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ON THE OSCILLATION OF HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS

ПРО ОСЦИЛЯЦІЮ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ВИЩОГО ПОРЯДКУ З ЗАПІЗНЕННЯМ

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The aim of this paper is to study the asymptotic properties and oscillation of the n-th order delay differential equation

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t)f\big(x(\tau(t))\big) = 0. \tag{E}$$

The results obtained are based on some new comparison theorems that reduce the problem of oscillation of an n-th order equation to that of the oscillation of one or more first order equations. We handle both the cases $\int^{\infty} r^{-1/\gamma}(t)\,dt = \infty$ and $\int^{\infty} r^{-1/\gamma}(t)\,dt < \infty$. The comparison principles simplify the analysis of equation (E).

Вивчено асимптотичні властивості та осциляцію диференціального рівняння n-го порядку з запізненням

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t)f\big(x(\tau(t))\big) = 0. \tag{E}$$

Отримані результати базуються на деяких нових теоремах порівняння, які зводять задачу про осциляцію рівняння n-го порядку до такої ж задачі для одного або кількох рівнянь першого порядку. Розглянуто обидва випадки: $\int^{\infty} r^{-1/\gamma}(t)\,dtt = \infty$ та $\int^{\infty} r^{-1/\gamma}(t)\,dt < \infty$. Теореми порівняння дозволяють спростити аналіз рівняння (E).

1. Introduction. In this paper, we examine the asymptotic and oscillatory behavior of solutions of the n-th order ($n \geq 3$) delay differential equation

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t)f\big(x(\tau(t))\big) = 0. \tag{E}$$

We assume that $q, \tau \in C([t_0,\infty))$, $r \in C^1([t_0,\infty))$, $f \in C((-\infty,\infty))$, and

(H_1) γ is the ratio of two odd positive integers;

(H_2) $r(t) > 0$, $r'(t) > 0$, and $q(t) > 0$;

(H_3) $\tau(t) \leq t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$, and $\tau(t)$ is nondecreasing;

(H_4) $xf(x) > 0$ for $x \neq 0$, $f(x)$ is nondecreasing, and

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for} \quad xy > 0.$$

By a *solution* of Eq. (E) we mean a function $x(t) \in C^{n-1}[T_x, \infty)$, $T_x \geq t_0$, for which $r(t)(x^{(n-1)}(t))^\gamma \in C^1[T_x, \infty)$, and $x(t)$ satisfies Eq. (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) that satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$ and we tacitly assume that Eq. (E) possesses such solutions. A solution of (E) is called *oscillatory* if it has arbitrarily large zeros on $[T_x, \infty)$ and it is said to be *nonoscillatory* otherwise. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Equation (E) and its special cases, especially for $n = 2$, has been studied by many authors (see, for example, [2–19]), mainly under the condition

$$\int_{t_0}^{\infty} r^{-1/\gamma}(s)\, ds = \infty. \tag{1.1}$$

There are comparatively fewer results (see, for example, [1] and [20]) for (E) in the case that

$$\int_{t_0}^{\infty} r^{-1/\gamma}(s)\, ds < \infty \tag{1.2}$$

holds. In this paper, we consider both possibilities.

If the gap between t and $\tau(t)$ is small, then there exists a nonoscillatory solution of (E), and so in this case our goal is to prove that every nonoscillatory solution of (E) tends to zero as $t \to \infty$. On the other hand, if the difference $t - \tau(t)$ is large enough, then we shall study the oscillation of (E). Our aim in this paper is to study both of these cases as well.

Various techniques have been used in investigating higher order differential equations. Our method here is based on establishing new comparison theorems that compare the n-th order equation (E) to one or a couple of first order delay differential equations in the sense that the oscillation of these first order equations imply the oscillation of Eq. (E). These comparison theorems greatly simplify the analysis of Eq. (E).

Remark 1. All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all sufficiently large t.

2. Main results. Our results make use of the following estimate that is due to Philos and Staikos (see [17, 18]).

Lemma A. *Let* $z \in C^k([t_0, \infty))$ *and assume that* $z^{(k)}$ *is of fixed sign and not identically zero on a subray of* $[t_0, \infty)$. *If, moreover,* $z(t) > 0$, $z^{(k-1)}(t)z^{(k)}(t) \leq 0$, *and* $\lim_{t\to\infty} z(t) \neq 0$, *then for every* $\delta \in (0, 1)$ *there exists* $t_\delta \geq t_0$ *such that*

$$z(t) \geq \frac{\delta}{(k-1)!}\, t^{k-1}\, |z^{(k-1)}(t)| \tag{2.1}$$

holds on $[t_\delta, \infty)$.

The positive solutions of (E) have the following structure.

Lemma 1. *If $x(t)$ is a positive solution of (E), then $r(t)\left[x^{(n-1)}(t)\right]^{\gamma}$ is decreasing, all derivatives $x^{(i)}(t)$, $1 \le i \le n-1$, are of constant signs, and $x(t)$ satisfies either*

$$x^{(n-1)}(t) > 0, \quad x^{(n)}(t) < 0 \tag{C_1}$$

or, if (1.2) holds,

$$x^{(n-2)}(t) > 0, \quad x^{(n-1)}(t) < 0. \tag{C_2}$$

Proof. Since $x(t)$ is a positive solution of (E), then it follows from (E) that

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' = -q(t)f\big(x(\tau(t))\big) < 0.$$

Thus, $r(t)\left[x^{(n-1)}(t)\right]^{\gamma}$ is decreasing, which implies that either $x^{(n-1)}(t) > 0$ or $x^{(n-1)}(t) < 0$. Note that the second case may occur only if (1.2) holds. Moreover, since $x(t) > 0$, it follows from $x^{(n-1)}(t) < 0$ that $x^{(n-2)}(t) > 0$.

On the other hand, if $x^{(n-1)}(t) > 0$, then using the fact that $r'(t) > 0$ in the expression

$$0 > \left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' = r'(t)\left[x^{(n-1)}(t)\right]^{\gamma} + r(t)\gamma\left[x^{(n-1)}(t)\right]^{\gamma-1}x^{(n)}(t),$$

we conclude that $x^{(n)}(t) < 0$. This completes the proof of the lemma.

We next give some criteria for excluding the possibility that cases (C_1) and (C_2) occur.

Theorem 1. *Let (1.1) hold. If for some constant $\delta \in (0,1)$, the first order delay differential equation*

$$y'(t) + q(t)f\left(\frac{\delta}{(n-1)!}\,\frac{\tau^{n-1}(t)}{r^{1/\gamma}(\tau(t))}\right) f\left(y^{1/\gamma}(\tau(t))\right) = 0, \tag{E_1}$$

is oscillatory, then

(i) for n even, (E) is oscillatory;

(ii) for n odd, every nonoscillatory solution $x(t)$ of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of (E), say $x(t) > 0$. It follows from Lemma 1 that $x(t)$ satisfies (C_1).

If n is even, then it is clear from (C_1) that $\lim_{t\to\infty} x(t) \ne 0$. Thus, it follows from Lemma A that for every $\delta \in (0,1)$

$$x(\tau(t)) \ge \frac{\delta}{(n-1)!}\,\frac{\tau^{n-1}(t)}{r^{1/\gamma}(t)}\left(r^{1/\gamma}(t)x^{(n-1)}(\tau(t))\right), \tag{2.2}$$

eventually. Using (2.2) in (E), we see that $y(t) = r(t)\left[x^{(n-1)}(t)\right]^{\gamma}$ is a positive solution of the delay differential inequality

$$y'(t) + q(t)f\left(\frac{\delta}{(n-1)!}\,\frac{\tau^{n-1}(t)}{r^{1/\gamma}(\tau(t))}\right) f\left(y^{1/\gamma}(\tau(t))\right) \le 0.$$

By Theorem 1 in [16], we conclude that the corresponding equation (E) also has a positive solution. This contradiction proves part *(i)* of the theorem.

Now assume that n is odd. We claim that $\lim_{t\to\infty} x(t) = 0$. If this is not the case, then proceeding exactly as in the proof of part (i), we again obtain that (E_1) has a positive solution. This contradiction proves part (ii) of the theorem.

Remark 2. It follows from the proof of Theorem 1 that the oscillation of (E_1) prevents case (C_1) of Lemma 1 from occurring provided that $\lim_{t\to\infty} x(t) \neq 0$.

Applying criteria for the oscillation of (E_1), we immediately obtain sufficient conditions for Cases (i) and (ii) of Theorem 1 to hold. We offer two such results.

Corollary 1. *Assume (1.1) holds,*

$$f(u^{1/\gamma})/u \geq 1 \quad \textit{for} \quad 0 < |u| \leq 1, \tag{2.3}$$

and for some $\delta \in (0,1)$,

$$\liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} q(s) f\left(\frac{\delta}{(n-1)!}\frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))}\right) ds > \frac{1}{\mathrm{e}}. \tag{2.4}$$

Then:

(i) if n is even, Eq. (E) is oscillatory;

(ii) if n is odd, every nonoscillatory solution $x(t)$ of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. First note that (2.4) implies

$$\int\limits_{t_0}^{\infty} q(s) f\left(\frac{\delta}{(n-1)!}\frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))}\right) ds = \infty.$$

By Theorem 1, it is sufficient to show that (E_1) is oscillatory. Assume to the contrary that (E_1) has an eventually positive solution $y(t)$. Then $y'(t) < 0$. We claim that $\lim_{t\to\infty} y(t) = 0$. If this is not the case, then there exists $\ell > 0$ such that $y(\tau(t)) > \ell$. Integrating (E_1) from t_1 to t, we have

$$y(t_1) = y(t) + \int\limits_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-1)!}\frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))}\right) f\left(y^{1/\gamma}(\tau(s))\right) ds \geq$$

$$\geq f\left(\ell^{1/\gamma}\right) \int\limits_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-1)!}\frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))}\right) ds.$$

Letting $t \to \infty$, we obtain a contradiction, so $\lim_{t\to\infty} y(t) = 0$. Thus $0 < y(t) \leq 1$ eventually. Using (2.3) in (E_1), it is easy to see that $y(t)$ is a positive solution of the differential inequality

$$y'(t) + q(t) f\left(\frac{\delta}{(n-1)!}\frac{\tau^{n-1}(s)}{r^{1/\gamma}(\tau(s))}\right) y(\tau(t)) \leq 0. \tag{2.5}$$

But, by Theorem 2.4.1 in [13], condition (2.4) ensures that inequality (2.5) has no positive solutions. This is a contradiction and completes the proof of the theorem.

A second such result is contained in the following corollary.

Corollary 2. *Let (1.1) hold and let β be the ratio of two odd positive integers with $\beta < \gamma$. If*

$$\limsup_{t\to\infty}\int\limits_{\tau(t)}^{t} q(s)\frac{\left(\tau^{n-1}(s)\right)^{\beta}}{r^{\beta/\gamma}(\tau(s))}\,ds > 0, \tag{2.6}$$

then the differential equation

$$\left(r(t)\left[x^{(n-1)}(t)\right]^{\gamma}\right)' + q(t)x^{\beta}(\tau(t)) = 0 \tag{E^{β}}$$

(i) for n even, is oscillatory;

(ii) for n odd, every nonoscillatory solution $x(t)$ satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. First note that (2.6) implies

$$\int\limits_{t_0}^{\infty} q(s)\frac{\left(\tau^{n-1}(s)\right)^{\beta}}{r^{\beta/\gamma}(\tau(s))}\,ds = \infty.$$

Taking into account Theorem 1, it is sufficient to show that Eq. (E_1), which now reduces to

$$y'(t) + \left(\frac{\delta}{(n-1)!}\right)^{\beta} q(t)\frac{\left(\tau^{n-1}(t)\right)^{\beta}}{r^{\beta/\gamma}(\tau(t))}\,y^{\beta/\gamma}(\tau(t)) = 0, \tag{E_1^{β}}$$

is oscillatory. Assume that (E_1^{β}) has an eventually positive solution $y(t)$. Similar to the proof of Corollary 1, we can show that $y(t)$ is decreasing and $\lim_{t\to\infty} y(t) = 0$. Integrating (E_1^{β}) from $\tau(t)$ to t, we obtain

$$0 = y(t) - y(\tau(t)) + \left(\frac{\delta}{(n-1)!}\right)^{\beta}\int\limits_{\tau(t)}^{t} q(s)\frac{\left(\tau^{n-1}(s)\right)^{\beta}}{r^{\beta/\gamma}(\tau(s))}y^{\beta/\gamma}(\tau(s))\,ds.$$

From the monotonicity of $y^{\beta/\gamma}(\tau(t))$, we have

$$\left(\frac{(n-1)!}{\delta}\right)^{\beta} y^{1-\beta/\gamma}(\tau(t)) \geq \int\limits_{\tau(t)}^{t} q(s)\frac{\left(\tau^{n-1}(s)\right)^{\beta}}{r^{\beta/\gamma}(\tau(s))}\,ds.$$

Taking the limit superior of both sides, we obtain a contradiction to (2.6), and this establishes the desired result.

Next, we turn our attention to the case where n is odd. Employing an additional condition, we are able to ensure the oscillation of all solutions of Eq. (E) for n odd. That is, we are able to eliminate the possibility that there are nonoscillatory solutions converging to zero. For convenience, we set

$$\xi_1(t) = \xi(t), \qquad \xi_{i+1}(t) = \xi_i\big(\xi(t)\big),$$
$$J_1(t) = \xi(t) - t, \qquad J_{i+1}(t) = \int\limits_t^{\xi(t)} J_i(s)\, ds,$$

where $\xi(t) \in C([t_0, \infty))$.

Theorem 2. *Let n be odd and (1.1) hold. Assume that $\xi(t) \in C([t_0, \infty))$ is such that*

$$\xi(t) \quad \textit{nondecreasing,} \quad \xi(t) > t, \quad \textit{and} \quad \xi_{n-1}(\tau(t)) < t. \tag{2.7}$$

If for some $\delta \in (0, 1)$, Eq. (E_1) is oscillatory and the equation

$$y'(t) + q(t) f\left(r^{-1/\gamma}\big(\xi_{n-1}(\tau(t)\big) J_{n-1}(\tau(t))\right) f\left(y^{1/\gamma}\big(\xi_{n-1}(\tau(t)\big)\right) = 0 \tag{E_2}$$

is also oscillatory, then Eq. (E) is oscillatory.

Proof. Assume to the contrary that $x(t)$ is a positive solution of (E). Then, by Theorem 1, the oscillation of (E) implies that $\lim_{t\to\infty} x(t) = 0$. Thus, in view of Lemma 1, $x(t)$ satisfies

$$(-1)^i x^{(i)}(t) > 0, \quad i = 1, 2, \ldots, n-1. \tag{2.8}$$

Consequently,

$$-x^{(n-2)}(t) \geq x^{(n-2)}(\xi(t)) - x^{(n-2)}(t) = \int\limits_t^{\xi(t)} x^{(n-1)}(s)\, ds \geq$$
$$\geq x^{(n-1)}(\xi(t))(\xi(t) - t) = x^{(n-1)}(\xi(t)) J_1(t).$$

The repeated integration of the previous inequalities from t to $\xi(t)$, yields

$$x(t) \geq x^{(n-1)}(\xi_{n-1}(t)) J_{n-1}(t)$$

or equivalently

$$x(\tau(t)) \geq \left[r^{1/\gamma}\big(\xi_{n-1}(\tau(t))\big) x^{(n-1)}(\xi_{n-1}(\tau(t)))\right] \frac{J_{n-1}(\tau(t))}{r^{1/\gamma}\big(\xi_{n-1}(\tau(t))\big)}.$$

Using the last inequality in (E), we see that $y(t) = r(t)\left[x^{(n-1)}(t)\right]^\gamma$ is a positive solution of the delay differential inequality

$$y'(t) + q(t) f\left(r^{-1/\gamma}\big(\xi_{n-1}(\tau(t)\big) J_{n-1}(\tau(t))\big)\right) f\left(y^{1/\gamma}\big(\xi_{n-1}(\tau(t))\big)\right) \leq 0.$$

It follows from Theorem 1 in [16], that the corresponding equation (E_2) also has a positive solution. This contradiction completes the proof.

Remark 3. Similar to Remark 2 above, the oscillation of Eq. (E_2) prevents case (C_1) of Lemma 1 from holding provided that $\lim_{t\to\infty} x(t) \neq 0$.

As an application of Theorem 2, we have the following corollary.

Corollary 3. *Let n be odd and (1.1), (2.3), and (2.4) hold for some $\delta \in (0,1)$. Assume that $\xi(t) \in C([t_0,\infty))$ is such that (2.7) is satisfied. If*

$$\liminf_{t\to\infty} \int\limits_{\xi_{n-1}(\tau(t))}^{t} q(s) f\left(r^{-1/\gamma}\big(\xi_{n-1}(\tau(s))\big) J_{n-1}(\tau(s))\right) ds > \frac{1}{\mathrm{e}}, \tag{2.9}$$

then (E) is oscillatory.

Proof. By Theorem 2, it is sufficient to show that both equations (E_1) and (E_2) are oscillatory. It follows from the proof of Corollary 1 that the oscillation of (E_1) is due to (2.4). Using arguments similar to those in the proof of Corollary 1, it can be shown that (2.9) guarantees the oscillation of (E_2). This proves the corollary.

We illustrate our results in the following examples.

Example 1. Consider the n-th order nonlinear differential equation

$$\left(t^3\left(x^{(n-1)}(t)\right)^3\right)' + \frac{b}{t^{3n-5}}\, x^3(\lambda t) = 0 \tag{2.10}$$

with $b > 0$ and $0 < \lambda < 1$. Condition (2.4) reduces to

$$\delta^3 b\lambda^{3n-6} \ln\frac{1}{\lambda} > \frac{\big((n-1)!\big)^3}{\mathrm{e}} \quad \text{for some} \quad \delta \in (0,1), \tag{2.11}$$

or simply

$$b\lambda^{3n-6} \ln\frac{1}{\lambda} > \frac{\big((n-1)!\big)^3}{\mathrm{e}} \tag{2.12}$$

since (2.12) implies (2.11). Hence, Corollary 1 guarantees that if (2.12) holds, then

(i) for n even, (2.10) is oscillatory;

(ii) for n odd, every nonoscillatory solution $x(t)$ of (2.10) satisfies $\lim_{t\to\infty} x(t) = 0$.

For $n = 3$ and $\beta > 0$ such that $3\beta^3(\beta+1)^4 = b\lambda^{-3\beta}$, one such solution is $x(t) = t^{-\beta}$.

Moreover, if n is odd, we set $\xi(t) = \alpha t$, where $\alpha = \dfrac{1+\lambda^{-1/(n-1)}}{2}$. Then condition (2.9) takes the form

$$\frac{b}{\alpha^{3n-3}\lambda^3}\left(\frac{(\lambda-1)(\lambda^2-1)\,\ldots\,(\lambda^{n-1}-1)\lambda^{n-1}}{(n-1)!}\right)^3 \ln\frac{1}{\alpha^{n-1}\lambda} > \frac{1}{\mathrm{e}}. \tag{2.13}$$

It follows from Corollary 3 that (2.10) is oscillatory even if n is odd, provided that both conditions (2.12) and (2.13) are satisfied.

Now, we turn our attention to the case where (1.2) holds. It is useful to observe that in this case, Eq. (E) may have a solution $x(t)$ with the property $x(t)x'(t) < 0$ no matter if n is even or odd.

Theorem 3. *Let (1.2) hold. If for some constant* $\delta \in (0,1)$ *and every* $t_1 \geq t_0$, *both the first order delay differential equations* (E_1) *and*

$$y'(t) + r^{-1/\gamma}(t) \left[\int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-2)!} \tau^{n-2}(s) \right) ds \right]^{1/\gamma} f^{1/\gamma}\left(y(\tau(t))\right) = 0 \qquad (E_3)$$

are oscillatory, then every nonoscillatory solution of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume to the contrary that $x(t)$ is a nonoscillatory solution of (E) such that $\lim_{t\to\infty} x(t) \neq 0$. We may assume that $x(t) > 0$. Lemma 1 implies that $x(t)$ satisfies either (C_1) or (C_2). On the other hand, it follows from the proof of Theorem 1 that the oscillation of (E_1) implies case (C_1) is not possible. We shall show that the oscillation of (E_3) excludes the case (C_2).

Lemma 1 gives the estimate

$$x(\tau(t)) \geq \frac{\delta}{(n-2)!} \tau^{n-2}(t)\, x^{(n-2)}(\tau(t)). \qquad (2.14)$$

Using (2.14) in (E) yields

$$\left(r(t) \left[x^{(n-1)}(t) \right]^{\gamma} \right)' + q(t) f\left(\frac{\delta}{(n-2)!} \tau^{n-2}(t) \right) f\left(x^{(n-2)}(\tau(t)) \right) \leq 0.$$

Integrating, we obtain

$$-r(t) \left[x^{(n-1)}(t) \right]^{\gamma} \geq \int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-2)!} \tau^{n-2}(s) \right) f\left(x^{(n-2)}(\tau(s)) \right) ds,$$

which in view of the monotonicity of $f\left(x^{(n-2)}(\tau(t))\right)$ gives

$$-x^{(n-1)}(t) \geq r^{-1/\gamma}(t) f^{1/\gamma}\left(x^{(n-2)}(\tau(t)) \right) \left[\int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-2)!} \tau^{n-2}(s) \right) ds \right]^{1/\gamma}.$$

Consequently, $y(t) = x^{(n-2)}(t)$ is a positive solution of the delay differential inequality

$$y'(t) + r^{-1/\gamma}(t) \left[\int_{t_1}^{t} q(s) f\left(\frac{\delta}{(n-2)!} \tau^{n-2}(s) \right) ds \right]^{1/\gamma} f^{1/\gamma}\left(y(\tau(t))\right) \leq 0.$$

By Theorem 1 in [16], the corresponding equation (E_3) also has a positive solution. This contradiction shows that $\lim_{t\to\infty} x(t) = 0$ and completes the proof of the theorem.

Remark 4. The oscillation of Eq. (E_3) prevents the case (C_2) in Lemma 1 from occurring provided that $\lim_{t\to\infty} x(t) \neq 0$.

Next, we eliminate the possibility that $\lim_{t\to\infty} x(t) = 0$ from Theorem 3 even if (1.2) holds. We shall consider another first order delay differential equation, namely,

$$y'(t) + r^{-1/\gamma}(t)\left[\int_{t_1}^{t} q(s)\,ds\right]^{1/\gamma} f^{1/\gamma}\big(J_{n-2}(\tau(t))\big) f^{1/\gamma}\left(y\big(\xi_{n-2}(\tau(t))\big)\right) = 0. \qquad (E_4)$$

Theorem 4. *Let (1.2) hold. Assume that for some $\delta \in (0,1)$ and every $t_1 \geq t_0$, both (E_1) and (E_3) are oscillatory. Assume further that there exists $\xi(t) \in C([t_0,\infty))$ such that*

(i) for n odd, (2.7) holds and (E_2) is oscillatory;

(ii) for n even, (E_4) is oscillatory for every $t_1 \geq t_0$, and

$$\xi(t) \quad \textit{nondecreasing}, \quad \xi(t) > t, \quad \textit{and} \quad \xi_{n-2}(\tau(t)) < t. \qquad (2.15)$$

Then Eq. (E) is oscillatory.

Proof. Assume that $x(t)$ is a positive solution of (E). It follows from the proofs of Theorems 1 and 3 that $\lim_{t\to\infty} x(t) = 0$. Then, in view of Lemma 1, $x(t)$ has to satisfy (2.8).

(i) If n is odd, it follows from the proof of Theorem 2 that (E) is oscillatory due to the oscillation of (E_2).

(ii) Assume that n is even. We shall show that (2.8) cannot hold. Proceeding exactly as in the proof of Theorem 2, we obtain

$$x(t) \geq x^{(n-2)}(\xi_{n-2}(t))J_{n-2}(t). \qquad (2.16)$$

On the other hand, an integration of (E) yields

$$-r(t)\left[x^{(n-1)}(t)\right]^{\gamma} \geq \int_{t_1}^{t} q(s)f(x(\tau(s)))\,ds \geq f(x(\tau(t)))\int_{t_1}^{t} q(s)\,ds.$$

That is,

$$-x^{(n-1)}(t) \geq r^{-1/\gamma}(t) f^{1/\gamma}(x(\tau(t)))\left[\int_{t_1}^{t} q(s)\,ds\right]^{1/\gamma},$$

which combined with (2.16) implies $y(t) = x^{(n-2)}(t)$ is a positive solution of the delay differential inequality

$$y'(t) + r^{-1/\gamma}(t)\left[\int_{t_1}^{t} q(s)\,ds\right]^{1/\gamma} f^{1/\gamma}\left(J_{n-2}(\tau(t))\right) f^{1/\gamma}\left(y\big(\xi_{n-2}(\tau(t))\big)\right) \leq 0.$$

Again by Theorem 1 in [16], the corresponding equation (E_2) must have a positive solution.

This completes the proof of the theorem.

Remark 5. The oscillation of (E_4) prevents the case (C_2) in Lemma 1 from holding provided that $\lim_{t\to\infty} x(t) \neq 0$.

Corollary 4. *Let (1.2) and (2.3) hold, and for some $\delta \in (0,1)$ and every $t_1 \geq t_0$, let both (2.4) and*

$$\liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} r^{-1/\gamma}(u) \left[\int\limits_{t_1}^{u} q(s) f\left(\frac{\delta}{(n-2)!} \tau^{n-2}(s) \right) ds \right]^{1/\gamma} du > \frac{1}{\mathrm{e}} \tag{2.17}$$

be satisfied. Then every nonoscillatory solution of (E) tend to zero as $t \to \infty$.

Assume, in addition, that there exists $\xi(t) \in C([t_0, \infty))$ such that:

(i) for n odd, (2.7) and (2.9) hold;

(ii) for n even, (2.15) holds and

$$\liminf_{t\to\infty} \int\limits_{\xi_{n-2}(\tau(t))}^{t} r^{-1/\gamma}(u) \left[\int\limits_{t_1}^{u} q(s)\, ds \right]^{1/\gamma} f^{1/\gamma}\big(J_{n-2}(\tau(u))\big)\, du > \frac{1}{\mathrm{e}}. \tag{2.18}$$

Then (E) is oscillatory.

Proof. Conditions (2.4), (2.9), (2.17), and (2.18) ensure that (E_1), (E_2), (E_3), and (E_4), respectively, are oscillatory. The assertion now follows from Theorems 3 and 4.

Example 2. Consider the n-th order nonlinear differential equation

$$\left(t^6 \left(x^{(n-1)}(t) \right)^3 \right)' + \frac{b}{t^{3n-8}} x^3(\lambda t) = 0 \tag{2.19}$$

with $b > 0$ and $0 < \lambda < 1$. Conditions (2.4) and (2.17) reduce to

$$b\lambda^{3n-9} \ln \frac{1}{\lambda} > \frac{\big((n-1)!\big)^3}{\mathrm{e}}, \tag{2.20}$$

$$b^{1/3}\lambda^{n-2} \ln \frac{1}{\lambda} > \frac{3^{1/3}(n-2)!}{\mathrm{e}}, \tag{2.21}$$

respectively. Corollary 4 guarantees that every nonoscillatory solution $x(t)$ of (2.19) tends to zero as $t \to \infty$ provided that both conditions (2.20) and (2.21) hold.

On the other hand, if n is odd, we set $\xi(t) = \alpha t$, where $\alpha = \dfrac{1+\lambda^{-1/(n-1)}}{2}$. Then condition (2.9) takes the form

$$\frac{b}{\alpha^{6n-6}\lambda^6} \left(\frac{(\lambda-1)(\lambda^2-1)\,\ldots\,(\lambda^{n-1}-1)\lambda^{n-1}}{(n-1)!} \right)^3 \ln \frac{1}{\alpha^{n-1}\lambda} > \frac{1}{\mathrm{e}}. \tag{2.22}$$

Conditions (2.20) – (2.22) imply Corollary 4 holds, so all solutions of Eq. (2.19) with n odd are oscillatory.

For n even, if we set $\xi(t) = \alpha t$, with $\alpha = \dfrac{1+\lambda^{-1/(n-2)}}{2}$, then condition (2.18) takes the form

$$b(\lambda-1)(\lambda^2-1)\,\ldots\,(\lambda^{n-2}-1)\lambda^{n-2}\ln\frac{1}{\alpha^{n-2}\lambda} > \frac{(n-2)!(3n-9)^{1/3}}{\mathrm{e}}. \tag{2.23}$$

It follows from Corollary 4 that Eq. (2.10) with n even is oscillatory if conditions (2.20), (2.21), and (2.23) are satisfied.

Example 3. Consider the fourth order delay differential equation

$$\left(\mathrm{e}^t x'''(t)\right)' + \frac{\mathrm{e}^{t-1/2}}{16}\,x(t-1) = 0, \quad t \ge 1. \tag{2.24}$$

This equation has been studied by Zhang et al. in [20]. They showed that every nonoscillatory solution of (2.24) tends to zero as $t \to \infty$. (This conclusion also follows from our Corollary 4 as well). In particular, $x(t) = \mathrm{e}^{-t/2}$ is a solution of (2.24). Now, we consider the more general differential equation

$$\left(\mathrm{e}^t x'''(t)\right)' + b\,\mathrm{e}^t x(t-1) = 0, \quad t \ge 1. \tag{2.25}$$

It is not difficult to verify that both (2.4) and (2.17) hold. If we set $\xi(t) = t/4$ then (2.18) takes the form

$$b > \frac{2^5}{\mathrm{e}},$$

which according to Corollary 4 yields the oscillation of (2.25). This is a new phenomena that does not appear to have been studied previously.

3. Summary. In this paper, we presented new comparison theorems for studying the asymptotic behavior and oscillation of Eq. (E) from the oscillation of a set of suitable first order delay differential equations. Thus, our method essentially simplifies the examination of higher order equations and what is more, it supports the value of continued research on first order delay differential equations. Our results here extend and complement many recent ones in the literature. Suitable illustrative examples were also provided.

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