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SOME METHODS OF COMPLEMENT OF WEAK REGULAR LINEAR EXTENSIONS OF DYNAMICAL SYSTEMS TO REGULAR

ДЕЯКІ МЕТОДИ ДОПОВНЕННЯ СЛАБКОРЕГУЛЯРНИХ ЛІНІЙНИХ РОЗШИРЕНЬ ДИНАМІЧНИХ СИСТЕМ ДО РЕГУЛЯРНИХ

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A method of completing certain weakly regular linear extensions of dynamical systems on the torus to the regular is presented.

Наведено метод доповнення деяких слабкорегулярних лінійних розширень динамічних систем на торі до регулярних.

1. Introduction. By examining the preservation issues of invariant tori of dynamical systems, it is important to solve the problem of the existence of the Green's function of linearized system. Such a system of differential equations is called linear extension of the dynamical system. If such a linear extension of a dynamical system has the Green's function, then the invariant torus of heterogeneous linear extensions can be written in an explicit integral form. This gives the opportunity to explore the smoothness of torus. If a homogeneous linear extension has many different Green's function, the test of smoothness of the invariant torus is rather difficult. Therefore, monographs [1, 3] propose a completion of the linear extension in the form of a triangle to the regular, enabling a Green's function for the initial linear extension to be an n-dimensional block in a $2n$-dimensional unique Green's function. This work is devoted to a more thorough study of the issue of complementing linear extensions in the triangle form to regular systems that have a unique Green's function.

2. The formulation and justification of the basic theorems. Consider a system of differential equations

$$\frac{d\phi}{dt} = a(\phi), \quad \frac{dx}{dt} = A(\phi)x, \tag{1}$$

where $\phi = (\phi_1, \ldots, \phi_m)$, $a(\phi) = (a_1(\phi), \ldots, a_m(\phi))$ is a continuous vector function, 2π-periodical with respect to each variable ϕ_j and satisfying the Lipschitz condition, thus belonging to the space $C_{\text{Lip}}(T_m)$, $x \in \mathbb{R}^n$, $A(\phi)$ is an $(n \times n)$-dimensional square matrix, continuous and 2π-periodical with respect to each variable ϕ_j, $A(\phi) \in C^0(T_m)$.

It is known (see [1, 2]) that the system (1) will have infinitely many different Green's function $G_0(\tau, \phi)$ with the exponential estimation $\|G_0(\tau, \phi)\| \leq K \exp\{-\gamma|\tau|\}$, $K, \gamma = \text{const} > 0$ if and only if there exists a quadratic form

$$V = \langle S(\phi)y, y\rangle, \tag{2}$$

$y \in \mathbb{R}^n$, with a continuously differentiable and 2π-periodical and symmetrical matrix of coefficients, $S(\phi) = S(\phi_1, \ldots, \phi_m)$, the derivative of which with respect to the adjoint system (1) at normal variable x, i.e., system of the form

$$\frac{d\phi}{dt} = a(\phi), \quad \frac{dy}{dt} = -A^T(\phi)y, \tag{3}$$

is positive definite:

$$\dot{V} = \left\langle \left[\sum_{j=1}^{m} \frac{\partial S(\phi)}{\partial \phi_j} a_j(\phi) - S(\phi)A^T(\phi) - A(\phi)S(\phi) \right] y, y \right\rangle \geq \|y\|^2, \tag{4}$$

and the matrix $S(\phi)$, for some values $\phi = \phi_0$, is degenerate,

$$\det S(\phi) = 0. \tag{5}$$

For example, the system of equations

$$\frac{d\phi}{dt} = (\sin\phi)^{2p-1}, \quad \frac{dx}{dt} = \mu x (\cos\phi)^{2l-1} \tag{6}$$

for any values p, l and real positive value μ has infinitely many different Green's function. We know this because the derivative of the function $V = -(\cos\phi)y^2$ with respect to the system,

$$\frac{d\phi}{dt} = (\sin\phi)^{2p-1}, \quad \frac{dy}{dt} = -\mu y (\cos\phi)^{2l-1}, \quad \mu > 0,$$

is positive definite, $\dot{V} = \left[(\sin\phi)^{2p} + \mu(\cos\phi)^{2l}\right] y^2$ and wherein $\cos\phi_0 = 0$, $\phi_0 = \dfrac{\pi}{2}$.

In the case when the system (1) has infinitely many different Green's functions, a study of the smoothness with respect to variables ϕ and the continuous dependence on the parameters of these functions causes some difficulties. In [1, 2] there was proposed a way to complement the system (1) with the new equation

$$\frac{d\phi}{dt} = a(\phi), \quad \frac{dx}{dt} = A(\phi)x, \quad \frac{dy}{dt} = x - A^T(\phi)y, \tag{7}$$

where $y \in \mathbb{R}^n$. It turns out that the complemented system (7) will always have a unique $(2n \times \times 2n)$-dimensional Green's function $\bar{G}_0(\tau, \phi)$ no matter that the system (1) has infinitely many different such functions or only a unique one. Wherein the derivative of the quadratic form:

$$W = 2p\langle x, y\rangle + \langle S(\phi)y, y\rangle, \tag{8}$$

with respect to the system (7) for a sufficiently large fixed values of the real-parameter $p > 0$ will be positive definite. Note that if the system (7) is generalized as follows:

$$\frac{d\phi}{dt} = a(\phi), \quad \frac{dx}{dt} = A(\phi)x, \quad \frac{dy}{dt} = x - A^T(\phi)y, \tag{9}$$

where $B(\phi)$ is an $(n \times n)$-dimensional continuous matrix ($B(\phi) \in C^0(T_m)$) that satisfies one of the conditions:

$$\langle B(\phi)x, x\rangle \geq \beta\|x\|^2, \quad \langle B(\phi)x, x\rangle \leq -\beta\|x\|^2, \tag{10}$$

where $\beta = \text{const} > 0$, then the derivative of the nondegenerate quadratic form (8) with respect to system (9) will also have a definite sign. By writing the Green's function of the system (9) in the block form,

$$\bar{G}_0(\tau, \phi) = \begin{bmatrix} G_0^{11}(\tau, \phi) & G_0^{12}(\tau, \phi) \\ G_0^{21}(\tau, \phi) & G_0^{22}(\tau, \phi) \end{bmatrix},$$

we note that the first $(n \times n)$-dimensional block $G_0^{11}(\tau, \phi)$ is the Green's function of the system (1).

Of course, if the system (1) is regular, i.e., has a unique Green's function, then the system (9) for any matrix $B(\phi) \in C^0(T_m)$ will also be regular, and if the system (1) has infinitely many different Green's functions, then the system (9) is not regular for every matrix $B(\phi) \in C^0(T_m)$. It turns out that there are systems (1) that have infinitely many different Green's functions and, at the same time, there are regular extended system (9) for certain matrix $B(\phi) \in C^0(T_m)$ that do not satisfy any of the conditions (10). Let's see this in the illustrating example

$$\frac{d\phi}{dt} = \sin\phi, \quad \frac{dx}{dt} = x\cos\phi, \quad \frac{dy}{dt} = x\sin\phi - y\cos\phi, \tag{11}$$

which is regular ($B(\phi) = \sin\phi$), because the derivative of the system (11) of the continuous quadratic form

$$W = x^2\cos\phi + 2xy\sin\phi - y^2\cos\phi \tag{12}$$

is positive definite, $\dot{W} = (\sin^2\phi + 2\cos^2\phi)(x^2+y^2) \geq x^2+y^2$. There is a problem of separating the class of regular systems in the form (9) with the matrix $B(\phi) \in C^0(T_m)$, which does not meet even one condition (10). This work is devoted to examinate that issue.

Let us consider the case $n = 1$, $A(\phi) = \lambda(\phi)$ is a continuous scalar function, 2π-periodical with respect to each variable ϕ_j, $j = \overline{1, m}$, and denote

$$\psi = k_1\phi_1 + \ldots + k_m\phi_m + \theta = \langle k, \phi\rangle + \theta, \tag{13}$$

where k_j are some integers, $k = (k_1, \ldots, k_m)$ a vector with integer coordinates, $|k| = |k_1| + \ldots$ $\ldots + |k_m|$, θ a constant.

The following theorem takes place.

Theorem 1. *Let for some integer vector k, $|k| > 0$ and a constant θ the following inequality be satisfied:*

$$\sigma = \langle k, a(\phi)\rangle\sin\psi + 2\lambda(\phi)\cos\psi > 0 \tag{14}$$

for all ϕ_j, $j = \overline{1, m}$. Then the system of equations

$$\frac{d\phi}{dt} = a(\phi), \quad \frac{dx}{dt} = \lambda(\phi)x, \quad \frac{dy}{dt} = \langle k, a(\phi)\rangle x - \lambda(\phi)y \tag{15}$$

is regular, and thus has only one (2×2)*-dimensional Green's function with the exponential estimation.*

Proof. Consider the quadratic form

$$W = x^2 \cos\psi + 2xy \sin\psi - y^2 \cos\psi \tag{16}$$

and write its derivative with respect to system (15), hence

$$\dot{W} = 2x\dot{x}\cos\psi + x^2(-\sin\psi)\dot{\psi} + 2\dot{x}y\sin\psi + 2x\dot{y}\sin\psi + 2xy(\cos\psi)\dot{\psi} - 2y\dot{y}\cos\phi +$$
$$+ y^2(\sin\psi)\dot{\psi} = 2x(\lambda x)\cos\psi + x^2(-\sin\psi)\langle k,a\rangle + 2(\lambda x)y\sin\psi + 2x(\langle k,a\rangle x - \lambda y)\sin\psi +$$
$$+ 2xy\cos\psi\langle k,a\rangle - 2y(\langle k,a\rangle x - \lambda y)\cos\psi - y^2(-\sin\psi)\langle k,a\rangle = \sigma(x^2+y^2),$$

where σ is the left-hand side of the inequality (14). Since the quadratic form (16) is nondegenerate, the system (15) is regular.

Theorem 1 is proved.

Remark 1. The inequality (14) shows that the scalar function $\langle k, a(\phi)\rangle$ needs to take a zero value for some $\phi = \phi_0$.

Consider the following examples.

Example 1. Let a differential system be given,

$$\frac{d\phi_1}{dt} = \sin\phi_1\cos\phi_2, \quad \frac{d\phi_2}{dt} = \cos\phi_1\sin\phi_2, \quad \frac{dx}{dt} = x\cos(\phi_1+\phi_2).$$

Choosing $k = (1,1)$, we get $\psi = \phi_1 + \phi_2$, $\lambda(\phi) = \cos\psi$, $\langle k,a\rangle = \sin\psi$ and the inequality (14) is fulfilled. The considered system has infinitely many Green's functions and the complemented system

$$\frac{d\phi_1}{dt} = \sin\phi_1\cos\phi_2, \quad \frac{d\phi_2}{dt} = \cos\phi_1\sin\phi_2, \quad \frac{dx}{dt} = x\cos\psi, \quad \frac{dy}{dt} = x\sin\psi - y\cos\psi,$$

has only one.

Note that, in the example, two inequalities are satisfied,

$$\langle k, a(\phi)\rangle \sin\psi \geqslant 0, \quad \lambda(\phi)\cos\psi \geq 0 \ \ \forall\psi \in \mathbb{R}, \tag{17}$$

which for the other examples may not occur.

Example 2. Let a system of differential equations be given,

$$\frac{d\phi_1}{dt} = 3\sin\phi_1\cos\phi_2, \quad \frac{d\phi_2}{dt} = 2\cos\phi_1\sin\phi_2, \quad \frac{dx}{dt} = x[n\cos(\phi_1-\phi_2) + \varepsilon\sin(\phi_1+\phi_2)],$$

where $n = 1, 2, \ldots$, $|\varepsilon| < 0,5$. Denoting $\psi = \phi_1 - \phi_2$, $k = (1;-1)$, we get

$$\langle k, a(\phi)\rangle \sin\psi = 2\sin^2\psi + \sin\phi_1\cos\phi_2\sin\psi,$$

$$2\lambda(\phi)\cos\psi = 2n\cos^2\psi + 2\varepsilon\cos\psi\sin(\phi_1+\phi_2).$$

This shows that inequalities (17) do not take place, but the inequality (14) will be satisfied. In this way the system of equations

$$\frac{d\phi_1}{dt} = 3\sin\phi_1\cos\phi_2, \quad \frac{d\phi_2}{dt} = 2\cos\phi_1\sin\phi_2,$$

$$\frac{dx}{dt} = [n\cos(\phi_1-\phi_2) + \varepsilon\sin(\phi_1+\phi_2)]x,$$

$$\frac{dy}{dt} = [3\sin\phi_1\cos\phi_2 - 2\cos\phi_1\sin\phi_2]x - [n\cos(\phi_1-\phi_2) + \varepsilon\sin(\phi_1+\phi_2)]y,$$

is regular for any natural value n and fulfilles the condition for $|\varepsilon| < 0,5$.

Remark 2. If we assume that in the system (15) $x, y \in \mathbb{R}^n$, so they are n-dimensional vectors, the notation generalizing of a quadratic form (16):

$$W = \|x\|^2\cos\psi + 2\langle x, y\rangle\sin\psi - \|y\|^2\cos\psi, \tag{18}$$

we easy see that the derivative $\dot{W}$ with respect to system (15) will also be positive definite.

Remark 3. If condition (14) are satisfied, the system of equations

$$\frac{d\phi}{dt} = a(\phi), \quad \frac{dx}{dt} = \lambda(\phi)x, \quad \frac{dy}{dt} = \langle k, a(\phi)\rangle Bx - \lambda(\phi)y, \quad x, y \in \mathbb{R}, \tag{19}$$

for any constant nondegenerate matrix B will be regular, because the derivative of nondegenerate quadratic form $\|Bx\|^2\cos\psi + 2\langle Bx, y\rangle\sin\psi - \|y\|^2\cos\psi$ with respect to the system (19) will be positive definite.

A generalization of quadratic form (18) is the following:

$$W = \langle S_1x_1, x_1\rangle\cos\psi + 2\langle S_{12}x_1, x_2\rangle\sin\psi - \langle S_2x_2, x_2\rangle\cos\psi, \tag{20}$$

where $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. The following statement ensures a lack of degeneration of the quadratic form (20).

Lemma 1. *Let symmetrical constant matrices S_1, S_2 in the quadratic form (20) be positive definite,*

$$\langle S_jx_j, x_j\rangle \geq \gamma\|x_j\|^2, \quad j = 1, 2, \quad x_j \in \mathbb{R}^{n_j}, \quad \gamma = \text{const} > 0. \tag{21}$$

Then the quadratic form (20) is nondegenerate for all values $\psi \in \mathbb{R}$ if and only if the following two conditions are satisfied

$$n_1 = n_2, \quad \det S_{12} \neq 0. \tag{22}$$

Proof. Let us write the symmetric matrix $\bar{S}(\psi)$ that corresponds to the quadratic form (19),

$$\bar{S}(\psi) = \begin{bmatrix} S_1 \cos\psi & S_{12}^T \sin\psi \\ S_{12} \sin\psi & -S_2 \cos\psi \end{bmatrix}, \tag{23}$$

and suppose that it is nondegenerate for all values $\psi \in \mathbb{R}$. By substituting the value $\psi = \frac{\pi}{2}$, we get $\bar{S}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & S_{12}^T \\ S_{12} & 0 \end{bmatrix}$. This matrix should also be nondegenerate, and it means that the square matrix $\bar{S}^2 = \begin{bmatrix} S_{12}^T S_{12} & 0 \\ 0 & S_{12} S_{12}^T \end{bmatrix}$ is a positive definite matrix and, consequently,

$$\begin{gathered} \left\langle S_{12}^T S_{12} x_1, x_1 \right\rangle \geq \varepsilon \|x_1\|^2, \quad x_j \in \mathbb{R}^{n_j}, \\ \left\langle S_{12} S_{12}^T x_2, x_2 \right\rangle \geq \varepsilon \|x_2\|^2, \quad \varepsilon = \text{const} > 0. \end{gathered} \tag{24}$$

Simultaneous fulfillment of two inequalities (24) is possible in this case, if the two systems of equations $S_{12}x_1 = 0$ and $S_{12}^T x_2 = 0$ have only zero solutions, which means that conditions (22) should be satisfied. Now we will show that fulfillment of conditions (21), (22) are sufficient for the matrix (23) to be nondegenerate for all $\psi \in \mathbb{R}$. Denote $n = n_1 = n_2$. Since the conditions (21) are fulfilled, there are nondegenerate matrices L_1, L_2, such that $L_1^T S_1 L_1 = I_n$, $L_2^T S_2 L_2 = I_n$. Of course, the matrix (23) is nondegenerate for all values $\psi \in \mathbb{R}$ only in the case where the matrix

$$\tilde{S}(\psi) = \begin{bmatrix} L_1^T & 0 \\ 0 & L_2^T \end{bmatrix} \begin{bmatrix} S_1 \cos\psi & S_{12}^T \sin\psi \\ S_{12} \sin\psi & -S_2 \cos\psi \end{bmatrix} \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} = \begin{bmatrix} I_n \cos\psi & \Theta^T \sin\psi \\ \Theta \sin\psi & -I_n \cos\psi \end{bmatrix}, \tag{25}$$

is nondegenerate, where $\Theta = L_2^T S_{12} L_1$. Let's see that the square matrix (25)

$$\tilde{S}^2(\psi) = \begin{bmatrix} I_n \cos^2\psi + \Theta^T\Theta \sin^2\psi & 0 \\ 0 & I_n \cos^2\psi + \Theta\Theta^T \sin^2\psi \end{bmatrix}, \tag{26}$$

is a positive definite matrix. In fact, since $\det S_{12} \neq 0$, then $\det\Theta = \det(L_2^T S_{12} L_1) \neq 0$, and this is the fulfillment of inequalities $\|\Theta x_1\| \geq \varepsilon\|x_1\|$, $\|\Theta^T x_2\| \geq \varepsilon\|x_2\|$, $\varepsilon = \text{const} > 0$. In this way we can write the estimate

$$\langle \tilde{S}^2(\psi)x, x \rangle = \|x_1\|^2 \cos^2\psi + \|\Theta x_1\|^2 \sin^2\psi + \|x_2\|^2 \cos^2\psi + \|\Theta^T x_2\|^2 \sin^2\psi \geq \varepsilon_0 \|x\|^2,$$

where $\varepsilon_0 = \min\{1, \varepsilon^2\}$. It follows that the matrices (26), (25), (23) are nondegenerate for all $\psi \in \mathbb{R}$.

Lemma 1 is proved.

Now we generalized the system of equations (19),

$$\frac{d\phi}{dt} = a(\phi), \quad \frac{dx_1}{dt} = \lambda(\phi)A(\phi)x, \quad \frac{dx_2}{dt} = \langle k, a(\phi)\rangle B(\phi)x_1 - \lambda(\phi)A^T(\phi)x_2, \tag{27}$$

where $x_j \in \mathbb{R}^n$, $\lambda(\phi)$ is a continuous scalar function that satisfies the inequality (14) and we will also explain the problem: for which $(n \times n)$-dimensional matrices $A(\phi)$, $B(\phi) \in C^0(T_m)$, the derivative of the nondegenerate quadratic form (20) with respect to the system (27) is positive definite. Remembering that the variable ψ is defined by equality (13), let us write the derivative

$$\dot{W} = 2\langle S_1x_1, A(\phi)x_1\rangle\lambda(\phi)\cos\psi - \langle S_1x_1, x_1\rangle\langle k, a(\phi)\rangle\sin\psi + 2\langle S_{12}A(\phi)x_1, x_2\rangle\lambda(\phi)\sin\psi +$$

$$+ 2\left\langle S_{12}x_1, \langle k, a(\phi)\rangle B(\phi)x_1 - \lambda(\phi)A^T(\phi)x_2\right\rangle\sin\psi + 2\langle S_{12}x_1, x_2\rangle\langle k, a(\phi)\rangle\cos\psi -$$

$$- 2\left\langle S_2x_2, \langle k, a(\phi)\rangle B(\phi)x_1 - \lambda(\phi)A^T(\phi)x_2\right\rangle\cos\psi + \langle S_2x_2, x_2\rangle\langle k, a(\phi)\rangle\sin\psi.$$

This shows that for the derivative $\dot{W}$ to be positive definite, one needs to assume fulfillment of the inequalities (14), (17), and the inequalities

$$\left\langle\left[S_1A(\phi) + A^T(\phi)S_1\right]x, x\right\rangle \geq \varepsilon_1\|x\|^2, \tag{28}$$

$$\left\langle\left[B^T(\phi)S_{12} + S_{12}^TB(\phi) - S_1\right]x, x\right\rangle \geq \varepsilon_2\|x\|^2, \tag{29}$$

$$S_{12}A(\phi) - A(\phi)S_{12} + S_{12} - S_2B(\phi) \equiv 0, \tag{30}$$

$$\left\langle\left[S_2A^T(\phi) + A(\phi)S_2\right]x, x\right\rangle \geq \varepsilon_3\|x\|^2, \tag{31}$$

where $\varepsilon_i = \text{const} > 0$, $i = \overline{1,3}$. In this way, we obtain the following conclusion.

Corollary 1. *If, in the quadratic form (20), two constant symmetrical matrices $S_j = S_j^T$ are positive-definite, and thus estimates (21) are true and, with those matrices and a constant nondegenerate matrix S_{12}, conditions (28) – (31) are fulfilled, then assuming that the inequalities (14), (17) are satisfied, the derivative of the quadratic form (20) with respect to system (27) is positive definite, so the system (27) is regular.*

Now, let us fix a quadratic form (20) and consider the set of equations of the form

$$\frac{d\phi}{dt} = b(\phi), \quad \frac{dx}{dt} = P(\phi)x, \quad x \in \mathbb{R}^{2n}, \tag{32}$$

such that the derivative of the the quadratic form with respect to the system (32) is positive definite. But it turns out that a vector function $b(\phi)$ can be choosen any $b(\phi) \in C_{\text{Lip}}(T_m)$, and the $(2n \times 2n)$-dimensional matrix $P(\phi) \in C^0(T_m)$ must be selected in the following form:

$$P(\phi) = \bar{S}^{-1}(\psi)\left[B(\psi) + M(\psi) - \frac{1}{2}\frac{d\bar{S}(\psi)}{d\psi}\langle k, b(\phi)\rangle\right], \tag{33}$$

where $B(\phi)$ is any symmetric, positive definite matrix and $M(\phi)$ is the antisymmetric matrix $M^T = -M$, $B(\phi)$, $M(\phi) \in C^0(\mathbb{R})$. We are interested in the question of whether there is a choice for the matrices $B(\phi)$ and $M(\phi)$ such that the matrix (33) has a block triangular form. In order to clarify this issue in more details, we consider the form of the matrix (33).

Let us write the inverse matrix $\bar{S}^{-1}(\psi)$ in the block form,

$$\bar{S}^{-1}(\psi) = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \tag{34}$$

and directly determine the $(n \times n)$-dimensional matrices X_{ij}. Hence

$$\begin{aligned} X_{11} &= [S_1 \cos^2\psi + S_{12}^T S_2^{-1} S_{12} \sin^2\psi]^{-1} \cos\psi = L_1 \cos\psi, \\ X_{12} &= [S_2 (S_{12}^T)^{-1} S_1 \cos^2\psi + S_{12} \sin^2\psi]^{-1} \sin\psi = L \sin\psi, \\ X_{21} &= [S_1 S_{12}^{-1} S_2 \cos^2\psi + S_{12}^T \sin^2\psi]^{-1} \sin\psi = L \sin\psi, \\ X_{22} &= -[S_2 \cos^2\psi + S_{12} S_1^{-1} S_{12}^T \sin^2\psi]^{-1} \cos\psi = L_2 \cos\psi. \end{aligned} \tag{35}$$

Choosing the matrices $B(\phi)$, $M(\phi)$ of the form

$$B(\phi) = \begin{bmatrix} B_1(\phi) & 0 \\ 0 & B_2(\phi) \end{bmatrix}, \quad M(\phi) = \begin{bmatrix} 0 & -\Phi^T(\phi) \\ \Phi(\phi) & 0 \end{bmatrix},$$

where the matrices $B_j(\phi)$ are positive definite and $\Phi(\phi)$ is any matrix, we write matrix (33) in the block form, $P(\phi) = \{P_{ij}(\phi)\}_{i,j=1}^2$, while we have

$$\begin{aligned} P_{11}(\phi) &= L_1 B_1(\phi) \cos\psi + L^T \Phi(\phi) \sin\psi + \frac{1}{2} \langle k, b(\phi) \rangle [L_1 S_1 - L^T S_{12}] \cos\psi \sin\psi, \\ P_{12}(\phi) &= -L_1 \Phi^T(\phi) \cos\psi + L^T B_2(\phi) \sin\psi - \frac{1}{2} \langle k, b(\phi) \rangle [L_1 S_{12}^T \cos^2\psi + L^T S_2 \sin^2\psi], \\ P_{21}(\phi) &= -L_2 \Phi(\phi) \cos\psi + L B_1(\phi) \sin\psi + \frac{1}{2} \langle k, b(\phi) \rangle [L_2 S_{12} \cos^2\psi + L S_1 \sin^2\psi], \\ P_{22}(\phi) &= -L_2 B_2(\phi) \cos\psi - L \Phi^T(\phi) \sin\psi + \frac{1}{2} \langle k, b(\phi) \rangle [L_2 S_2 - L S_{12}^T] \cos\psi \sin\psi. \end{aligned}$$

We explain the conditions under which there are matrices $\Phi(\phi)$, $B_2(\phi)$ such that $P_{12}(\phi) \equiv 0$. Wherein the matrix $B_2(\phi)$ to be symmetric and positive definite.

The equation $P_{12}(\phi) \equiv 0$ takes the form

$$-L_1 \Phi^T(\phi) \cos\psi + L^T B_2(\phi) \sin\psi = \frac{1}{2} \langle k, b(\phi) \rangle [L_1 S_{12}^T \cos^2\psi + L^T S_2 \sin^2\psi]. \tag{36}$$

Note that when the scalar function

$$\rho(\phi) = \frac{1}{2} \langle k, b(\phi) \rangle \tag{37}$$

identically equals zero or is identically equal to a certain constant, or in general, preserve the sign for all $\phi \in T_m$, equation (36) has no solutions such that the matrix $B_2(\phi)$ is positive definite. Indeed, substituting in (36) the values $\psi = \pm\frac{\pi}{2}$, we obtain

$$B_2(\phi)|_{\psi=\frac{\pi}{2}} = \rho(\phi)|_{\psi=\frac{\pi}{2}}\, S_2, \quad - B_2(\phi)|_{\psi=-\frac{\pi}{2}} = \rho(\phi)|_{\psi=-\frac{\pi}{2}}\, S_2.$$

Since the constant matrix S_2 is positive definite, the matrix $B_2(\phi)$ can be positively defined only under the condition that the scalar function (37) changes sign, while we must $\rho(\phi)|_{\psi=\frac{\pi}{2}} > 0$, $\rho(\phi)|_{\psi=-\frac{\pi}{2}} < 0$. Recall that the variable ψ is related to the variables ϕ_j in terms of the linear relation (13)

$$\psi = k_1\phi_1 + \ldots + k_m\phi_m + \theta.$$

Now suppose that the scalar function (37) has the form

$$\rho(\phi) = \frac{1}{2}\,\langle k, b(\phi)\rangle = \bar{\rho}(\phi)\,\sin^{2p-1}\psi, \quad p \in \mathbb{N}, \tag{38}$$

with a coefficient $\bar{\rho}(\phi) > 0$. We will find a solution $(\Phi^T(\phi), B_2(\phi))$ of the equation (36) with a positive definite matrix $B_2(\phi)$. For this purpose, we place matrix $\bar{B}_2(\phi)$ in equation (36) as follows: $B_2(\phi) = \bar{B}_2(\phi)\,[\sin^{2p}\psi + \cos^{2p}\psi]$ and get

$$-L_1\Phi^T(\phi)\cos\psi + L^T\bar{B}_2(\phi)[\sin^{2p}\psi + \cos^{2p}\psi]\sin\psi = \bar{\rho}(\phi)L_1S_{12}^T\cos^2\psi\sin^{2p+1}\psi +$$
$$+ \bar{\rho}(\phi)L^TS_2\sin^{2p+1}\psi.$$

We can see here that the resulting equation will be satisfied when the following two equations take place:

$$\bar{B}_2(\phi) = \bar{\rho}(\phi)S_2,$$
$$-L_1\Phi^T + L^T\bar{B}_2(\phi)\cos^{2p-1}\psi\sin\psi - \bar{\rho}(\phi)L_1S_{12}^T\cos\psi\sin^{2p-1}\psi = 0.$$

Thus if the scalar function (37) in equation (36) has the form (38), this equation always has a solution

$$B_2(\phi) = \bar{\rho}(\phi)S_2\,[\sin^{2p}\psi + \cos^{2p}\psi],$$
$$\Phi^T = \bar{\rho}(\phi)L_1^{-1}L^TS_2\cos^{2p-1}\psi\sin\psi - \bar{\rho}(\phi)S_{12}^T\cos\psi\sin^{2p-1}\psi.$$

Summing up the considerations, we can write them in the form of the following theorem.

Theorem 2. *Let a nondegenerate quadratic form (20) be given,*

$$W = \langle S_1x_1, x_1\rangle\cos\psi + 2\langle S_{12}x_1, x_2\rangle\sin\psi - \langle S_2x_2, x_2\rangle\cos\psi,$$

with constant $(n \times n)$-dimensional matrices S_j, S_{12} that satisfy the conditions (21), (22), $\psi = = k_1\phi_1 + \ldots + k_m\phi_m + \theta$, and suppose that a set of systems (32) for which the derivative with respect to them is positive definite. Then, under the condition (38), within these systems there are systems with block the triangular matrix P

$$P = \begin{bmatrix} P_1(\phi) & 0 \\ P_{21}(\phi) & P_2(\phi) \end{bmatrix}, \tag{39}$$

where P_j *are* $(n \times n)$*-dimensional matrices. If the condition (38) will be replaced with one of the following conditions:* $\langle k, b(\phi)\rangle \equiv \mathrm{const}$, *or* $\langle k, b(\phi)\rangle > 0$, *or* $\langle k, b(\phi)\rangle < 0$, *then the set of systems (32) does not contain a block triangular matrix* P *in the form (39).*

3. Conclusions. The regularity property of certain classes of linear extensions of dynamical systems on the m-dimensional torus with the block triangular matrix in normal variables was examined. The studies make it possible to explain smoothness of the Green's function with respect to parameters for weakly regular linear extensions of dynamical systems, as well as to clarify the continuous differentiability of invariant torus for disordered systems. Research are related to the nonlinear multifrequency oscillations. The trajectories of the function that describes the vibrations are located on the multidimensional toroidal surfaces. Problem of preserving such toroidal surfaces under small disturbances is important and currently not resolved completely. A contribution to the mathematical solution of this issue is this work.

1. *Mitropolskij Yu. A., Samoilenko A. M., Kulyk V. L.* Investigation of dichotomy of linear systems of differential equations using Lyapunov functions. — Kiev: Naukova Dumka, 1990.

2. *Kulik V. L.* On some classes of regular linear extensions of dynamical systems on a torus // Ukr. Math. J. — 1994. — **46**, № 11. — P. 1479 – 1485.

3. *Mitropolsky Yu. A., Samoilenko A. M., Kulyk V. L.* Dichotomies and stability in nonautonomous linear systems. — London: Taylor & Francis Inc., 2003.

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