU + 2XX_x - 2UX_u = \lambda(uX_{xx} - 2uU_{xu} - 2U_x),\\
&\lambda(uU_{uu} + U_u + u^{-1}U) + 2(XX_u - \lambda u X_{xu}) = 0.
\end{aligned}$$

A simple computation yields the general solution of the above system. It splits into two inequivalent classes

$$X = \frac{1}{3}k_1 x + k_2, \quad U = \frac{2}{3}k_1 u; \tag{7}$$

$$X = (k_1 + t)^{-1}\left(\frac{1}{3}(k_2+1)x + k_3\right), \quad U = (k_1+t)^{-1}\frac{(2k_2-1)u}{3}. \tag{8}$$

Here k_1, k_2, k_3 are arbitrary real constants.

Now inserting the formulae (7) into (6) under $T = 1$ yields an operator that is the linear combination of Lie symmetry operators

$$T = \partial_t, \quad P = \partial_x, \quad D_2 = x\partial_x + 2u\partial_u.$$

Next inserting the formulae (8) into (6) and multiplying by $k_1 + t$ yield an operator that is a linear combination of Lie symmetry operators

$$T = \partial_t, \quad P = \partial_x, \quad D_1 = t\partial_t - u\partial_u, \quad D_2 = x\partial_x + 2u\partial_u.$$

Case 2. PME (5) is conditionally-invariant with respect to the operator

$$Q = \partial_x + \eta(t,x,u)\partial_u, \tag{9}$$

iff the following equation holds

$$L(\eta) \equiv \eta_t - \lambda\Big(3\eta\eta_x + u\eta_{xx} + 2u\eta\eta_{xu} + \eta^2(u\eta_{uu} + 2\eta_u)\Big) = 0. \tag{10}$$

Let us make in the above equation the substitution

$$\eta(t,x,u) = -\frac{w_x}{w_u}, \tag{11}$$

where $w = w(t, x, u)$ is a new dependent variable. As a result, we get

$$L\left(-\frac{w_x}{w_u}\right) = (w_u)^{-2}\left(w_x\frac{\partial}{\partial u} - w_u\frac{\partial}{\partial x}\right)\Big[w_t - \lambda u(w_{xx} - 2(w_u)^{-1}w_x w_{xu} + (w_u)^{-2}w_x^2 w_{uu}) + \lambda(w_u)^{-1}w_x^2\Big]. \qquad (12)$$

Combining equations (10) and (12) we conclude that there exists such function $f(t, w)$ that

$$w_t - \lambda u(w_{xx} - 2(w_u)^{-1}w_x w_{xu} + (w_u)^{-2}w_x^2 w_{uu}) + \lambda(w_u)^{-1}w_x^2 = f(t, w).$$

Making in the above equation the hodograph transformation

$$t = y_0, \quad x = y_1, \quad u = U(y_0, y_1, y_2), \quad w = y_2 \qquad (13)$$

yields that the function $U(t, x, u)$ obeys the nonlinear PDE

$$-\frac{1}{U_{y_2}}(U_{y_0} - \lambda(UU_{y_1y_1} + U_{y_1}^2)) = f(y_0, y_2).$$

At last, making the change of variables

$$z_0 = y_0, \quad z_1 = y_1, \quad z_2 = \Omega(y_0, y_2), \quad V = U, \qquad (14)$$

where Ω is a first integral of the first-order PDE

$$\Omega_{y_0} + f(y_0, y_2)\Omega_{y_2} = 0 \qquad (15)$$

reduces the equation obtained to the porous medium equation (5) for the function $V = V(z_0, z_1, z_2)$

$$V_{z_0} = \lambda(VV_{z_1z_1} + V_{z_1}^2). \qquad (16)$$

Note that the variable z_2 is parametrical.

Thus integration of the determining equation for the coefficient of infinitesimal operator (9) is equivalent to solving the initial equation (5). By this very reason, conditional symmetries of the form (9) obtained by guesswork in [20] yield solutions that are nothing else than group-invariant solutions. Consider an illustrative example. Let us take as $V(z_0, z_1, z_2)$ a solution of (16) invariant with respect to the displacement group by the variable z_0, namely,

$$V(z_0, z_1, z_2)^2 = r_1(z_2)z_1 + r_2(z_2),$$

where r_1, r_2 are arbitrary smooth functions. Making the changes of variables (14) and (13) we rewrite the above expression as follows

$$u^2 = r_1(w(t,x,u))x + r_2(w(t,x,u)). \tag{17}$$

This equality determines implicitly the function $w(t,x,u)$. Computing the partial derivatives w_x and w_u yields the form of $\eta(t,x,u)$

$$\eta(t,x,u) = -\frac{w_x}{w_u} = \frac{r_1(w(t,x,u))}{2u}.$$

Hence we get the form of the conditional symmetry operator admitted by the initial porous medium equation (5)

$$Q = \partial_x - \frac{r_1(w(t,x,u))}{2u}\partial_u,$$

where $w(t,x,u)$ is given by (17). Evidently, this operator does not belong to the symmetry algebra of equation (5) and, consequently, is a conditional symmetry for (5). However, if we will integrate the equation for characteristics

$$u_x = \frac{r_1(w(t,x,u))}{2u}$$

we will get a solution that is invariant with respect to the group of displacements by x. Indeed, the above equation is equivalent to the following PDE:

$$(u^2)_x = r_1(w(t,x,u)).$$

Differentiating it with respect to x yields

$$(u^2)_{xx} = r_1'(w_x + w_u u_x) = r_1'\left(-\frac{r_1}{r_1'x + r_2'} + \frac{2uu_x}{r_1'x + r_2'}\right) = 0.$$

Consequently, the solution corresponding to the conditional symmetry Q belongs to the class

$$u^2(t,x) = R_1(t)x + R_2(t).$$

Inserting this expression into PME yields that $R_1' = R_2' = 0$, which is the same as what was to be proved.

As proved in [21] the same assertion holds for the linear heat equation $u_t = u_{xx}$. Namely, it has been shown that investigating its conditional symmetry within the class of operators (9) is equivalent to solving the heat equation. In view of this fact one gets suspicious that a statement of this kind should hold for a more general class of PDEs. It is indeed the case for arbitrary order evolution type equations in one spatial variable

$$u_t = F(t, x, u, u_1, u_2, \ldots, u_n), \tag{18}$$

where $u_i = \partial^i u / \partial x^i,\ i = 1, 2, \ldots, n.$

Theorem 1 *Let an equation of the form (18) be conditionally-invariant with respect to the operator (9). Then the determining equation for the function $\eta(t, x, u)$ is equivalent to the initial equation (18).*

We give here a principal scheme of the proof omitting the secondary technical details. The criterion for the equation (18) to be conditionally invariant with respect to the operator (9) reads

$$D_t \eta - D_x G \,\Big|_{u_t = G} = 0. \tag{19}$$

Here $G = G(t, x, u)$ is obtained from F by replacing the derivatives u_1, u_2, …, u_n by their expressions via the function $\eta(t, x, u)$ and its derivatives with the use of the side condition $u_x = \eta(t, x, u)$ and its differential consequences. The symbols D_t, D_x stand for the total derivatives with respect to t and x correspondingly, i.e.

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u}, \quad D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u}.$$

Relation (19) is rewritten to become

$$\eta_t + G\eta_u - \left(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} \right) G = 0.$$

Now we make the substitution (11) in the above equation which yields the following PDE:

$$
\begin{aligned}
&w_u^{-2}(w_x w_{ut} - w_u w_{xt}) + w_u^{-2}(w_x w_{uu} - w_u w_{xu})H \\
&\quad -w_u^{-1}\left(w_x\frac{\partial}{\partial x} - w_u\frac{\partial}{\partial u}\right)H = 0.
\end{aligned} \tag{20}
$$

Here H is the function of t, x, w and of the derivatives of w obtained from G via the substitution (11).

Equation (20) is rewritten in the following equivalent form:

$$
w_u^{-2}\left(w_x\frac{\partial}{\partial x} - w_u\frac{\partial}{\partial u}\right)(w_t + w_u H) = 0,
$$

whence we conclude that there exists smooth function $f(t, w)$ such that

$$
w_t + w_u H = f(t, w).
$$

Making in the equation obtained the hodograph transformation (13) yields

$$
U_{y_0} - F(y_0, y_1, U, U_1, \ldots, U_n) = -f(y_0, y_2)U_{y_2}. \tag{21}
$$

At last, changing the independent variables according to formulae (14), (15) yields that the function $U = U(z_0, z_1, z_2)$ is a solution of the equation $U_{z_0} - F(z_0, z_1, U, U_1, \ldots, U_n) = 0$ which contains the variable z_2 as a parameter. As the last equation up to notations coincides with the initial equation (9), the theorem is proved.

3 Some conclusions

Thus an extension of the classical Lie reduction scheme in order to include into consideration conditional symmetries is inefficient for the case of PME in one spatial dimension. So there is an evident need for more general (or simply different) approaches to constructing its general solutions. One of the possibilities is utilizing high-order conditional symmetries of PDEs introduced independently by Zhdanov and Fushchych [22, 23] and Fokas and Liu [24]. These symmetries provide a theoretical background for a procedure of 'nonlinear separation of variables' suggested by Galaktionov [25]. As we believe, in this way it might be possible to construct principally new exact solutions of PME. For example, using a technique developed in [23] we have established that the porous medium equation with a source

$$u_t = \lambda(uu_{xx} + u_x^2) + au^2 + bu + c, \tag{22}$$

where a, b, c are arbitrary constants is conditionally-invariant with respect to the Lie-Bäcklund vector field

$$Q = (u_{xxx} + \frac{a}{2\lambda}u_x)\frac{\partial}{\partial u} + \ldots .$$

This symmetry can be utilized in order to reduce PDE (22) to *three* ordinary differential equations and thus get its new non-Lie solutions.

The fact that PDEs under consideration are of parabolic type is crucial for holding no-go theorems like Theorem 1. Consider a hyperbolic type equation, say, the nonlinear wave equation

$$u_{tt} - u_{xx} = F(u). \tag{23}$$

Let us study its conditional invariance within the class of operators (9). A simple computation yields that each nonlinear PDE of the form (23) admitting an operator (9) is equivalent to the weakly nonlinear wave equation

$$u_{tt} - u_{xx} = \lambda u \ln u. \tag{24}$$

Furthermore, the conditional symmetry operator admitted by (24) takes the form $Q = \partial_x - a(x)u$, where $a(x)$ is a solution of the nonlinear ordinary differential equation

$$a'' + 2aa' + \lambda a = 0. \tag{25}$$

Note that the above equation is integrable by quadratures. Its particular solution $a(x) = -\frac{1}{2}\lambda(x + \mathrm{const})$ gives rise to an invariant solution of (24)

$$u(t,x) = \exp\left\{\frac{\lambda}{4}(t^2 - (x + \mathrm{const})^2)\right\}.$$

All other solutions of (25) yields non-Lie solutions of the wave equation with the logarithmic nonlinearity (24). These solutions have the form

$$u(t,x) = \exp\left\{\int\limits_{x_0}^{x} a(y)dy\right\}\varphi(t),$$

the function $\varphi(t)$ satisfying the nonlinear ordinary differential equation

$$\varphi'' = (C + \lambda \ln \varphi)\varphi.$$

Here $C = a'(x) + a^2(x) + \int\limits_{x_0}^{x} a(y)dy$ is the first integral of the ordinary differential equation (25).

So for the hyperbolic type equations the method of conditional symmetries is quite efficient and yields new exact solutions. A principal difference between the cases of hyperbolic and parabolic type PDEs is that in the case of the latter it is possible to eliminate all derivatives using the equation, side condition $u_x = \eta(t, x, u)$ and its differential consequences. This means that there is nothing to split by and the system of determining equations reduces to one PDE. And it is only natural that this equation is equivalent to solving the initial equation whose conditional symmetry is investigated. We guess that the same assertion holds for multi-dimensional evolution type PDEs

$$u_t = F(t, \vec{x}, u, \underset{1}{u}, \ldots, \underset{N}{u}), \tag{26}$$

where $\vec{x} = (x_1, \ldots, x_n)$ and the symbol $\underset{k}{u}$ stands for the set of kth order derivatives of the function $u(t, \vec{x})$ with respect to the spatial variables $x_1, \ldots, x_n$. Namely, the problem of describing conditional symmetries of PDE (26) within the class of operators

$$Q_i = \partial_{x_i} - \eta_i(t, \vec{x}, u)$$

is equivalent to integrating (26).

Note that the problem of symmetry reduction of the three-dimensional PME

$$u_t = \lambda\Delta(u^2)$$

has been completely solved in [26]. We intend to devote one of our future publications to investigating conditional symmetries of PME in three spatial dimensions.

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