# On algebraic classification of Hermitian quasi-exactly solvable matrix Schrödinger operators on line

Stanislav Spichak* and Renat Zhdanov†

Institute of Mathematics, 3 Tereshchenkivska Street, 252004 Kyiv, Ukraine

**Abstract**

We construct six multi-parameter families of Hermitian quasi-exactly solvable matrix Schrödinger operators in one variable. The method for finding these operators relies heavily upon a special representation of the Lie algebra $o(2,2) \cong sl(2) \bigoplus sl(2)$ whose representation space contains an invariant finite-dimensional subspace. Furthermore, we select those quasi-exactly solvable matrix models that have square integrable eigenfunctions on $\mathbf{R}$. These models are in direct analogy with the quasi-exactly solvable scalar Schrödinger operators obtained by Turbiner and Ushveridze.

## 1 Introduction

In the papers [1, 2] we have extended the Turbiner-Shifman approach [3]–[5] (see, also [6, 7]) to the construction of quasi-exactly solvable (QES) models on line for the case of matrix Hamiltonians. We remind that originally their method was applied to scalar one-dimensional stationary Schrödinger equations. Later on it was extended to the case of multi-dimensional scalar stationary Schrödinger equations [8]–[10] (see, also [11]).

The procedure of constructing a QES matrix (scalar) model is based on the concept of a Lie-algebraic Hamiltonian. We call a second-order operator in one variable Lie-algebraic if the following requirements are met:

- The Hamiltonian is a quadratic form with constant coefficients of first-order operators $Q_1, Q_2, \ldots, Q_n$ forming a Lie algebra $g$;
- The Lie algebra $g$ has a finite-dimensional invariant subspace $\mathcal{I}$ of the whole representation space.

Now if a given Hamiltonian $H[x]$ is Lie-algebraic, then after being restricted to the space $\mathcal{I}$ it becomes a matrix operator $\mathcal{H}$ whose eigenvalues and eigenvectors are computed in a purely algebraic way. This means that the Hamiltonian $H[x]$ is quasi-exactly solvable (for further details on scalar QES models see [11]).

*Electronic mail: spichak@apmat.freenet.kiev.ua

†Electronic mail: renat@imath.kiev.ua

It should be noted that there exist alternative approaches to constructing matrix QES models [12]–[17]. The principal idea of these is fixing the form of basis elements of the invariant space $\mathcal{I}$. They are chosen to be polynomials in $x$. This assumption leads to a challenging problem of classification of superalgebras by matrix-differential operators in one variable [17].

We impose no *a priori* restrictions on the form of basis elements of the space $\mathcal{I}$. What is fixed is the class to which the basis elements of the Lie algebra $g$ should belong. Following [1, 2] we choose this class $\mathcal{L}$ as the set of matrix differential operators of the form

$$\mathcal{L} = \{Q : \quad Q = a(x)\partial_x + A(x)\}\,. \tag{1}$$

Here $a(x)$ is a smooth real-valued function and $A(x)$ is an $N \times N$ matrix whose entries are smooth complex-valued functions of $x$. Hereafter we denote $d/dx$ as $\partial_x$.

Evidently, $\mathcal{L}$ can be treated as an infinite-dimensional Lie algebra with a standard commutator as a Lie bracket. Given a subalgebra $\langle Q_1, Q_2, \ldots, Q_n\rangle$ of the algebra $\mathcal{L}$, whose representation space contains a finite-dimensional invariant subspace, we can easily construct a QES matrix model. To this end we compose a bilinear combination of the operators $Q_1, Q_2, \ldots, Q_n$ (one of them may be the unit $N \times N$ matrix $I$) with constant complex coefficients $\alpha_{jk}$ and get

$$H[x] = \left(\sum_{j,k=1}^{n} \alpha_{jk} Q_j Q_k\right). \tag{2}$$

So there arises a natural problem of classification of subalgebras of the algebra $\mathcal{L}$ within its inner automorphism group. The problem of classification of inequivalent realizations of Lie algebras by first-order differential operators in one and two variables has been solved in a full generality by Lie itself [18, 19] (see, also [20]). However, the classification problem for the case when $A(x) \neq f(x)I$ with a scalar function $f(x)$ is open by now. In the paper [2] we have classified realizations of the Lie algebras of the dimension up to three by the operators belonging to $\mathcal{L}$ with an arbitrary $N$. Next, fixing $N = 2$ we have selected those giving rise to QES matrix Hamiltonians $H[x]$. It occurs that the only three-dimensional algebra that meets this requirement is the algebra $sl(2)$ (which is fairly easy to predict taking into account the scalar case!). This yields the two families of $2 \times 2$ QES models, one of them under proper restrictions giving rise to the well-known family of scalar QES Hamiltonians (for more details, see [2]).

As is well-known a physically meaningful QES matrix Schrödinger operator has to be Hermitian. This requirement imposes restrictions on the choice of QES models which somehow were beyond considerations of our previous papers [1, 2]. It should be noted that a problem of reducing QES scalar operator to a Hermitian form is fairly trivial and solved straightforwardly by rearranging a dependent variable and making an appropriate gauge transformation of the wave function. However, for the case of matrix QES first- or second-order operators the problem of transforming these to Hermitian Schrödinger forms becomes a non-trivial one and requires very involved calculations. It occurs that in contrast to the scalar case, not every second-order matrix QES operator can be reduced to a Hermitian form. One of the principal aims of the present paper is developing a systematic algebraic procedure for constructing QES Hermitian matrix Schrödinger operators

$$\hat{H}[x] = \partial_x^2 + V(x). \tag{3}$$

This requires a slight modification of the algebraic procedure used in [2]. We consider as an algebra $g$ the direct sum of two $sl(2)$ algebras which is equivalent to the algebra $o(2,2)$. The necessary algebraic structures are introduced in Section 2. The next Section is devoted to constructing in a regular way Hermitian QES matrix Schrödinger operators on line which is a core result of the paper. We give the list of thus obtained QES models in Section 4.

A more strong constraint imposed on the QES Schrödinger operators is that the basis elements of invariant space $\mathcal{I}$ must be square integrable on $\mathbf{R}$. A detailed study of this problem for the case of scalar QES Schrödinger operators has been carried out recently in [21]. Using the above mentioned results we have constructed in the present paper several classes of QES matrix Schrödinger operators (Hamiltonians) having finite-dimensional invariant spaces whose basis elements are square integrable on $\mathbf{R}$. Since these Hamiltonians are, in our opinion, the most important result of the paper we list them below without giving derivation details which are based on tedious calculations of Sections 2–4. The models 1–4 given below are particular cases of the more general Hermitian QES Schrödinger operators constructed in Section 4.

**Model 1.** $(\hat{H}[y]+E)\psi(y)=0$, where

$$\hat{H}[y]=\partial_y^2-\frac{y^6}{256}+\frac{4m-1}{16}y^2-\frac{1}{4}y^2\sigma_3-\sigma_1.$$

This model corresponds to Case 1 of Section 4, where $\alpha_1=1, \beta_0=\frac{1}{2}, \beta_2=-1$ and the remaining coefficients are equal to zero. The invariant space $\mathcal{I}$ of this operator has the dimension $2m$ and is spanned by the vectors

$$\begin{aligned}\vec{f}_j &= \exp\left(-\frac{y^4}{64}\right)\left(\frac{y}{2}\right)^{2j}\vec{e}_1,\\ \vec{g}_k &= \exp\left(-\frac{y^4}{64}\right)\left(m\left(\frac{y}{2}\right)^{2k}\vec{e}_2-k\left(\frac{y}{2}\right)^{2k-2}\vec{e}_1\right),\end{aligned}$$

where $j=0,\ldots,m-2,\ k=0,\ldots,m,\ \vec{e}_1=(1,0)^T, \vec{e}_2=(0,1)^T$ and $m$ is an arbitrary natural number.

It is not difficult to verify that the basis vectors of the invariant space $\mathcal{I}$ are square integrable on the interval $(-\infty,+\infty)$. One more remark is that there exists an analogous QES scalar Schrödinger operator whose invariant space has square integrable basis vectors (see, for more details [3, 22]).

**Model 2.** $(\hat{H}[y]+E)\psi(y)=0$, where

$$\begin{aligned}\hat{H}[y] &= \partial_y^2-\frac{1}{4}-\frac{1}{4}\exp(-2y)+m\exp(-y)+\frac{1}{2}\exp(2y)\\ &+\left[m\frac{\sqrt{3}+1}{2}\sin(\sqrt{2}e^y)-\frac{\sqrt{6}}{2}\cos(\sqrt{2}e^y)-\exp(-y)\sin(\sqrt{2}e^y)\right]\sigma_1\\ &+\left[m\frac{\sqrt{3}+1}{2}\cos(\sqrt{2}e^y)+\frac{\sqrt{6}}{2}\sin(\sqrt{2}e^y)-\exp(-y)\cos(\sqrt{2}e^y)\right]\sigma_3.\end{aligned}$$

This model corresponds to Case 3 of Section 4, where $\alpha_2=1, \beta_2=-1, \gamma_1=-1$ and the remaining coefficients are equal to zero. The invariant space $\mathcal{I}$ of this operator has the dimension $2m$ and is spanned by the vectors

$$\begin{aligned}\vec{f}_j &= U^{-1}(y)\exp(-jy)\vec{e}_1,\\ \vec{g}_k &= U^{-1}(y)\left(m\exp(-ky)\vec{e}_2-k\exp(-(k-1)y)\vec{e}_1\right),\end{aligned}$$

where $j=0,\dots,m-2, k=0,\dots,m$, $m$ is an arbitrary natural number and

$$\begin{aligned} U^{-1}(y) &= \frac{1}{2\sqrt{2}}\exp\left(-\frac{y}{2}\right)\exp\left(-\frac{1}{2}e^{-y}\right) \\ &\times(\sqrt{3}+\sqrt{2}-\sigma_3)\left[\cos(\sqrt{2}e^y)+\frac{i\sqrt{3}\sigma_2-\sigma_1}{\sqrt{2}}\sin(\sqrt{2}e^y)\right]. \end{aligned}$$

The basis vectors of the invariant space $\mathcal{I}$ are square integrable. Indeed, the functions $\vec{f}_j(y)$ and $\vec{g}_k(y)$ behave asymptotically as $\exp\left(-\frac{(2j+1)y}{2}\right)$ and $\exp\left(-\frac{(2k+1)y}{2}\right)$, correspondingly, with $y\to+\infty$. Furthermore, they behave as $\exp\left(-\frac{(2j+1)y}{2}\right)\exp\left(-\frac{1}{2}e^{-y}\right)$ and $\exp\left(-\frac{(2k+1)y}{2}\right)$ $\times\exp\left(-\frac{1}{2}e^{-y}\right)$, correspondingly, with $y\to-\infty$. This means that they vanish rapidly provided $y\to\pm\infty$.

**Model 3.** $(\hat{H}[y]+E)\psi(y)=0$, where

$$\begin{aligned} \hat{H}[y] &= \partial_y^2+\frac{1}{4\sinh^2 y}\left[-\frac{1}{4}\cosh^4 y+(2m-1)\cosh^3 y-2\cosh^2 y\right. \\ &\left.-\,2m\cosh y+2\right]+\frac{1}{2}\left(2m-1-\cosh y\right)\sigma_3-\frac{1}{2}\sigma_1+\frac{1}{2}. \end{aligned}$$

This model corresponds to Case 1 of Section 4, where $\alpha_0=-1,\alpha_2=1,\beta_1=1,\beta_2=-\frac{1}{2}$ and the remaining coefficients are equal to zero. The invariant space $\mathcal{I}$ of this operator has the dimension $2m$ and is spanned by the vectors

$$\begin{aligned} \vec{f}_j &= U^{-1}(y)\exp(-jy)\vec{e}_1, \\ \vec{g}_k &= U^{-1}(y)\left(m\exp(-ky)\vec{e}_2-k\exp(-(k-1)y)\vec{e}_1\right), \end{aligned}$$

where $j=0,\dots,m-2, k=0,\dots,m$, $m$ is an arbitrary natural number and

$$U^{-1}(y) = \exp\left(-\frac{\cosh y}{4}\right)\left|\tanh\frac{y}{2}\right|^{-1/4}.$$

It is straightforward to check that the basis vectors of the invariant space $\mathcal{I}$ are square integrable on the interval $(-\infty,+\infty)$.

**Model 4.** $(\hat{H}[y]+E)\psi(y)=0$, where

$$\begin{aligned} \hat{H}[y] &= \partial_y^2-\frac{y^2}{16}+\frac{5m^2-2m}{4m^2y^2}+\frac{(2m-1)(4m-1)}{m\rho}\sin\left(-\frac{\rho}{2}\ln|y|\right)\sigma_1 \\ &-\frac{4m-1}{2}\sqrt{\frac{2m-1}{2m}}\cos\left(-\frac{\rho}{2}\ln|y|\right)\sigma_3+\frac{1}{2} \end{aligned}$$

and $\rho=\frac{\sqrt{16m^2-8m}}{m}$.

This model corresponds to Case 6.1 of Section 4, where $\alpha_1=4,\beta_0=4,\beta_1=-1,\gamma_2=\frac{4m-1}{m},\gamma_3=\frac{1}{m}$ and the remaining coefficients are equal to zero. The invariant space $\mathcal{I}$ of this operator has the dimension $2m$ and is spanned by the vectors

$$\begin{aligned} \vec{f}_j &= U^{-1}(y)\exp(-jy)\vec{e}_1, \\ \vec{g}_k &= U^{-1}(y)\left(m\exp(-ky)\vec{e}_2-k\exp(-(k-1)y)\vec{e}_1\right), \end{aligned}$$

where $j = 0, \dots, m-2, k = 0, \dots, m$, $m$ is an arbitrary natural number and

$$U^{-1}(y) \;=\; |y|^{1/2} \exp\left(-\frac{y^2}{8}\right) \left[\cos\left(\frac{\rho}{4} \ln|y|\right) + \frac{i(4m-1)\sigma_2 + \sigma_3}{\sqrt{16m^2 - 8m}} \sin\left(\frac{\rho}{4} \ln|y|\right)\right] \Lambda$$

with $\Lambda = 1 + i(\sqrt{16m^2 - 8m} + 4m - 1)\sigma_1$.

The basis vectors of the invariant space $\mathcal{I}$ are evidently square integrable on the interval $(-\infty, +\infty)$.

## 2 Extension of the algebra $sl(2)$

Following [1, 2] we consider the realization of the algebra $sl(2)$

$$[Q_-,\ Q_+] = 2Q_0, \quad [Q_\pm,\ Q_0] = \pm Q_\pm \tag{4}$$

having the basis elements

$$Q_- = \partial_x, \quad Q_0 = x\partial_x - \frac{m-1}{2} + S_0, \quad Q_+ = x^2\partial_x - (m-1)x + 2S_0 x + S_+, \tag{5}$$

where $S_0 = \sigma_3/2$, $S_+ = (i\sigma_2 + \sigma_1)/2$, $\sigma_k$ are the $2 \times 2$ Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $m \geq 2$ is an arbitrary natural number. This representation gives rise to a family of QES models and furthermore the algebra (5) has the following finite-dimensional invariant space

$$\begin{aligned} &\mathcal{I}_{sl(2)} = \mathcal{I}_1 \bigoplus \mathcal{I}_2 = \langle \vec{e}_1, x\vec{e}_1, \dots, x^{m-2}\vec{e}_1 \rangle \bigoplus \\ &\langle m\vec{e}_2, \dots, mx^j\vec{e}_2 - jx^{j-1}\vec{e}_1, \dots, mx^m\vec{e}_2 - mx^{m-1}\vec{e}_1 \rangle. \end{aligned} \tag{6}$$

Since the spaces $\mathcal{I}_1, \mathcal{I}_2$ are invariant with respect to an action of any of the operators (5), the above representation is reducible. A more serious trouble is that it is not possible to construct a QES operator, that is equivalent to a Hermitian Schrödinger operator, by taking a bilinear combination (2) of operators (5) with coefficients being complex numbers. To overcome this difficulty we use the idea indicated in [2] and let the coefficients of the bilinear combination (2) to be constant $2 \times 2$ matrices. To this end we introduce a wider Lie algebra and add to the algebra (5) the following three matrix operators:

$$R_- = S_-, \quad R_0 = S_- x + S_0, \quad R_+ = S_- x^2 + 2S_0 x + S_+, \tag{7}$$

where $S_\pm = (i\sigma_2 \pm \sigma_1)/2$. Note that the matrices $S_-, S_0, S_+$ satisfy the commutation relations of the algebra $sl(2)$ (4).

It is straightforward to verify that the space (6) is invariant with respect to an action of a linear combination of the operators (7). Consider next the following set of operators:

$$\langle T_\pm = Q_\pm - R_\pm,\ T_0 = Q_0 - R_0,\ R_\pm,\ R_0,\ I \rangle, \tag{8}$$

where $Q$ and $R$ are operators (5) and (7), respectively, and $I$ is a unit $2 \times 2$ matrix. By a direct computation we check that the operators $T_\pm, T_0$ as well as the operators $R_\pm, R_0$, fulfill the commutation relations of the algebra $sl(2)$. Furthermore any of the operators $T_\pm, T_0$ commutes with any of the operators $R_\pm, R_0$. Consequently, operators (8) form the Lie algebra

$$sl(2) \bigoplus sl(2) \bigoplus I \cong o(2,2) \bigoplus I.$$

In a sequel we denote this algebra as $g$.

The Casimir operators of the Lie algebra $g$ are multiples of the unit matrix

$$C_1 = T_0^2 - T_+T_- - T_0 = \left(\frac{m^2-1}{4}\right) I, \quad K_2 = R_0^2 - R_+R_- - R_0 = \frac{3}{4}I.$$

Using this fact it can be shown that the representation of $g$ realized on the space $\mathcal{I}_{sl(2)}$ is irreducible.

One more remark is that the operators (8) satisfy the following relations:

$$\begin{aligned} &R_-^2 = 0, \quad R_0^2 = \tfrac{1}{4}, \quad R_+^2 = 0, \\ &\{R_-, R_0\} = 0, \quad \{R_+, R_0\} = 0, \quad \{R_-, R_+\} = -1, \\ &R_-R_0 = \tfrac{1}{2}R_-, \quad R_0R_+ = \tfrac{1}{2}R_+, \quad R_-R_+ = R_0 - \tfrac{1}{2}. \end{aligned} \tag{9}$$

Here $\{Q_1, Q_2\} = Q_1Q_2 + Q_2Q_1$. One of the consequences of this fact is that the algebra $g$ may be considered as a superalgebra which shows and evident link to the results of the paper [17].

# 3 The general form of the Hermitian QES operator

Using the commutation relations of the Lie algebra $g$ together with relations (9) one can show that any bilinear combination of the operators (8) is a linear combination of twenty one (basis) quadratic forms of the operators (8). Composing this linear combination yields all QES models which can be obtained with the help of our approach. However the final goal of the paper is not to get some families of QES matrix second-order operators as such but to get QES Schrödinger operators (3). This means that it is necessary to transform bilinear combination (2) to the standard form (3). What is more, it is essential that the corresponding transformation should be given by explicit formulae, since we need to write down explicitly the matrix potential $V(x)$ of thus obtained QES Schrödinger operator and the basis functions of its invariant space.

The general form of QES model obtainable within the framework of our approach is as follows

$$H[x] = \xi(x)\partial_x^2 + B(x)\partial_x + C(x), \tag{10}$$

where $\xi(x)$ is some real-valued function and $B(x), C(x)$ are matrix functions of the dimension $2\times2$. Let $U(x)$ be an invertible $2\times2$ matrix-function satisfying the system of ordinary differential equations

$$U'(x) = \frac{1}{2\xi(x)}\left(\frac{\xi'(x)}{2} - B(x)\right)U(x), \tag{11}$$

and the function $f(x)$ be defined by the relation

$$f(x) = \pm \int \frac{dx}{\sqrt{\xi(x)}}. \tag{12}$$

Equations (11), (12) ensure absence of terms with the first derivatives in a transformed Hamiltonian, so that the change of variables reducing (10) to the standard form (3) reads as

$$\begin{aligned} x &\to y = f(x), \\ H[x] &\to \hat{H}[y] = \hat{U}^{-1}(y) H[f^{-1}(y)] \hat{U}(y), \end{aligned} \tag{13}$$

where $f^{-1}$ stands for the inverse of $f$ and $\hat{U}(y) = U(f^{-1}(y))$.

Performing the transformation (13) yields the Schrödinger operator

$$\hat{H}[y] = \partial_y^2 + V(y) \tag{14}$$

with

$$\begin{aligned} V(y) = & \left\{ U^{-1}(x) \left[ -\tfrac{1}{4\xi} B^2(x) - \tfrac{1}{2} B'(x) + \tfrac{\xi'}{2\xi} B(x) + C(x) \right] U(x) \right. \\ & \left. + \frac{\xi''}{4} - \frac{3\xi'^2}{16\xi} \right\} \Bigg|_{x=f^{-1}(y)}. \end{aligned} \tag{15}$$

Hereafter, the notation $\{W(x)\}_{x=f^{-1}(y)}$ means that we should replace $x$ with $f^{-1}(y)$ in the expression $W(x)$.

Furthermore, if we denote the basis elements of the invariant space (6) as $\vec{f}_1(x), \ldots, \vec{f}_{2m}(x)$, then the invariant space of the operator $\hat{H}[y]$ takes the form

$$\hat{\mathcal{I}}_{sl(2)} = \left\langle \hat{U}^{-1}(y) \vec{f}_1(f^{-1}(y)), \ldots, \hat{U}^{-1}(y) \vec{f}_{2m}(f^{-1}(y)) \right\rangle. \tag{16}$$

In view of the remark made at the beginning of this section we are looking for such QES models that the transformation law (13) can be given explicitly. This means that we should be able to construct a solution of system (11) in an explicit form. To achieve this goal we select from the above mentioned set of twenty one linearly independent quadratic forms of operators (8) those ones whose linear combinations give rise to Hamiltonians (10) with $B(x) = f(x) + \sum\limits_{i=1}^{3} \gamma_i \sigma_i$, where $f(x)$ is a complex-valued scalar function and $\gamma_i$ are complex constants. It turns out that the corresponding bilinear combinations of operators (8) form twelve-dimensional vector space whose basis elements can be chosen as follows

$$\begin{aligned} & A_0 = \partial_x^2, \ A_1 = x\partial_x^2, \quad A_2 = x^2\partial_x^2 + (m-1)\sigma_3, \\ & B_0 = \partial_x, \ B_1 = x\partial_x + \frac{\sigma_3}{2}, \quad B_2 = x^2\partial_x - (m-1)x + \sigma_3 x + \sigma_1, \\ & C_1 = \sigma_1\partial_x + \frac{m}{2}\sigma_3, \quad C_2 = i\sigma_2\partial_x + \frac{m}{2}\sigma_3, \quad C_3 = \sigma_3\partial_x, \\ & D_1 = x^3\partial_x^2 - 2\sigma_1 x\partial_x + (3m - m^2 - 3)x + (2m-3)x\sigma_3 + (4m-4)\sigma_1, \\ & D_2 = x^3\partial_x^2 - 2i\sigma_2 x\partial_x + (3m - m^2 - 3)x + (2m-3)x\sigma_3 + (4m-4)\sigma_1, \\ & D_3 = 2\sigma_3 x\partial_x + (1-2m)\sigma_3. \end{aligned} \tag{17}$$

However, in the present paper we study systematically the first nine quadratic forms from the above list and exclude the quadratic forms $D_1, D_2, D_3$ from further considerations.

Thus the general form of the Hamiltonian to be considered in a sequel is as follows

$$\begin{aligned} H[x] &= \sum_{\mu=0}^{2}(\alpha_\mu A_\mu + \beta_\mu B_\mu) + \sum_{i=1}^{3}\gamma_i C_i = (\alpha_2 x^2 + \alpha_1 x + \alpha_0)\partial_x^2 \\ &+(\beta_2 x^2 + \beta_1 x + \beta_0 + \gamma_1\sigma_1 + i\gamma_2\sigma_2 + \gamma_3\sigma_3)\partial_x + \beta_2\sigma_3 x \\ &-\beta_2(m-1)x + \beta_2\sigma_1 + \left[\alpha_2(m-1) + \frac{\beta_1}{2} + \frac{m}{2}(\gamma_1+\gamma_2)\right]\sigma_3. \end{aligned} \tag{18}$$

Here $\alpha_0, \alpha_1, \alpha_2$ are arbitrary real constants and $\beta_0, \ldots, \gamma_3$ are arbitrary complex constants.

If we denote

$$\begin{aligned} &\tilde{\gamma}_1 = \gamma_1,\ \tilde{\gamma}_2 = i\gamma_2,\ \tilde{\gamma}_3 = \gamma_3,\ \delta = 2\alpha_2(m-1) + \beta_1 + m(\gamma_1+\gamma_2), \\ &\xi(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0,\ \eta(x) = \beta_2 x^2 + \beta_1 x + \beta_0, \end{aligned} \tag{19}$$

then the general solution of system (11) reads as

$$U(x) = \xi^{1/4}(x)\exp\left[-\frac{1}{2}\int\frac{\eta(x)}{\xi(x)}dx\right]\exp\left[-\frac{1}{2}\tilde{\gamma}_i\sigma_i\int\frac{1}{\xi(x)}dx\right]\Lambda, \tag{20}$$

where $\Lambda$ is an arbitrary constant invertible $2\times 2$ matrix. Performing the transformation (13) with $U(x)$ being given by (20) reduces QES operator (18) to a Schrödinger form (14), where

$$\begin{aligned} V(y) &= \left\{\frac{1}{4\xi}\Lambda^{-1}\{-\eta^2 + 2\xi'\eta - 2\xi\eta' - 4\beta_2(m-1)x\xi - \tilde{\gamma}_i^2 \right. \\ &+2(\xi'-\eta)\tilde{\gamma}_i\sigma_i + 4\beta_2\xi\tilde{U}^{-1}(x)\sigma_1\tilde{U}(x) + (4\beta_2 x + 2\delta)\xi \\ &\left.\times\tilde{U}^{-1}(x)\sigma_3\tilde{U}(x)\}\Lambda + \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16\xi}\right\}\Bigg|_{x=f^{-1}(y)}. \end{aligned} \tag{21}$$

Here $\xi, \eta$ are functions of $x$ defined in (19), $f^{-1}(y)$ is the inverse of $f(x)$ which is given by (12) and

$$\tilde{U}(x) = \exp\left[-\frac{1}{2}\tilde{\gamma}_i\sigma_i\int\frac{1}{\xi(x)}dx\right].$$

The requirement of hermiticity of the Schrödinger operator (14) is equivalent to the requirement of hermiticity of the matrix $V(y)$. To select from the multi-parameter family of matrices (21) Hermitian ones we will make use of the following technical lemmas.

**Lemma 1** *The matrices $z\sigma_a, w(\sigma_a \pm i\sigma_b), a \neq b$, with $\{z, w\} \subset \mathbf{C}, z \notin \mathbf{R}, w \neq 0$ cannot be reduced to Hermitian matrices with the help of a transformation*

$$A \to A' = \Lambda^{-1}A\Lambda, \tag{22}$$

*where $\Lambda$ is an invertible constant $2\times 2$ matrix.*

**Proof.** It is sufficient to prove the statement for the case $a = 1, b = 2$, since all other cases are equivalent to this one. Suppose the inverse, namely that there exists a transformation (22) transforming the matrix $z\sigma_1$ to a Hermitian matrix $A'$. As $\mathrm{tr}\,(z\sigma_1) = \mathrm{tr}\,A' = 0$, the matrix $A'$ has the form $\alpha_i\sigma_i$ with some real constants $\alpha_i$. Next, from the equality $\det(z\sigma_1) = \det A'$ we get $z^2 = \alpha_i^2$. The last relation is in contradiction to the fact that $z \notin \mathbf{R}$. Consequently, the matrix $z\sigma_1$ cannot be reduced to a Hermitian matrix with the aid of a transformation (22).

Let us turn now to the matrix $w(\sigma_1 + i\sigma_2)$. Taking a general form of the matrix $\Lambda$

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we represent (22) as follows

$$A' = \Lambda^{-1} w(\sigma_1 + i\sigma_2)\Lambda = \frac{2w}{\delta}\begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix}, \qquad \delta = \det\Lambda.$$

The conditions of hermiticity of the matrix $A'$ read

$$\frac{w}{\delta}cd = \frac{\bar{w}}{\bar{\delta}}\bar{c}\bar{d}, \qquad \frac{-w}{\delta}c^2 = \frac{\bar{w}}{\bar{\delta}}\bar{d}^2,$$

where the bar over a symbol stands for the complex conjugation.

It follows from the second relation that $c, d$ can vanish only simultaneously which is impossible in view of the fact that the matrix $\Lambda$ is invertible. Consequently, the relation $cd \neq 0$ holds. Hence we get

$$\frac{-d}{c} = \frac{\bar{c}}{\bar{d}} \leftrightarrow |c|^2 + |d|^2 = 0.$$

This contradiction proves the fact that the matrix $w(\sigma_1 + i\sigma_2)$ cannot be reduced to a Hermitian form.

As the matrix $\sigma_1 + i\sigma_2$ is transformed to become $\sigma_1 - i\sigma_2$ with the use of an appropriate transformation (22), the lemma is proved.

**Lemma 2** *Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$ be complex vectors and $\vec{\sigma}$ be the vector whose components are the Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$. Then the following assertions holds true.*

1. *A non-zero matrix $\vec{a}\vec{\sigma}$ is reduced to a Hermitian form with the help of a transformation (22) iff $\vec{a}^2 > 0$ (this inequality means, in particular, that $\vec{a}^2 \in \mathbf{R}$);*

2. *Non-zero matrices $\vec{a}\vec{\sigma}, \vec{b}\vec{\sigma}$ with $\vec{b} \neq \lambda\vec{a}$, $\lambda \in \mathbf{R}$, are reduced simultaneously to Hermitian forms with the help of a transformation (22) iff*

$$\vec{a}^2 > 0, \quad \vec{b}^2 > 0, \quad (\vec{a} \times \vec{b})^2 > 0;$$

3. *Matrices $\vec{a}\vec{\sigma}, \vec{b}\vec{\sigma}, \vec{c}\vec{\sigma}$ with $\vec{a} \neq \vec{0}$, $\vec{b} \neq \lambda\vec{a}$, $\vec{c} \neq \mu\vec{b}$, $\{\lambda, \mu\} \subset \mathbf{R}$ are reduced simultaneously to Hermitian forms with the help of a transformation (22) iff*

$$\vec{a}^2 > 0, \quad \vec{b}^2 > 0, \quad (\vec{a} \times \vec{b})^2 > 0,$$
$$\left\{\vec{a}\vec{c}, \quad \vec{b}\vec{c}, \quad (\vec{a} \times \vec{b})\vec{c}\right\} \subset \mathbf{R}.$$

*Here we designate the scalar product of vectors $\vec{a}, \vec{b}$ as $\vec{a}\vec{b}$ and the vector product of these as $\vec{a} \times \vec{b}$.*

**Proof.** Let us first prove the necessity of the assertion 1 of the lemma. Suppose that the non-zero matrix $\vec{a}\vec{\sigma}$ can be reduced to a Hermitian form. We will prove that hence it follows the inequality $\vec{a}^2 > 0$.

Consider the matrices:

$$\Lambda_{ij}(a,b) = \begin{cases} 1 + \epsilon_{ijk} \dfrac{\sqrt{a^2+b^2} - b}{a} i\sigma_k, & a \neq 0, \\ 1, & a = 0, \end{cases} \tag{23}$$

where $(i,j,k)$ = cycle $(1,2,3)$. It is not difficult to verify that these matrices are invertible, provided

$$\sqrt{a_i^2 + a_j^2} \neq 0. \tag{24}$$

Given the condition (24), the following relations hold

$$\sigma_l \to \Lambda_{ij}^{-1}(a,b)\ \sigma_l\ \Lambda_{ij}(a,b) = \begin{cases} \sigma_k, & l = k, \\ \dfrac{b\sigma_i + a\sigma_j}{\sqrt{a^2+b^2}}, & l = i, \\ \dfrac{-a\sigma_i + b\sigma_j}{\sqrt{a^2+b^2}}, & l = j. \end{cases} \tag{25}$$

As $\vec{a}$ is a non-zero vector, there exists at least one pair of the indices $i, j$ such that $a_i^2 + a_j^2 \neq 0$. Applying the transformation (25) with $a = a_i, b = a_j$ we get

$$\vec{a}\vec{\sigma} \to \vec{a}'\vec{\sigma} = \sqrt{a_i^2 + a_j^2}\,\sigma_j + a_k\sigma_k \tag{26}$$

(no summation over the indices $i, j, k$ is carried out). As the direct check shows, the quantity $\vec{a}^2$ is invariant with respect to transformation (25), i.e. $\vec{a}^2 = \vec{a}'^2$.

If $\vec{a}^2 = 0$, then $a'^2_j + a'^2_k = 0$, or $a'_i = \pm i a'_k$. Hence by force of Lemma 1 it follows that the matrix (26) cannot be reduced to a Hermitian form. Consequently, $\vec{a}^2 \neq 0$ and the relation $a'^2_j + a'^2_k \neq 0$ holds true. Applying transformation (25) with $a = \sqrt{a_i^2 + a_j^2},\ b = a_k$ we get

$$\vec{a}'\vec{\sigma} \to \sqrt{\vec{a}^2}\,\sigma_k. \tag{27}$$

Due to Lemma 1, if the number $\sqrt{\vec{a}^2}$ is complex, then the above matrix cannot be transformed to a Hermitian matrix. Consequently, the relation $\vec{a}^2 > 0$ holds true.

The sufficiency of the assertion 1 of the lemma follows from the fact that, given the condition $\vec{a}^2 > 0$, the matrix (27) is Hermitian.

Now we will prove the necessity of the assertion 2 of the lemma. First of all we note that due to assertion 1, $\vec{a}^2 >, \vec{b}^2 > 0$. Next, without loss of generality we can again suppose that $a_i^2 + a_j^2 \neq 0$. Taking the superposition of two transformations of the form (25) with $a = a_i, b = a_j$ and $a = \sqrt{a_i^2 + a_j^2}, b = a_k$ yields

$$\begin{aligned} \Lambda_{ij}(a_i,a_j)\Lambda_{jk}(\sqrt{a_i^2+a_j^2}, a_k) = 1 + i\epsilon_{ijk}\frac{\sqrt{\vec{a}^2} - a_k}{\sqrt{a_i^2+a_j^2}}\sigma_i \\ + i\epsilon_{ijk}\frac{\sqrt{a_i^2+a_j^2} - a_j}{a_i}\sigma_k - i\epsilon_{ijk}\frac{\sqrt{a_i^2+a_j^2} - a_j}{a_i}\frac{\sqrt{\vec{a}^2} - a_k}{\sqrt{a_i^2+a_j^2}}\sigma_j \end{aligned} \tag{28}$$

(here the finite limit exists when $a_i \to 0$). Using this formula and taking into account (25) yield

$$\vec{a}\vec{\sigma} \to \sqrt{\vec{a}^2}\sigma_k, \quad \vec{b}\vec{\sigma} \to \vec{b}'\vec{\sigma} = \frac{b_i a_j - b_j a_i}{\sqrt{a_i^2 + a_j^2}}\sigma_i + \frac{a_k \vec{a}\vec{b} - b_k \vec{a}^2}{\sqrt{\vec{a}^2}\sqrt{a_i^2 + a_j^2}}\sigma_j + \frac{\vec{a}\vec{b}}{\sqrt{\vec{a}^2}}\sigma_k. \tag{29}$$

Let us show that the necessary condition for the matrices $\sqrt{\vec{a}^2}\sigma_k$, $\vec{b}'\vec{\sigma}$ to be reducible to Hermitian forms simultaneously reads as $\vec{a}\vec{b} \in \mathbf{R}$. Indeed, as the matrices $\vec{b}'\vec{\sigma}, \sigma_k$ are simultaneously reduced to Hermitian forms, the matrix $\vec{b}'\vec{\sigma} + \lambda\sigma_k$ can be reduced to a Hermitian form with any real $\lambda$. Hence, in view of the assertion 1 we conclude that

$$b'^2_i + b'^2_j + (b'_k + \lambda)^2 > 0, \tag{30}$$

where $\lambda$ is an arbitrary real number. The above equality may be valid only when $b'_k = \frac{\vec{a}\vec{b}}{\sqrt{\vec{a}^2}} \in \mathbf{R}$.

Choosing $\lambda = -b'_k$ in (30) yields that $b'^2_i + b'^2_j > 0$. Since $b'^2_i + b'^2_j = (\vec{a} \times \vec{b})^2$, hence we get the desired inequality $(\vec{a} \times \vec{b})^2 > 0$. The necessity is proved.

In order to prove the sufficiency of the assertion 2, we consider transformation (25) with

$$a = \frac{b_i a_j - b_j a_i}{\sqrt{a_i^2 + a_j^2}}, \qquad b = \frac{a_k \vec{a}\vec{b} - b_k \vec{a}^2}{\sqrt{\vec{a}^2}\sqrt{a_i^2 + a_j^2}}. \tag{31}$$

This transformation leaves the matrix $\sqrt{\vec{a}^2}\sigma_k$ invariant, while the matrix $\vec{b}'\vec{\sigma}$ (29) transforms as follows

$$\vec{b}'\vec{\sigma} \to \vec{b}''\vec{\sigma} = \frac{\sqrt{(\vec{a} \times \vec{b})^2}}{\sqrt{\vec{a}^2}}\sigma_j + \frac{\vec{a}\vec{b}}{\sqrt{\vec{a}^2}}\sigma_k, \tag{32}$$

whence it follows the sufficiency of the assertion 2.

The proof of the assertion 3 of the lemma is similar to one of the assertion 2. The first three conditions are obtained with account of the assertion 2. A sequence of transformations (25) with $a, b$ of the form (28), (31) transforms the matrix $\vec{c}\vec{\sigma}$ to become

$$\vec{c}\vec{\sigma} \to \vec{c}\,''\vec{\sigma} = \frac{\epsilon_{ijk}\vec{a}(\vec{c} \times \vec{b})}{\sqrt{(\vec{c} \times \vec{b})^2}}\sigma_i + \frac{(\vec{a} \times \vec{b})(\vec{a} \times \vec{c})}{\sqrt{(\vec{c} \times \vec{b})^2}\sqrt{\vec{a}^2}}\sigma_j + \frac{\vec{a}\vec{c}}{\sqrt{\vec{a}^2}}\sigma_k. \tag{33}$$

Using the standard identities for the mixed vector products we establish that the coefficients by the matrices $\sigma_i, \sigma_j, \sigma_k$ are real if and only if the relations

$$\left\{\vec{a}\vec{c},\ \vec{b}\vec{c},\ (\vec{a} \times \vec{b})\vec{c}\right\} \subset \mathbf{R}$$

hold true. This completes the proof of Lemma 2.

# 4 QES matrix models

Lemma 2 plays the crucial role when reducing operators (18) to Hermitian forms. This is done as follows. Firstly, we reduce QES operator (18) to the Schrödinger form

$$\partial_y^2 + f(y)\vec{a}\vec{\sigma} + g(y)\vec{b}\vec{\sigma} + h(y)\vec{c}\vec{\sigma} + r(y).$$

Note that when performing a change of variables (13) we firstly transform the function and after that make the change of the dependent variable.

In the above formulae $f, g, h, r$ are some linearly-independent real-valued functions and $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$ are complex constant vectors whose components depend on the parameters $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$. Next, using Lemma 2 we obtain the conditions for the parameters $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ that provide a simultaneous reducibility of the matrices $\vec{a}\vec{\sigma}, \vec{b}\vec{\sigma}, \vec{c}\vec{\sigma}$ to Hermitian forms. Then, making use of formulae (23), (28), (31) we find the form of the matrix $\Lambda$. Formulae (27), (32), (33) yield explicit forms of the transformed matrices $\vec{a}\vec{\sigma}, \vec{b}\vec{\sigma}, \vec{c}\vec{\sigma}$ and, consequently, the Hermitian form of the matrix potential $V(y)$.

Applying this classification scheme we have described all possible values of parameters $\alpha_\mu$, $\beta_\mu$, $\gamma_i$ enabling reducibility of operator $H[x]$ (18) to a Hermitian Schrödinger operator. As a result, we have arrived at the six inequivalent classes of Schrödinger operators (3) with a Hermitian matrix $V(x)$. This, in its turn, yields a complete description of QES matrix models (18) that can be reduced to Hermitian Schrödinger matrix operators. We give below the final results, namely, the restrictions on the choice of parameters and the explicit forms of the QES Hermitian Schrödinger operators and then consider in some detail derivation of the corresponding formulae for one of the six inequivalent cases. In the formulae below we denote the disjunction of two statements $A$ and $B$ as $[A]\bigvee[B]$.

**Case 1.** $\tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma}_3 = 0$ and

$$[\beta_0,\ \beta_1,\ \beta_2 \in \mathbf{R}]\bigvee[\beta_2 = 0,\ \beta_1 = 2\alpha_2,\ \beta_0 = \alpha_1 + i\mu,\ \mu \in \mathbf{R}];$$

$$\begin{aligned}
\hat{H}[y] &= \partial_y^2 + \Bigg\{\frac{1}{4(\alpha_2 x^2 + \alpha_1 x + \alpha_0)}\{-\beta_2^2 x^4 - [2\beta_1\beta_2 + 4\alpha_2\beta_2(m-1)]x^3 \\
&\quad + \left[2\alpha_2\beta_1 - 2\alpha_1\beta_2 - \beta_1^2 - 2\beta_0\beta_2 - 4\alpha_1\beta_2(m-1)\right]x^2 \\
&\quad + [4\alpha_2\beta_0 - 2\beta_0\beta_1 - 4m\alpha_0\beta_2]\,x + 2\alpha_1\beta_0 - 2\alpha_0\beta_1 - \beta_0^2 \\
&\quad + 4\beta_2(\alpha_2 x^2 + \alpha_1 x + \alpha_0)\sigma_1 + (4\beta_2 x + 2\delta)(\alpha_2 x^2 + \alpha_1 x + \alpha_0)\sigma_3\} \\
&\quad + \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)}\Bigg\}\Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= 1.
\end{aligned}$$

Hereafter, we denote the inverse of the function

$$y = f(x) \equiv \int \frac{dx}{\sqrt{\alpha_2 x^2 + \alpha_1 x + \alpha_0}}, \tag{34}$$

as $f^{-1}(y)$.

**Case 2.** $\beta_2, \delta = 0$ and

$$\begin{aligned}
&2\alpha_2\beta_1 - \beta_1^2 \in \mathbf{R},\ 2\alpha_2\beta_0 - \beta_0\beta_1 \in \mathbf{R},\ 2\alpha_1\beta_0 - 2\beta_1\alpha_0 - \beta_0^2 - \tilde{\gamma}_i^2 \in \mathbf{R}, \\
&\left[(2\alpha_2 - \beta_1)^2\tilde{\gamma}_i^2 > 0\right]\bigvee[2\alpha_2 - \beta_1 = 0],\ \left[(\alpha_1 - \beta_0)^2\tilde{\gamma}_i^2 > 0\right]\bigvee[\alpha_1 - \beta_0 = 0];
\end{aligned}$$

$$
\begin{aligned}
\hat{H}[y] &= \partial_y^2 + \left\{ \frac{1}{4(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \left\{ \beta_1(2\alpha_2 - \beta_1)x^2 + 2\beta_0(2\alpha_2 - \beta_1)x \right.\right. \\
&\quad \left. + 2\alpha_1\beta_0 - 2\beta_1\alpha_0 - \beta_0^2 - \tilde{\gamma}_i^2 + [2(2\alpha_2 - \beta_1)x + 2(\alpha_1 - \beta_0)]\sqrt{\tilde{\gamma}_i^2}\sigma_3 \right\} \\
&\quad \left. + \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \right\}\Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= \Lambda_{12}(\tilde{\gamma}_1, \tilde{\gamma}_2)\Lambda_{23}(\sqrt{\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2}, \tilde{\gamma}_3), \ \ \tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 \neq 0.
\end{aligned}
$$

(If $\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 = 0$, then one can choose another matrix $\Lambda$ (27) with $\tilde{\gamma}_i^2 + \tilde{\gamma}_j^2 \neq 0$).
**Case 3.** $\alpha_2 \neq 0, \beta_2 \neq 0$ and

$$
\left[ \begin{array}{l} \{\beta_2, \gamma_1\} \subset \mathbf{Re}, \ \gamma_3 = 0, \ \gamma_2 = \sqrt{\gamma_1^2 - 2\alpha_2\gamma_1}, \alpha_2\gamma_1 < 0, \\ \beta_1 = 2\alpha_2 + \beta_2 \dfrac{\alpha_1}{\alpha_2}, \ \beta_0 = \alpha_1 + \beta_2 \dfrac{\alpha_0}{\alpha_2} \end{array} \right];
$$

$$
\begin{aligned}
\hat{H}[y] &= \partial_y^2 + \left\{ \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} + \frac{1}{4(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \left\{ -\beta_2^2 x^4 \right.\right. \\
&\quad - \left[ 2\beta_2^2 \frac{\alpha_1}{\alpha_2} + 4\alpha_2\beta_2 m \right] x^3 - \left[ \frac{\beta_2^2}{\alpha_2^2}(\alpha_1^2 + 2\alpha_0\alpha_2) + 2\alpha_1\beta_2(1 + 2m) \right] x^2 \\
&\quad - \left[ \frac{2\alpha_1\beta_2(\alpha_1\alpha_2 + \alpha_0\beta_2)}{\alpha_2^2} + 4\alpha_0\beta_2 m \right] x + \alpha_1^2 - \beta_2^2 \frac{\alpha_0^2}{\alpha_2^2} - 4\beta_2 \frac{\alpha_0\alpha_1}{\alpha_2} - 4\alpha_0\alpha_2 - 2\alpha_2\gamma_1 \\
&\quad + 4\beta_2 x(\alpha_2 x^2 + \alpha_1 x + \alpha_0) \left[ \sin\left( \theta(y)\sqrt{-2\alpha_2\gamma_1} \right) \sigma_1 + \cos\left( \theta(y)\sqrt{-2\alpha_2\gamma_1} \right) \sigma_3 \right] \\
&\quad + 2(\alpha_2 x^2 + \alpha_1 x + \alpha_0) \left[ \frac{\sin\left( \theta(y)\sqrt{-2\alpha_2\gamma_1} \right)}{\sqrt{-2\alpha_2\gamma_1}} \left( \delta\sqrt{-2\alpha_2\gamma_1}\sigma_1 - 2\beta_2\sqrt{\gamma_1^2 - 2\alpha_2\gamma_1}\sigma_3 \right) \right. \\
&\quad \left.\left.\left. + \cos\left( \theta(y)\sqrt{-2\alpha_2\gamma_1} \right) \left( \frac{2\beta_2\sqrt{\gamma_1^2 - 2\alpha_2\gamma_1}}{\sqrt{-2\alpha_2\gamma_1}}\sigma_1 + \delta\sigma_3 \right) \right] \right\} \right\}\Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= 1 + \left( \sqrt{1 - \frac{2\alpha_2}{\gamma_1}} - \sqrt{\frac{-2\alpha_2}{\gamma_1}} \right) \sigma_3,
\end{aligned}
$$

where the function $\theta = \theta(y)$ is defined as follows

$$
\theta(y) = -\left\{ \int \frac{dx}{\alpha_2 x^2 + \alpha_1 x + \alpha_0} \right\}\Bigg|_{x=f^{-1}(y)}. \tag{35}
$$

**Case 4.** $\alpha_2 \neq 0, \beta_2 = 0.$

**Subcase 4.1.** $\delta \neq 0$, $\gamma_1, \gamma_2$ do not vanish simultaneously and

$$
\gamma_1^2 - \gamma_2^2 < 0, \quad \gamma_3 = i\mu, \quad \{\mu, \delta\} \subset \mathbf{R}, \quad i(\alpha_1 - \beta_0) \in \mathbf{R}, \quad \beta_1 = 2\alpha_2;
$$

$$
\begin{aligned}
\hat{H}[y] &= \partial_y^2 + \Bigg\{\frac{\alpha_2}{2} - \frac{3(2\alpha_2 x+\alpha_1)^2}{16(\alpha_2x^2+\alpha_1x+\alpha_0)} + \frac{1}{4\xi}\Bigg\{-\beta_0^2+2\alpha_1\beta_0-2\alpha_0\beta_1-\tilde{\gamma}_i^2 \\
&+2(\alpha_2x^2+\alpha_1x+\alpha_0)\Bigg[\delta\sqrt{\gamma_2^2-\gamma_1^2}\sigma_1\frac{\sin\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)}{\sqrt{-\tilde{\gamma}_i^2}} \\
&+\frac{-i\delta\gamma_3\sqrt{\gamma_2^2-\gamma_1^2}\sigma_2+\delta(\gamma_1^2-\gamma_2^2)\sigma_3}{\tilde{\gamma}_i^2}\cos\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)\Bigg] + \Bigg[\frac{2\delta\alpha_2\gamma_3}{\tilde{\gamma}_i^2}x^2 \\
&+\frac{2\delta\alpha_1\gamma_3}{\tilde{\gamma}_i^2}x + \frac{(2\alpha_1-2\beta_0)\tilde{\gamma}_i^2+2\delta\alpha_0\gamma_3}{\tilde{\gamma}_i^2}\Bigg]\left(i\sqrt{\gamma_2^2-\gamma_1^2}\sigma_2+\gamma_3\sigma_3\right)\Bigg\}\Bigg\}\Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= \Lambda_{21}(i\gamma_1,\gamma_2).
\end{aligned}
$$

**Subcase 4.2.** $\delta\neq 0,\ \gamma_1=\gamma_2=0,\ \gamma_3\neq 0$ and

$$\{\delta,\ \beta_1(2\alpha_2-\beta_1),\ \beta_0(2\alpha_2-\beta_1),\ -\beta_0^2+2\alpha_1\beta_0-2\alpha_0\beta_1,\gamma_3(2\alpha_2-\beta_1),\ \gamma_3(\alpha_1-\beta_0)\}\subset\mathbf{R};$$

$$
\begin{aligned}
\hat{H}[y] &= \partial_y^2 + \Bigg\{\frac{\alpha_2}{2} - \frac{3(2\alpha_2x+\alpha_1)^2}{16(\alpha_2x^2+\alpha_1x+\alpha_0)} + \frac{1}{4(\alpha_2x^2+\alpha_1x+\alpha_0)}\{\beta_1(2\alpha_2-\beta_1)x^2 \\
&+2\beta_0(2\alpha_2-\beta_1)x-\beta_0^2+2\alpha_1\beta_0-2\beta_1\alpha_0-\gamma_3^2 \\
&+\left[2\delta\alpha_2x^2+2x((2\alpha_2-\beta_1)\gamma_3+\delta\alpha_1)+2(\alpha_1-\beta_0)\gamma_3+2\delta\alpha_0\right]\sigma_3\}\Bigg\}\Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= 1,
\end{aligned}
$$

**Case 5.** $\alpha_2=0,\ \beta_2\neq 0$ and

$$
\begin{aligned}
&\alpha_1\neq 0,\ \gamma_1^2-\gamma_2^2<0,\ \tilde{\gamma}_i^2<0,\ \gamma_3=\frac{\tilde{\gamma}_i^2}{2\alpha_1}, \\
&\{\beta_0,\ \beta_1,\ \beta_2,\ \gamma_2,\ \delta(\gamma_2^2-\gamma_1^2)+2\beta_2\gamma_1\gamma_3\}\subset\mathbf{R}, \\
&\{i(2\alpha_0\beta_2\gamma_3-\beta_1\tilde{\gamma}_i^2+2\beta_2\alpha_1\gamma_1+\delta\alpha_1\gamma_3),\ i((\alpha_1-\beta_0)\tilde{\gamma}_i^2+2\beta_2\alpha_0\gamma_1+\delta\alpha_0\gamma_3)\}\subset\mathbf{R};
\end{aligned}
$$

$$
\begin{aligned}
\hat{H}[y] &= \partial_y^2 + \Bigg\{-\frac{3\alpha_1^2}{16(\alpha_1x+\alpha_0)} + \frac{1}{4(\alpha_1x+\alpha_0)}\Bigg\{-\beta_2^2x^4-2\beta_1\beta_2x^3 \\
&+\left[(2-4m)\alpha_1\beta_2-\beta_1^2-2\beta_0\beta_2\right]x^2-\left[2\beta_0\beta_1+4m\alpha_0\beta_2\right]x \\
&+2\alpha_1\beta_0-2\alpha_0\beta_1-\beta_0^2-\tilde{\gamma}_i^2+4x(\alpha_1x+\alpha_0)\Bigg[\beta_2\sqrt{\gamma_2^2-\gamma_1^2}\sigma_1\frac{\sin\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)}{\sqrt{-\tilde{\gamma}_i^2}} \\
&+\frac{\beta_2\sqrt{(\gamma_1^2-\gamma_2^2)\tilde{\gamma}_i^2}}{\tilde{\gamma}_i^2}\sigma_3\cos\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)\Bigg]+2(\alpha_1x+\alpha_0) \\
&\times\Bigg[\left(\frac{\delta(\gamma_2^2-\gamma_1^2)+2\beta_2\gamma_1\gamma_3}{\sqrt{\gamma_2^2-\gamma_1^2}}\sigma_1-\frac{2\beta_2\gamma_2\tilde{\gamma}_i^2}{\sqrt{(\gamma_1^2-\gamma_2^2)\tilde{\gamma}_i^2}}\sigma_3\right)\frac{\sin\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)}{\sqrt{-\tilde{\gamma}_i^2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{2\beta_2\gamma_2}{\sqrt{\gamma_2^2-\gamma_1^2}}\sigma_1+\frac{\delta(\gamma_1^2-\gamma_2^2)-2\beta_2\gamma_1\gamma_3}{\sqrt{(\gamma_1^2-\gamma_2^2)\tilde{\gamma}_i^2}}\sigma_3\right)\cos\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)\Bigg] \\
&+\Bigg[x\frac{4\alpha_0\beta_2\gamma_3-2\beta_1\tilde{\gamma}_i^2+4\alpha_1\beta_2\gamma_1+2\delta\alpha_1\gamma_3}{\tilde{\gamma}_i^2} \\
&+\frac{(2\alpha_1-2\beta_0)\tilde{\gamma}_i^2+4\alpha_0\beta_2\gamma_1+2\delta\alpha_0\gamma_3}{\tilde{\gamma}_i^2}\Bigg]\left(-i\sqrt{-\tilde{\gamma}_i^2}\sigma_2\right)\Bigg\}\Bigg\}\Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= \Lambda_{21}(i\gamma_1,\gamma_2)\Lambda_{23}\left(-i\gamma_3\sqrt{\gamma_2^2-\gamma_1^2},\gamma_1^2-\gamma_2^2\right).
\end{aligned}
$$

**Case 6.** $\alpha_2=0,\ \beta_2=0.$

**Subcase 6.1.** $\delta\neq 0$, $\tilde{\gamma}_1,\tilde{\gamma}_2$ do not vanish simultaneously and

$$
\begin{aligned}
&\tilde{\gamma}_i^2<0,\ \{\delta^2(\gamma_1^2-\gamma_2^2)<0,\ \beta_0,\ \beta_1\}\subset\mathbf{R}, \\
&\{i(-\beta_1\tilde{\gamma}_i^2+\delta\alpha_1\gamma_3),\ i((\alpha_1-\beta_0)\tilde{\gamma}_i^2+\delta\alpha_0\gamma_3)\}\subset\mathbf{R};
\end{aligned}
$$

$$
\begin{aligned}
\hat{H}[y] &= \partial_y^2+\Bigg\{-\frac{3\alpha_1^2}{16(\alpha_1x+\alpha_0)}+\frac{1}{4(\alpha_1x+\alpha_0)}\Bigg\{-\beta_1^2x^2-2\beta_0\beta_1x+2\alpha_1\beta_0 \\
&-2\alpha_0\beta_1-\beta_0^2-\tilde{\gamma}_i^2+2(\alpha_1x+\alpha_0)\Bigg[\delta\sqrt{\gamma_2^2-\gamma_1^2}\sigma_1\frac{\sin\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)}{\sqrt{-\tilde{\gamma}_i^2}} \\
&+\frac{\delta(\gamma_1^2-\gamma_2^2)}{\sqrt{(\gamma_1^2-\gamma_2^2)\tilde{\gamma}_i^2}}\sigma_3\cos\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)\Bigg]+\Bigg[x\frac{-2\beta_1\tilde{\gamma}_i^2+2\delta\alpha_1\gamma_3}{\tilde{\gamma}_i^2} \\
&+\frac{(2\alpha_1-2\beta_0)\tilde{\gamma}_i^2+2\delta\alpha_0\gamma_3}{\tilde{\gamma}_i^2}\Bigg]\left(-i\sqrt{-\tilde{\gamma}_i^2}\sigma_2\right)\Bigg\}\Bigg\}\Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= \Lambda_{21}(i\gamma_1,\gamma_2)\Lambda_{23}\left(-i\gamma_3\sqrt{\gamma_2^2-\gamma_1^2},\gamma_1^2-\gamma_2^2\right).
\end{aligned}
$$

**Subcase 6.2.**

$$
\begin{aligned}
&\gamma_1=\gamma_2=0,\ \gamma_3\neq 0,\ \{\beta_1^2,\ \beta_0\beta_1\}\subset\mathbf{R}, \\
&\{-\beta_1\gamma_3+\delta\alpha_1,\ (\alpha_1-\beta_0)\gamma_3+\delta\alpha_0,\ -\beta_0^2+2\alpha_1\beta_0-2\alpha_0\beta_1\}\subset\mathbf{R};
\end{aligned}
$$

$$
\begin{aligned}
\hat{H}[y] &= \partial_y^2+\Bigg\{-\frac{3\alpha_1^2}{16(\alpha_1x+\alpha_0)} \\
&+\frac{1}{4(\alpha_1x+\alpha_0)}\{-\beta_1^2x^2-2\beta_0\beta_1x+2\alpha_1\beta_0-2\alpha_0\beta_1-\beta_0^2-\gamma_3^2 \\
&+2(\alpha_1x+\alpha_0)[2x\beta_1(\alpha_1-\gamma_3)+2(\alpha_1-\beta_0)\gamma_3+2\beta_1\alpha_0]\sigma_3\}\Bigg\}\Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= 1.
\end{aligned}
$$

In the above formulae $\tilde{\gamma}_i^2$ stands for $\tilde{\gamma}_1^2+\tilde{\gamma}_2^2+\tilde{\gamma}_3^2$.

The whole procedure of derivation of the above formulae is very cumbersome. That is why we restrict ourselves to indicating the principal steps of the derivation of the corresponding formulae for the case when $\alpha_2 \neq 0, \beta_2 \neq 0$ omitting the secondary details. It is not difficult to prove that $\tilde{\gamma}_i^2 \neq 0$. Indeed, suppose that the relation $\tilde{\gamma}_i^2 = 0$ holds and consider the expression $\Omega = U^{-1}(x)\sigma_3 U(x)$ from (21). Making use of the Campbell-Hausdorff formula we get

$$\Omega = \sigma_3 + \theta(i\gamma_1\sigma_2 + \gamma_2\sigma_1) - \frac{\theta^2}{2}\gamma_3\tilde{\gamma}_i\sigma_i,$$

where $\theta$ is the function (35). Considering the coefficient at $\theta^2$ yields that $\gamma_3 = 0$ (otherwise using Lemma 2 we get the inequality $\gamma_3^2\tilde{\gamma}_i^2 \neq 0$ that contradicts to the assumption $\tilde{\gamma}_i^2 = 0$). Since the matrix coefficient at $\theta$ has to be Hermitian, we get $\tilde{\gamma}_i^2 = \gamma_1^2 - \gamma_2^2 < 0$. This contradiction proves that $\tilde{\gamma}_i^2 \neq 0$. In view of this inequality we can represent the matrix potential (21) as follows

$$\begin{aligned}
V(y) \;=\; & \left\{ \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \right. \\
& + \frac{1}{4(\alpha_2 x^2 + \alpha_1 x + \alpha_0)}\Lambda^{-1}\Big\{ -\beta_2^2 x^4 - [2\beta_1\beta_2 + 4\alpha_2\beta_2(m-1)]x^3 \\
& + \Big[2\alpha_2\beta_1 - 2\alpha_1\beta_2 - \beta_1^2 - 2\beta_0\beta_2 - 4\alpha_1\beta_2(m-1)\Big]x^2 \\
& + [4\alpha_2\beta_0 - 2\beta_0\beta_1 - 4m\alpha_0\beta_2]\,x + 2\alpha_1\beta_0 - 2\alpha_0\beta_1 - \beta_0^2 - \tilde{\gamma}_i^2 \\
& + 4x(\alpha_2 x^2 + \alpha_1 x + \alpha_0)\Big[\beta_2\gamma_3(\tilde{\gamma}_i^2)^{-1}\tilde{\gamma}_i\sigma_i + \beta_2(\gamma_2\sigma_1 + i\gamma_1\sigma_2)(\tilde{\gamma}_i^2)^{-1/2}\sinh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right) \\
& + [\beta_2(-\gamma_1\gamma_3\sigma_1 - i\gamma_2\gamma_3\sigma_2 + (\gamma_1^2 - \gamma_2^2)\sigma_3)](\tilde{\gamma}_i^2)^{-1}\cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)\Big] \qquad (36) \\
& + 2(\alpha_2 x^2 + \alpha_1 x + \alpha_0)\Big[(\delta\gamma_2\sigma_1 + i(\delta\gamma_1 - 2\beta_2\gamma_3)\sigma_2 - 2\beta_2\gamma_2\sigma_3)(\tilde{\gamma}_i^2)^{-1/2}\sinh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right) \\
& + [(2\beta_2(\gamma_3^2 - \gamma_2^2) - \delta\gamma_1\gamma_3)\sigma_1 - i(2\beta_2\gamma_1\gamma_2 + \delta\gamma_2\gamma_3)\sigma_2 + (\delta(\gamma_1^2 - \gamma_2^2) - 2\beta_2\gamma_1\gamma_3)\sigma_3] \\
& (\tilde{\gamma}_i^2)^{-1}\cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)\Big] + [(-2\beta_2\tilde{\gamma}_i^2 + 4\alpha_2\beta_2\gamma_1 + 2\delta\alpha_2\gamma_3)x^2 + ((4\alpha_2 - 2\beta_1)\tilde{\gamma}_i^2 \\
& \left. + 4\alpha_1\beta_2\gamma_1 + 2\delta\alpha_1\gamma_3)x + \frac{(2\alpha_1 - 2\beta_0)\tilde{\gamma}_i^2 + 4\alpha_0\beta_2\gamma_1 + 2\delta\alpha_0\gamma_3}{\tilde{\gamma}_i^2}](\tilde{\gamma}_i^2)^{-1}\tilde{\gamma}_i\sigma_i\Big\}\Lambda\ \right\}\Bigg|_{x=f^{-1}(y)},
\end{aligned}$$

where $\theta = \theta(y)$ is given by (35).

Let us first suppose that $\gamma_1, \gamma_2$ do not vanish simultaneously. We will prove that hence it follows that $\tilde{\gamma}_i^2 \in \mathbf{R}$. Consider the (non-zero) matrix coefficient at $4x\xi\cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)$ in the expression (36) and suppose that $\sqrt{\tilde{\gamma}_i^2} = a + ib$, with some non-zero real numbers $a$ and $b$. Now it is easy to prove that $\cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right) = f(x) + ig(x)$, where $f, g$ are linearly-independent real-valued functions. Considering the matrix coefficients of $f(x)$, $g(x)$ we see that in order to reduce the matrix (36) to a Hermitian form we should reduce to Hermitian forms the matrices $A$, $iA$ which is impossible. This contradiction proves that $\tilde{\gamma}_i^2 \in \mathbf{R}$.

Consider next the non-zero matrix coefficients of $4x\xi \dfrac{\sinh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)}{\sqrt{\tilde{\gamma}_i^2}}$, $4x\xi\cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)$ in (36). These coefficients can be represented in the form $\vec{a}\vec{\sigma}$, $\vec{b}\vec{\sigma}$, where

$$\vec{a} = \beta_2(\gamma_2, i\gamma_1, 0),\ \vec{b} = \beta_2(-\gamma_1\gamma_3, -i\gamma_2\gamma_3, \gamma_1^2 - \gamma_2^2),$$

and, what is more,

$$\vec{a}\times\vec{b} = \beta_2^2(\gamma_1^2 - \gamma_2^2)(i\gamma_1, -\gamma_2, i\gamma_3).$$

Applying Lemma 2 yields

$$\beta_i \in \mathbf{R},\quad \gamma_3 = i\mu,\ \mu\in\mathbf{R},\quad \gamma_1^2 - \gamma_2^2 < 0.$$

Next we turn to the matrix coefficient of $2\xi\dfrac{\sinh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)}{\sqrt{\tilde{\gamma}_i^2}}$ which is of the form $\vec{c}\vec{\sigma}$ with $\vec{c} = (\delta\gamma_2, i(\delta\gamma_1 - 2\beta_2\gamma_3), -2\beta_2\gamma_2)$. Making use of the assertion 3 of Lemma 2 we obtain the conditions

$$\{\gamma_1,\ \gamma_2\}\subset\mathbf{R},\quad [\gamma_1 = 0]\bigvee[\gamma_3 = 0].$$

Considering in a similar way the matrix coefficient of $2\xi\cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)$ yields the following restrictions on the coefficients $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$:

$$\left[\{\beta_2, \gamma_1\}\subset\mathbf{R},\ \gamma_3 = 0, \gamma_2 = \sqrt{\gamma_1^2 - 2\alpha_2\gamma_1},\ \alpha_2\gamma_1 < 0, \beta_1 = 2\alpha_2 + \beta_2\frac{\alpha_1}{\alpha_2},\ \beta_0 = \alpha_1 + \beta_2\frac{\alpha_0}{\alpha_2}\right].$$

As a result we get the formulae of Case 2.

One can prove in an analogous way that, provided $\gamma_1 = \gamma_2 = 0, \gamma_3 \neq 0$, the matrix (36) cannot be reduced to a Hermitian form.

A further restriction narrowing the choice of QES matrix Hamiltonians is a requirement that the basis elements of the corresponding invariant space should be square integrable on the interval $(-\infty, \infty)$. For example, if we put in Case I $\alpha_1 = 1, \beta_0 = \frac{1}{2}, \beta_2 = -1$, the remaining coefficients being equal to zero, then we arrive at the model 1 from the list of QES Hamiltonians given in Introduction. The remaining models given there are obtained in an analogous way.

# 5 Some conclusions

A principal aim of the paper is to give a systematic algebraic treatment of Hermitian QES Hamiltonians within the framework of the approach to constructing QES matrix models suggested in our papers [1, 2]. The whole procedure is based on a specific representation of the algebra $o(2,2)$ given by formulae (5), (7), (8). Making use of the fact that the representation space of the algebra (8) has a finite-dimensional invariant subspace (6) we have constructed in a systematic way six multi-parameter families of Hermitian QES Hamiltonians on line. Due to computational reasons we do not present here a systematic description of Hermitian QES Hamiltonians with potentials depending on elliptic functions.

The problem of constructing all Hermitian QES Hamiltonians of the form (18) having square integrable eigenfunctions is also beyond the scope of the present paper. We restricted our analysis of this problem to giving several examples of such Hamiltonians postponing its further investigation for our future publications.

A very interesting problem is a comparison of the results of the present paper based on structure of representation space of the representation (5), (7), (8) of the Lie algebra $o(2,2)$ to those of the paper [17], where some superalgebras of matrix-differential operators come into play. The link to the results of [17] is provided by the fact that the Lie algebra $o(2,2)$ has a structure of a superalgebra. This is a consequence of the fact that operators (8) fulfill identities (9).

One more challenging problem is a utilization of the obtained results for integrating multi-dimensional Pauli equation with the help of the method of separation of variables. As an intermediate problem to be solved within the framework of the method in question is a reduction of the Pauli equation to four second-order systems of ordinary differential equations with the help of a separation Ansatz. The next step is studying whether the corresponding matrix-differential operators belong to one of the six classes of QES Hamiltonians constructed in Section 4.

Investigation of the above enumerated problems is in progress now and we hope to report the results obtained in one of our future publications.

# Acknowledgments

It is a pleasure for the authors to thank the referees for valuable comments and suggestions. This work is partially supported by the DFFD Foundation of Ukraine (project 1.4/356).

# References

[1] Zhdanov R Z 1997 *Phys. Lett. B* **405** 253

[2] Zhdanov R Z 1997 *J. Phys. A: Math. Gen.* **30** 8761

[3] Turbiner A V 1988 *Commun. Math. Phys.* **118** 467

[4] Shifman M A 1989 *Int. J. Mod. Phys.* **5A** 2897

[5] Shifman M A and Turbiner A V *Commun. Math. Phys.* **126** 347

[6] Zaslavskii O B and Ulyanov V V 1984 *Sov. Phys. JETR* **60** 991

[7] Zaslavskii O B and Ulyanov V V 1987 *Theor. Math. Phys.* **71** 520

[8] González–López A, Kamran N and Olver N 1991 *J. Phys. A: Math. Gen.* **24** 3995

[9] González–López A, Kamran N and Olver N 1994 *Commun. Math. Phys.* **159** 503

[10] González–López A, Kamran N and Olver P 1994 *Contemp. Math.* **160** 113

[11] Ushveridze A G 1993 *Quasi–exactly Solvable Models in Quantum Mechanics* (Bristol: IOP Publishing).

[12] Turbiner A V 1992 Preprint ETH-TH/92-34

[13] Turbiner A V 1994 *Contemp. Math.* **160** 263

[14] Brihaye Y and Kosinski Y 1995 *J. Math. Phys.* **36** 4340

[15] Brihaye Y, Giller S, Gonera C and Kosinski P, *J. Math. Phys.* **36** 4340

[16] Finkel F, González–López A and Rodríguez M A 1995 Preprint hep-th/9509057

[17] Finkel F, González–López A and Rodríguez M A 1997 *J. Phys. A: Math. Gen.* **30** 6879

[18] Lie S 1924 in: *Gesammelte Abhandlungen* (Leipzig: B.G. Teubner) Vol.5, 767–773.

[19] Lie S 1924 in: *Gesammelte Abhandlungen* (Leipzig: B.G. Teubner) Vol.6, 1–94.

[20] González–López A, Kamran N and Olver P 1992 *Proc. London Math. Soc.* **64** 339

[21] González-López A, Kamran N and Olver P 1993 *Comm. Math. Phys.* **153** 117

[22] Ushveridze A G 1988 *Sov. Phys. - Leb. Inst. Rep.* **2** 50