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UNIDIRECTIONAL SYNCHRONIZATION OF TWO MULTISCROLL CHAOTIC SYSTEMS USING NONLINEAR CONTROL TECHNIQUE

ОДНОСПРЯМОВАНА СИНХРОНІЗАЦІЯ ДВОХ БАГАТОВИМІРНИХ СКРОЛІНГОВИХ ХАОТИЧНИХ СИСТЕМ З ВИКОРИСТАННЯМ ТЕХНІКИ НЕЛІНІЙНОГО КЕРУВАННЯ

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The dynamics of nonlinear systems and especially chaotic ones have attracted increasing attention in recent years. In this paper, we consider a new chaotic system that recently introduced in the literature. This system exhibits several various behaviors such as two, three, and four scrolls. Using a nonlinear control methodology, we synchronize a unidirectional coupling structure for the two chaotic systems. Numerical simulations are used to support the theoretical analysis. Additionally we see the robustness of the system in the presence of a noise in simulation.

В останні роки зростає увага до динаміки нелінійних систем, особливо до хаотичної динаміки. У статті розглянуто нову хаотичну систему, що нещодавно з'явилась у літературі. Ця система проявляє декілька поведінок, таких як дво-, три- та чотиривимірні скролінги. Використовуючи техніку нелінійного керування, отримано односпрямовану синхронізацію сполученої структури для двох хаотичних систем. Чисельне моделювання підтверджує теоретичні дослідження. Також було спостережено стійкість системи відносно наявності шуму при моделюванні.

1. Introduction. Chaos theory and its related technology have gradually become well known as a promising research field with significant impacts on an increasing number of novel, potentially attractive, time- and energy-critical engineering applications. Among different effort in the field, chaos control has wide applications in diverse fields. Chaos control was developed by Grebogi, Ott and Yorke in the recent years [1]. With different applications, effective methods such as adaptive method [2], back-stepping design [3], time-delay feedback control [4], active control [5], and nonlinear control [6] were devised to synchronize and control various chaotic systems.

Many mathematical definitions of chaos exist but roughly, it may be described as a type of dynamic behavior with the following characteristics [7]: extreme sensitivity to changes in initial conditions, random-like behavior, and deterministic motion. Traditionally we have several types of chaotic systems in practice. A regular chaotic system has one positive Lyapunov exponent. Systems with more than one positive Lyapunov exponent are called *hyperchaotic* and reveal more complicated dynamics that chaotic systems do.

Generally we have three problems in chaos literature: suppression [8 – 10], chaotization [11 – 13], and synchronization. In the following we describe the synchronization briefly.

The important class of the control objectives corresponds to the problem of synchronization. Synchronization finds important applications in vibration technology [7], communications [12], biology and ecology [13], and many others. Numerous publications on control of synchronization of the chaotic processes and their application in the data transmission systems appeared during nineties [7].

In this paper we consider a new chaotic system which has been recently introduced in the scientific community of chaos. This system is presented and analyzed in [14]. We will make a unidirectional coupling structure for two such systems that are run in different initial conditions. We call this structure as master-slave structure. Bases on the supersensitivity to initial conditions property for chaotic systems, a small discrepancy between initial conditions makes the chaotic system behavior unpredictable in the long run. Thus designing a synchronization scheme via a control law may be useful in various applied fields. One of these fields is secure chaos-based communication systems that use the synchronization structure as its important parts. Using a nonlinear technique the slave trajectories are forced to track the master trajectories asymptotically. Such type of synchronization is called as complete synchronization.

The plan of this paper is as follows. Section 2 briefly presents an introduction to the novel chaotic system. Section 3 studies the synchronization scheme using nonlinear control techniques and simulation results are presented in Section 4. Conclusions in Section 5 close the paper.

2. System description. Consider a three-dimensional autonomous system, which was proposed in [14]. This system has very rich nonlinear dynamics, including chaos, period doubling bifurcations, and others. Moreover, this system can generate two-scroll chaotic attractors. We mean by two-scroll attractors, any set that is dense in the bounded region of attraction [15]. However, by varying a single parameter, a new three-scroll chaotic attractor is detected in the novel three-dimensional smooth system. This three-scroll chaotic attractor evolves into a four-scroll chaotic attractor in some way.

The chaotic system is described as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} y - ax + byz \\ cy - xz + z \\ dxy - hz \end{pmatrix}, \tag{1}$$

where $[x(t), y(t), z(t)]^T \in X \subset \mathbf{R}^3$ is the state vector, X is a well-defined subspace of $\mathbf{R}^3$, and $\{a, b, c, d, h\} \in \mathbf{R}^+$ are some positive constants.

For finding the equilibria of the proposed system it is enough to equate the left-hand side of (1) to zero. The equilibria are:

$$\begin{aligned}
&Q_1 : (0\,0\,0), \\
&Q_2 : \left(\frac{d + \sqrt{\Delta}}{2d} \; \frac{h}{b} \left(\frac{-1 + \sqrt{1 + \Lambda}}{d + \sqrt{\Delta}} \right) \; \frac{-1 + \sqrt{1 + \Lambda}}{2b} \right), \\
&Q_3 : \left(\frac{d + \sqrt{\Delta}}{2d} \; \frac{h}{b} \left(\frac{-1 - \sqrt{1 + \Lambda}}{d + \sqrt{\Delta}} \right) \; \frac{-1 - \sqrt{1 + \Lambda}}{2b} \right), \\
&Q_4 : \left(\frac{d - \sqrt{\Delta}}{2d} \; \frac{h}{b} \left(\frac{-1 + \sqrt{1 + \Lambda}}{d - \sqrt{\Delta}} \right) \; \frac{-1 + \sqrt{1 + \Gamma}}{2b} \right),
\end{aligned} \tag{2}$$

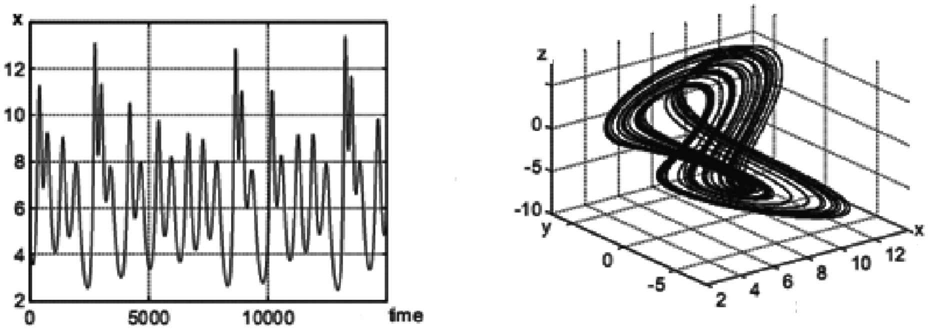

Fig. 1. Chaotic attractor of system with $(a, b, c, d, h) = (3, 2.7, 4.7, 2, 9)$, (2-scroll) with initial conditions $(x_0, y_0, z_0) = (5, -2, 1)$.

$$Q_5 : \left(\frac{d - \sqrt{\Delta}}{2d} \; \frac{h}{b} \left(\frac{-1 - \sqrt{1 + \Lambda}}{d - \sqrt{\Delta}} \right) \; \frac{-1 - \sqrt{1 + \Gamma}}{2b} \right),$$

where

$$\Delta = d^2 + 4chd, \quad \Lambda = \frac{2ab}{h} \left(d + 2ch + \sqrt{\Delta} \right), \quad \Gamma = \frac{2ab}{h} \left(d + 2ch - \sqrt{\Delta} \right).$$

The Jacobian matrix for (1) evaluated in the equilibrium point $Q_i : (x^*, y^*, z^*)$, $i = 1, 2, \ldots, 5$, where x^*, y^*, and z^* are the coordination of the equilibrium, can be calculated as follows:

$$J = \begin{pmatrix} -a & 1 + bz^* & by^* \\ -z^* & c & 1 - x^* \\ dy^* & dx^* & -h \end{pmatrix}. \tag{3}$$

It has been shown that for the following quantities there exist various behaviors [14]:

$$\begin{aligned} &\text{(i) } (a, b, c, d, h) = (3, 2.7, 4.7, 2, 9), \\ &\text{(ii) } (a, b, c, d, h) = (3, 2.7, 1.7, 2, 9), \\ &\text{(iii) } (a, b, c, d, h) = (3, 2.7, 3.9, 2, 9), \end{aligned} \tag{4}$$

where (i) – (iii) correspond to two, three, and four scrolls respectively. As can be seen the coefficient c is variable in these categories.

The numerical simulations for these three chaotic attractors are depicted in Figs. 1, 2, and 3.

3. Synchronization scheme. Consider the master-slave synchronization scheme of two autonomous different fractional order chaotic systems:

master

$$\dot{x}(t) = f(x),$$

slave

$$\dot{y}(t) = g(y) + u,$$

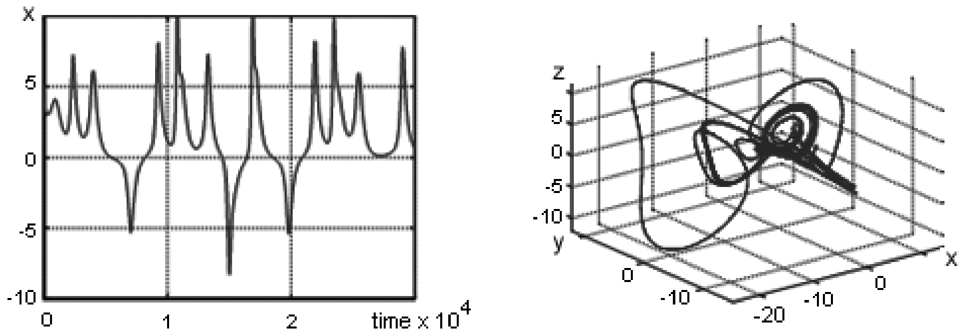

Fig. 2. Chaotic attractor of system with $(a, b, c, d, h) = (3, 2.7, 1.7, 2, 9)$, (3-scroll) with initial conditions $(x_0, y_0, z_0) = (5, -2, 1)$.

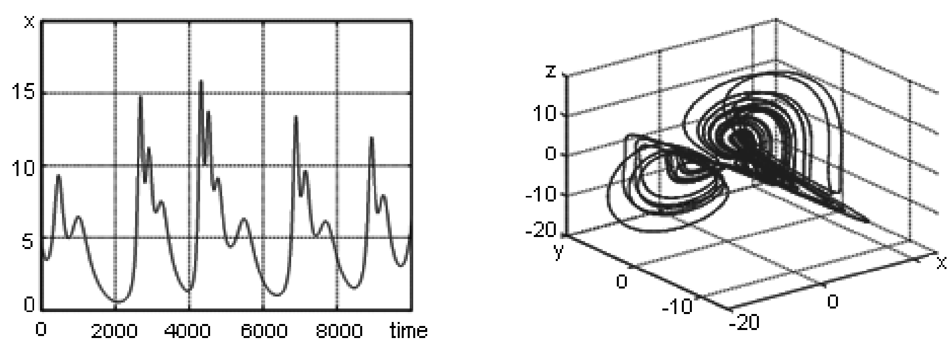

Fig. 3. Chaotic attractor of system with $(a, b, c, d, h) = (3, 2.7, 3.9, 2, 9)$, (4-scroll) with initial conditions $(x_0, y_0, z_0) = (5, -2, 1)$.

where $x, y \in X \subset \mathbf{R}^n$ represent the states of the drive and the response systems, respectively. Moreover, $f : \mathbf{R}^n \to \mathbf{R}^n$, $g : \mathbf{R}^n \to \mathbf{R}^n$ are the vector fields of the drive and response systems respectively. The aim is to choose a suitable control function $u = (u_1, u_2, \ldots, u_n)^T$ such that the states of the drive and response systems are synchronized asymptotically, i.e., $\lim_{t\to\infty} \|y(t) - x(t)\| = 0$. Such convergence between master and slave state is called asymptotic synchronization scheme.

For the system (1), we construct the following master-slave structure as follows in which the subscripts m and s stand for master and slave systems, respectively:

master system

$$\begin{pmatrix} \dot{x}_m \\ \dot{y}_m \\ \dot{z}_m \end{pmatrix} = \begin{pmatrix} y_m - ax_m + by_m z_m \\ cy_m - x_m z_m + z_m \\ dx_m y_m - hz_m \end{pmatrix}, \tag{5}$$

$$(x_m(0), y_m(0), z_m(0)) = (x_{m0}, y_{m0}, z_{m0})$$

and slave system

$$\begin{pmatrix} \dot{x}_s \\ \dot{y}_s \\ \dot{z}_s \end{pmatrix} = \begin{pmatrix} y_s - ax_s + by_s z_s \\ cy_s - x_s z_s + z_s \\ dx_s y_s - hz_s. \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad (6)$$

$$(x_s(0), y_s(0), z_s(0)) = (x_{s0}, y_{s0}, z_{s0}),$$

where and are nonlinear controllers to be designed such that the two chaotic system can be synchronized. Notice that the structure of both systems (master and slave) is the same. For achieving this goal, let us define the error variables as follows:

$$\begin{aligned} e_1(t) &= x_s(t) - x_m(t), \\ e_2(t) &= y_s(t) - y_m(t), \\ e_e(t) &= z_s(t) - z_m(t). \end{aligned} \qquad (7)$$

Substituting Eqs. (5) and (6) in (7) we have

$$\begin{aligned} \dot{e}_1 &= e_2 - ae_1 + by_s e_3 + bz_m e_2 + u_1, \\ \dot{e}_2 &= ce_2 + e_3 - x_s e_3 - z_m e_1 + u_2, \\ \dot{e}_3 &= -he_3 + dx_s e_2 + dy_m e_1 + u_3. \end{aligned} \qquad (8)$$

Now we are ready to express the main theorem of the manuscript.

Theorem 1. *Systems (5) and (6) will approach global and exponential asymptotical synchronization for any initial condition if the control law is selected as follows:*

$$\begin{aligned} u_1 &= -e_2(1 + (b-1)z_m), \\ u_2 &= -e_3(1 - (d-1)x_s) - (1+c)e_2, \\ u_3 &= -e_1(by_s + dy_m). \end{aligned} \qquad (9)$$

Proof. Consider the following Lyapunov function:

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2\right). \qquad (10)$$

With the control law given in Eq. (9), the time derivative of the Lyapunov function along the trajectories of system (8) is:

$$\begin{aligned} \dot{V}(t) &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 = e_1(e_2 - ae_1 + by_s e_3 + bz_m e_2 + u_1) + \\ &+ e_2(ce_2 + e_3 - x_s e_3 - z_m e_1 + u_2) + \\ &+ e_3(-he_3 + dx_s e_2 + dy_m e_1 + u_3) = -ae_1^2 - e_2^2 - he_3^2. \end{aligned} \qquad (11)$$

As mentioned in Eq. (4), all system parameters are positive. Therefore

$$\dot{V}(t) = -ae_1^2 - e_2^2 - he_3^2 < 0 \tag{12}$$

which guarantees the exponential and global stability of Eq. (8). This means that $\lim_{t\to\infty} e_i(t) = 0$, for $i = 1, 2, 3$. Thus we can conclude the complete asymptotical synchronization of (5) and (6).

Using a similar approach one can prove the following theorem.

Theorem 2. *Systems (5) and (6) will approach global and exponential asymptotical synchronization for any initial condition if the control law is selected as follows:*

$$\begin{aligned} u_1 &= -e_2(1 + (b-1)z_s), \\ u_2 &= -e_3(1 + (d+1)x_m) - (1+c)e_2, \\ u_3 &= -e_1(by_m + dy_s). \end{aligned} \tag{13}$$

In the next section we examine the proposed method numerically.

4. Simulation results. In this section, to demonstrate the effectiveness of the proposed methods, we will present the numerical results for synchronizing chaotic systems (5) and (6) under the control laws provided in Theorems 1 and 2. We selected the initial conditions $(x_{m0}, y_{m0}, z_{m0}) = (2, -2, 1)$ for the master system and $(x_{s0}, y_{s0}, z_{s0}) = (6, -1.7, 3)$ for the slave system when we use the control law (10); $(x_{m0}, y_{m0}, z_{m0}) = (-6, 2, 1)$ and $(x_s0), y_{s0}, z_{s0}) = (-6, -1.7, 3)$ for slave system when the control law is as (13). Note that we have chosen the initial conditions very different intentionally. Driving the simulations we turn on the control at the instant $t = 2$. Numerical results for application of Theorem 1, and Theorem 2 are depicted in Fig. 4, and 5 respectively. As can be seen after an initial transient, synchronization is achieved completely and the errors converge to zero asymptotically.

For investigating the robustness of the designed system we insert a band-limited white noise to the master signals. The band-limited white noise is a distributed random numbers that are generated by using Matlab codes. Albeit the primary difference between this function and the internal-predefined Random Number block is that the band-limited white noise function produces output at a specific sample rate, which is related to the correlation time of the noise.

Theoretically, continuous white noise has a correlation time of 0, a flat power spectral density (PSD), and a covariance of infinity. In practice, physical systems are never disturbed by white noise, although white noise is a useful theoretical approximation when the noise disturbance has a correlation time that is very small relative to the natural bandwidth of the system. In what follows we repeat the simulation for the given synchronization scheme by using control law (13). We consider power noise (the height of the PSD of the white noise) as 0.01, 0.03, and 0.02 for x_m, y_m, and z_m respectively. In this case we can observe the robustness of the systems (Fig. 6).

An important point that must be noted is that the above theorems have been derived based on general values for parameters. Therefore the proposed method is valuable for synchronizing the chaotic systems where they exhibit 2-scrolls, 3-scrolls or 4-scrolls. The above simulations can be done for the case when both master and slave systems have 2-scroll chaotic attractors.

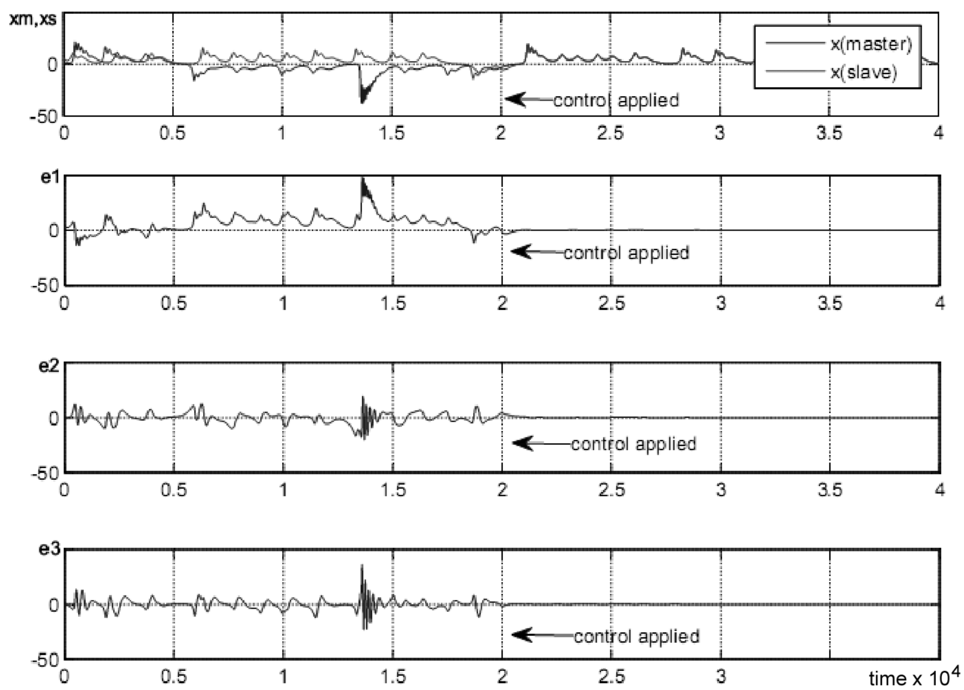

Fig. 4. Time responses for synchronization scheme based on control law (9) with initial conditions $(x_{m0}, y_{m0}, z_{m0}) = (2, -2, 1)$ and $(x_{s0}, y_{s0}, z_{s0}) = (6, -1.7, 3)$.

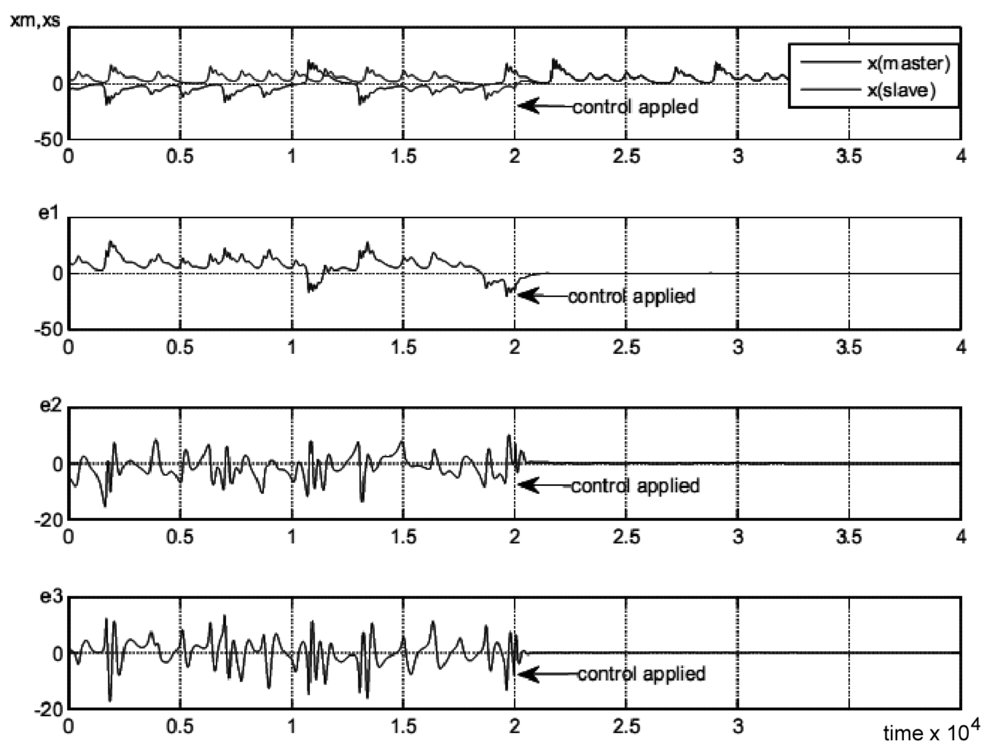

Fig. 5. Time responses for synchronization scheme based on control law (13) with initial conditions $(x_{m0}, y_{m0}, z_{m0}) = (-6, 2, 1)$ and $(x_{s0}, y_{s0}, z_{s0}) = (6, -1.7, 3)$.

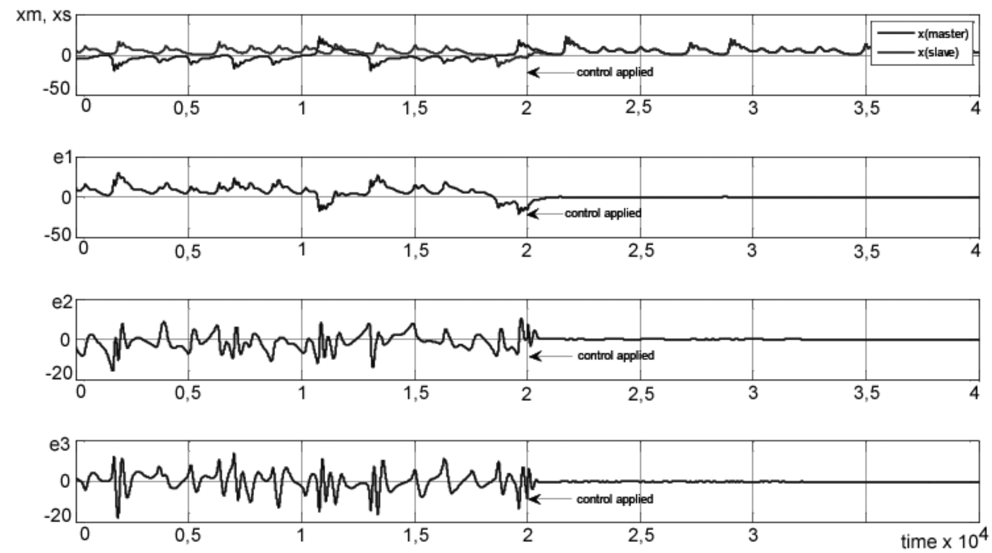

Fig. 6. Time responses for synchronization scheme based on control law (13) with initial conditions $(x_{m0}, y_{m0}, z_{m0}) = (-6, 2, 1)$ and $(x_{s0}, y_{s0}, z_{s0}) = (6, -1.7, 3)$. in the presence of noise.

5. Conclusion. This work discusses a nonlinear control scheme for synchronizing two chaotic systems. Indeed the chaotic system is a novel chaotic system that recently has been introduced in the research societies. The main feature of this system is its interesting behavior. Indeed the system presents various chaotic attractors such as 2, 3, or 4 scrolls. Based on Lyapunov method, we developed two theorems which proposed two control laws that guarantee the asymptotical stability of the error dynamics which means the synchronization is achieved completely. Our proposed method is valuable for each case: both master and slave systems are 2-scrolls, or 3-scrolls, or 4-scrolls. Numerical simulations have been used for clarify the effectiveness of the proposed control laws. It has been shown numerically the proposed techniques are robust in the presence of noisy signals which are gotten form the master system.

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