UDC 517.9

OSCILLATION AND NONOSCILLATION OF SECOND ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS*

ОСЦИЛЯЦІЙНІ ТА НЕОСЦИЛЯЦІЙНІ НАПІВЛІНІЙНІ ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ ДРУГОГО ПОРЯДКУ

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We consider the oscillation and nonoscillation of second order half-linear differential equation. By using some new technique, we establish new oscillation and nonoscillation criteria which extend and improve some known results for second order linear differential equation.

Розглянуто осциляційні та неосциляційні напівлінійні диференціальні рівняння другого порядку. З використанням нової техніки встановлено нові критерії наявності та відсутності осциляцій. Ці критерії поширюють та покращують відомі результати для лінійних диференціальних рівнянь другого порядку.

1. Introduction. Consider the second order half-linear differential equation

$$(|u'(t)|^{\alpha-1}u'(t))' + p(t)|u(t)|^{\alpha-1}u(t) = 0, \tag{1}$$

where $\alpha > 0$ is a constant, $p \in C([0,+\infty), [0,+\infty))$ is an integrable function.

During the last three decades, investigation of oscillation and nonoscillation of second order half-linear differential equations has been attracting attention of numerous researchers. The reader is referred to the monographs by Agarwal, Grace and O'Regan [1, 2], Dosly and Rehak [3], papers [5–18], and references therein.

By a solution of (1) is meant a function $u \in C^1[T_u,\infty)$, $T_u \geq 0$, which has the property $|u'|^{\alpha-1}u' \in C^1[T_u,\infty)$ and satisfies the equation for all $t \geq T_u$. We consider only those solutions $u(t)$ of (1) which satisfy $\sup\{|u(t)| : t \geq T\} > 0$ for all $T \geq T_u$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory.

The purpose of this paper is to establish new oscillation and nonoscillation criteria of Eq. (1) which extend and improve some criteria of linear differential equation in the references.

We can easily show that if for some $\lambda < \alpha$ the integral $\int^{+\infty} s^{\lambda}p(s)\,ds$ diverges, then equa-

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tion (1) is oscillatory. Therefore, we shall always assume below that

$$\int^{+\infty} s^{\lambda} p(s) ds < +\infty \quad \text{for} \quad \lambda < \alpha.$$

2. Main results. Introduce the notations

$$h_{\lambda}(t) = t^{\alpha-\lambda} \int_{t}^{\infty} s^{\lambda} p(s)\, ds \quad \text{for} \quad t > 0 \quad \text{and} \quad \lambda < \alpha,$$

$$h_{\lambda}(t) = t^{\alpha-\lambda} \int_{t_1}^{t} s^{\lambda} p(s) ds \quad for \quad t > 0 \quad \text{and} \quad \lambda > \alpha, \tag{2}$$

$$p_*(\lambda) = \liminf_{t\to+\infty} h_{\lambda}(t), \quad p^*(\lambda) = \limsup_{t\to+\infty} h_{\lambda}(t).$$

The following lemmas will be useful for establishing oscillation criteria for Eq. (1). The first one is a well-known inequality which is due to Hardy et al. [4].

Lemma 1 [4]. *If X and Y are nononegative, then*

$$X^q + (q-1)Y^q \geq qXY^{q-1} \quad for \quad q > 1, \tag{3}$$

where the equality holds if and only if $X = Y$.

Lemma 2. *Let equation (1) be nonoscillatory. Then there exits $t_0 > 0$ such that the equation*

$$\rho' + p(t) + \alpha\rho^{1+1/\alpha} = 0 \tag{4}$$

has a solution $\rho\colon [t_0, +\infty) \to [0, +\infty)$, moreover,

$$\rho(t_0+) = +\infty, \quad (t - t_0)(\rho(t))^{\frac{1}{\alpha}} < 1 \quad for \quad t_0 < t < +\infty, \tag{5}$$

$$\lim_{t\to+\infty} t^{\lambda}(\rho(t))^{\frac{1}{\alpha}} = 0 \quad for \quad \lambda < 1, \tag{6}$$

and

$$\liminf_{t\to+\infty} t^{\alpha}\rho(t) \geq A, \quad \limsup_{t\to+\infty} t^{\alpha}\rho(t) \leq B, \tag{7}$$

where

$$\begin{aligned} A &= \min\{r \in R^+ | p_*(0) - r + r^{1+1/\alpha} \leq 0\}, \\ B &= \max\{R_1 \in R^+ | p_*(1+\alpha) - \alpha R_1 + \alpha R_1 A^{\frac{1}{\alpha}} \leq 0\}. \end{aligned} \tag{8}$$

Proof. Since equation (1) is nonoscillatory, there exists $t_0 > 0$ such that the solution $u(t)$ of equation (1) under the initial conditions $u(t_0) = 0$, $u'(t_0) = 1$ satisfies the inequalities

$$u(t) > 0, \quad u'(t) \geq 0 \quad \text{for} \quad t_0 < t < +\infty.$$

Clearly, the function $\rho(t) = (u'(t)/u(t))^\alpha$ for $t_0 < t < +\infty$ is a solution of equation (4), and $\lim_{t\to t_0+} \rho(t) = +\infty$. From (4) we have

$$\frac{-\rho'(t)}{\alpha(\rho(t))^{1+1/\alpha}} > 1 \quad \text{for} \quad t_0 < t < +\infty.$$

Integrating the above inequality from t_0 to t, we obtain $(t - t_0)(\rho(t))^{1/\alpha} < 1$ for $t_0 < t < +\infty$. In particular, equality (6) holds for any $\lambda < 1$.

Now, we show that inequalities (7) are valid. Assume $p_*(0) \neq 0$ and $p_*(1+\alpha) \neq 0$ (inequalities (7) are trivial, otherwise). We introduce the notation

$$r = \liminf_{t\to+\infty} t^\alpha \rho(t), \quad R_1 = \limsup_{t\to+\infty} t^\alpha \rho(t).$$

From (4) we easily find that for any $t_1 > t_0$

$$t^\alpha \rho(t) = t^\alpha \int_t^{+\infty} p(s)\, ds + t^\alpha \alpha \int_t^{+\infty} (\rho(s))^{1+\frac{1}{\alpha}}\, ds. \tag{9}$$

Multiplying equality (4) by $t^{1+\alpha}$, integrating it from t_1 to t, we get

$$\int_{t_1}^{t} \rho'(s) s^{1+\alpha}\, ds + \int_{t_1}^{t} p(s) s^{1+\alpha}\, ds + \int_{t_1}^{t} \alpha\rho(s)^{1+\frac{1}{\alpha}} s^{1+\alpha}\, ds = 0.$$

By simple calculation, we have

$$\rho(t)t^{1+\alpha} - \rho(t_1)t_1^{1+\alpha} - \int_{t_1}^{t} \rho(s)s^\alpha(1+\alpha)\, ds + \int_{t_1}^{t} p(s)s^{1+\alpha}\, ds + \int_{t_1}^{t} \alpha\rho(s)^{1+\frac{1}{\alpha}} s^{1+\alpha}\, ds = 0.$$

Hence, we obtain

$$t^\alpha \rho(t) = \frac{t_1^{1+\alpha}\rho(t_1)}{t} - \frac{1}{t}\int_{t_1}^{t} p(s)s^{1+\alpha}\, ds +$$

$$+ \frac{1}{t}\int_{t_1}^{t} s^\alpha \rho(s)[1 + \alpha - \alpha s(\rho(s))^{\frac{1}{\alpha}}]\, ds \quad \text{for} \quad t_1 < t < +\infty. \tag{10}$$

Using Lemma 1 with $X = 1, Y = s(\rho(s))^{\frac{1}{\alpha}}$, we have that

$$(1+\alpha)s^{\alpha}\rho(s) - \alpha s^{1+\alpha}(\rho(s))^{1+1/\alpha} \leq 1.$$

Hence, for $t_1 < t < +\infty$,

$$t^{\alpha}\rho(t) \leq \frac{t_1^{1+\alpha}\rho(t_1)}{t} - \frac{1}{t}\int_{t_1}^{t} p(s)s^{1+\alpha}\,ds + \frac{t-t_1}{t}.$$

Therefore, (9) and (10) imply that $r \geq p_*(0)$ and $R_1 \leq 1 - p_*(1+\alpha)$ respectively.

It is easily seen that for any $0 < \varepsilon < \min\{r, 1-R_1, R_1\}$ there exists $t_\varepsilon > t_1$ such that for $t_\varepsilon < t < +\infty$,

$$r - \varepsilon < t^{\alpha}\rho(t) < R_1 + \varepsilon,$$

$$t^{\alpha}\int_{t}^{+\infty} p(s)ds > p_*(0) - \varepsilon,$$

and

$$\frac{1}{t}\int_{t_1}^{t} s^{1+\alpha}p(s)ds > p_*(1+\alpha) - \varepsilon.$$

Taking into account the above argument, for $t_\varepsilon < t < +\infty$, from (9) and (10), we have that

$$t^{\alpha}\rho(t) > p_*(0) - \varepsilon + (r-\varepsilon)^{1+\frac{1}{\alpha}},$$

$$t^{\alpha}\rho(t) < \frac{t_\varepsilon^{1+\alpha}\rho(t_\varepsilon)}{t} - p_*(1+\alpha) + (R_1+\varepsilon)[(1+\alpha) - \alpha(r-\varepsilon)^{\frac{1}{\alpha}}].$$

Hence

$$r \geq p_*(0) + r^{1+\frac{1}{\alpha}}, \quad R_1 \leq -p_*(1+\alpha) + R_1[(1+\alpha) - \alpha r^{\frac{1}{\alpha}}].$$

By the definition of A in (8), we can get $r \geq A$, $R_1 \leq -p_*(1+\alpha) + R_1[(1+\alpha) - \alpha A^{\frac{1}{\alpha}}]$, thus $R_1 \leq B$, where B is defined by equalities (8). Hence (7) holds.

For the completion of the picture we give a proposition, which was proved by Kusano, Naito and Ogata in [9].

Proposition [9]. *If* $p_*(0) > \dfrac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}$, *then every solution of equation (1) is oscillatory.*

Theorem 1. *If* $p_*(1+\alpha) > \left(\dfrac{\alpha}{\alpha+1}\right)^{\alpha+1}$, *then every solution of equation (1) is oscillatory.*

Proof. Assume that equation (1) is nonoscillatory. From (10), we have

$$t^{\alpha}\rho(t) = \frac{t_1^{1+\alpha}\rho(t_1)}{t} - \frac{1}{t}\int_{t_1}^{t} p(s)s^{1+\alpha}\,ds + \frac{1}{t}\int_{t_1}^{t} s^{\alpha}\rho(s)\,ds + \frac{1}{t}\int_{t_1}^{t} s^{\alpha}\rho(s)[\alpha - \alpha s(\rho(s))^{\frac{1}{\alpha}}]\,ds,$$

we can get $s^{\alpha}\rho(s)[\alpha - \alpha s(\rho(s))^{\frac{1}{\alpha}}] \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$, so we obtain

$$t^{\alpha}\rho(t) < \frac{t_{\varepsilon}^{1+\alpha}\rho(t_{\varepsilon})}{t} - p_*(1+\alpha) + (R_1+\varepsilon)(t-t_1)/t + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{t-t_1}{t}\right).$$

If $t \to +\infty$, we get

$$p_*(1+\alpha) \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

which contradicts the condition of Theorem 1 and so the proof is complete.

Theorem 2. *Assume that* $p_*(0) \le \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}$. *If for some* $\lambda < \alpha$,

$$p^*(\lambda) > \frac{\lambda^{1+\alpha}}{(1+\alpha)^{1+\alpha}(\alpha-\lambda)} + B, \tag{11}$$

then equation (1) is oscillatory.

Proof. Assume the contrary. Let equation (1) be nonoscillatory. Then, according to Lemma 2, equation (4) has a solution $\rho : [t_0 + \infty) \to [0, +\infty)$ satisfying condition (5) – (7). Suppose $\lambda < \alpha$. Because of (7) we have that for any $\varepsilon > 0$ there exists $t_{\varepsilon} > t_0$ such that

$$t^{\alpha}\rho(t) < B + \varepsilon \quad \text{for} \quad t_{\varepsilon} < t < +\infty.$$

Multiplying equality (4) by t^{λ}, integrating it from t to $+\infty$, and taking into account (5) – (7), we get

$$\int_t^{+\infty} s^{\lambda}p(s)ds = -\int_t^{+\infty} s^{\lambda}\rho'(s)\,ds - \int_t^{+\infty} \alpha s^{\lambda}(\rho(s))^{1+\frac{1}{\alpha}}\,ds =$$

$$= t^{\lambda}\rho(t) + \int_t^{+\infty} \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} s^{\lambda-\alpha-1}\,ds -$$

$$- \int_t^{+\infty} \left(\alpha s^{\lambda}(\rho(s))^{1+\frac{1}{\alpha}} + \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} s^{\lambda-\alpha-1} - \lambda s^{\lambda-1}\rho(s)\right) ds.$$

Using Lemma 1 with

$$X = \frac{\lambda s^{\frac{\lambda}{1+\alpha}-1}}{1+\alpha}, \quad Y = s^{\frac{\lambda}{1+\alpha}}(\rho(s))^{\frac{1}{\alpha}},$$

we have that

$$\alpha s^{\lambda}(\rho(s))^{1+\frac{1}{\alpha}} + \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} s^{\lambda-\alpha-1} - \lambda s^{\lambda-1}\rho(s) \ge 0.$$

Hence,

$$\int\limits_t^{+\infty} s^\lambda p(s)\,ds \le t^{\lambda-\alpha}\left(t^\alpha\rho(t) + t^{\alpha-\lambda}\int\limits_t^{+\infty}\frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}}s^{\lambda-\alpha-1}ds\right) <$$

$$< t^{\lambda-\alpha}\left(B+\varepsilon+\frac{\lambda^{\alpha+1}}{(\alpha-\lambda)(\alpha+1)^{\alpha+1}}\right),$$

and so we have $p^*(\lambda) \le \dfrac{\lambda^{\alpha+1}}{(\alpha-\lambda)(\alpha+1)^{\alpha+1}} + B$, which contradicts inequality (11).

Theorem 2 is proved.

Theorem 3. *Assume that* $p_*(1+\alpha) \le \left(\dfrac{\alpha}{\alpha+1}\right)^{\alpha+1}$. *If for some* $\lambda > \alpha$

$$p^*(\lambda) > \frac{\lambda^{1+\alpha}}{(1+\alpha)^{1+\alpha}(\lambda-\alpha)} - A, \tag{12}$$

then equation (1) is oscillatory.

Proof. Assume the contrary. Let equation (1) be nonoscillatory. Then, according to Lemma 2, equation (4) has a solution $\rho : (t_0, +\infty) \to [0, +\infty)$ satisfying condition (5) and (7). Suppose $\lambda > \alpha$. Because of (7) we have that for any $\varepsilon > 0$ there exists $t_\varepsilon > t_0$ such that

$$t^\alpha\rho(t) > A-\varepsilon \quad \text{for} \quad t_\varepsilon < t < +\infty.$$

Multiplying equality (4) by t^λ, integrating it from t_ε to t, and taking into account (5) and (7), we get

$$\int\limits_{t_\varepsilon}^{t} s^\lambda p(s)\,ds \le t^{\lambda-\alpha}\left(-t^\alpha\rho(t) + t^{\alpha-\lambda}t_\varepsilon^\lambda\rho(t_\varepsilon) + t^{\alpha-\lambda}\int\limits_{t_\varepsilon}^{t}\frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}}s^{\lambda-\alpha-1}\,ds\right) <$$

$$< t^{\lambda-\alpha}\left(-A+\varepsilon+\frac{\lambda^{\alpha+1}}{(\lambda-\alpha)(\alpha+1)^{\alpha+1}}+\frac{t_\varepsilon^\lambda\rho(t_\varepsilon)}{t^{\lambda-\alpha}}\right),$$

hence we have $p^*(\lambda) \le \dfrac{\lambda^{\alpha+1}}{(\lambda-\alpha)(\alpha+1)^{\alpha+1}} - A$, which contradicts inequality (12).

Corollary 1. *Let either*

$$\lim_{\lambda\to\alpha-}(\alpha-\lambda)p^*(\lambda) > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \tag{13}$$

or

$$\lim_{\lambda\to\alpha+}(\lambda-\alpha)p^*(\lambda) > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}}. \tag{14}$$

Then equation (1) is oscillatory.

Corollary 2. *For some* $\lambda \neq \alpha$ *let*

$$|\lambda-\alpha|p_*(\lambda) > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}}. \tag{15}$$

Then equation (1) is oscillatory.

To convince ourselves that Corollary 1 is valid, let us note that (13) ((14)) implies

$$\lim_{\lambda\to\alpha-}\left\{(\alpha-\lambda)p^*(\lambda)-\frac{\lambda^{\alpha+1}}{(1+\alpha)^{1+\alpha}}-(\alpha-\lambda)B\right\}\geq 0$$

$$\left(\lim_{\lambda\to\alpha+}\left\{(\lambda-\alpha)p^*(\lambda)-\frac{\lambda^{\alpha+1}}{(1+\alpha)^{1+\alpha}}-(\lambda-\alpha)A\right\}\geq 0\right).$$

Consequently, (11) ((12)) is fulfilled for some $\lambda < \alpha$ ($\lambda > \alpha$). Thus, according to Theorems 2 and 3, respectively, equation (1) is oscillatory. As for Corollary 2, taking into account that the mapping $\lambda \mapsto (\alpha-\lambda)p_*$ for $\lambda < \alpha$ ($\lambda \mapsto (\lambda-\alpha)p_*$ for $\lambda > \alpha$) is nondecreasing (nonincreasing), we easily find from (15) that (13) ((14)) is fulfilled for some λ.

Theorem 4. *Assume that* $p_*(0) \leq \dfrac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$ *and* $p_*(1+\alpha) \leq \left(\dfrac{\alpha}{\alpha+1}\right)^{\alpha+1}$. *Moreover, let either*

$$p^*(\lambda) > \frac{B(\lambda-\alpha A^{\frac{1}{\alpha}})}{\alpha-\lambda}+B \tag{16}$$

for some $\lambda < \alpha$, *or*

$$p^*(\lambda) > \frac{B(\lambda-\alpha A^{\frac{1}{\alpha}})}{\alpha-\lambda}-A \tag{17}$$

for some $\lambda > \alpha$. *Then equation (1) is oscillatory.*

Proof. Assume the contrary. Let equation (1) be nonoscillatory. Then according to Lemma 2, equation (4) has a solution $\rho : (t_0, +\infty) \to [0, +\infty)$ satisfying conditions (5)–(7). On account of (7), (8), for any $0 < \varepsilon$ there exists $t_\varepsilon > t_0$ such that

$$A-\varepsilon < t^\alpha\rho(t) < B+\varepsilon \quad \text{for} \quad t_\varepsilon < t < +\infty.$$

Multiplying equality (4) by t^λ, integrating it from t to $+\infty$ (from t_ε to t), and taking into account (5)–(7), we easily find that

$$t^{\alpha-\lambda}\int_t^{+\infty} s^\lambda p(s)\,ds = t^\alpha\rho(t)+t^{\alpha-\lambda}\int_t^{+\infty} s^{\lambda-\alpha-1}s^\alpha\rho(s)\left(\lambda-\alpha s(\rho(s))^{\frac{1}{\alpha}}\right)ds <$$

$$< B+\varepsilon+\frac{(B+\varepsilon)[\lambda-\alpha(A-\varepsilon)^{\frac{1}{\alpha}}]}{\alpha-\lambda} \quad \text{for} \quad t_\varepsilon < t < +\infty$$

$$\left(t^{\alpha-\lambda} \int\limits_{t_\varepsilon}^{t} s^\lambda p(s)\,ds < \varepsilon - A + \frac{(B+\varepsilon)[\lambda - \alpha(A-\varepsilon)^{\frac{1}{\alpha}}]}{\lambda-\alpha} + t^{\alpha-\lambda} t_\varepsilon^\lambda \rho(t_\varepsilon) \text{ for } t_\varepsilon < t < +\infty \right).$$

This implies

$$p^*(\lambda) \le \frac{B(\lambda - \alpha A^{\frac{1}{\alpha}})}{\alpha-\lambda} + B$$

$$\left(p^*(\lambda) \le \frac{B(\lambda - \alpha A^{\frac{1}{\alpha}})}{\lambda-\alpha} - A \right),$$

which contradicts condition (16) ((17)).

Theorem 5. *Assume that* $p_*(0) \le \dfrac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$ *and* $p_*(1+\alpha) \le (\frac{\alpha}{1+\alpha})^{1+\alpha}$. *Moreover, let*

$$p_*(\lambda) > B - \frac{\alpha A^{1+\frac{1}{\alpha}}}{\alpha-\lambda} + \frac{\lambda B}{\alpha-\lambda} \tag{18}$$

for some $0 < \lambda < \alpha$, *or*

$$p_*(\lambda) > -A - \frac{\alpha A^{1+\frac{1}{\alpha}}}{\lambda-\alpha} + \frac{\lambda B}{\lambda-\alpha} \tag{19}$$

for some $0 < \alpha < \lambda$. *Then equation (1) is oscillatory.*

Proof. Assume the contrary. Let equation (1) be nonoscillatory. Then according to Lemma 2, equation (4) has a solution $\rho : (t_0, +\infty) \to [0, +\infty)$ satisfying conditions (5) – (7). Suppose $0 < \lambda < \alpha$ $(0 < \alpha < \lambda)$. Multiplying equation (4) by t^λ, integrating it from t to $+\infty$ (from t_ε to t), and taking into account (6), we easily obtain

$$\begin{gathered} t^\alpha \rho(t) = h_\lambda(t) - \lambda t^{\alpha-\lambda} \int\limits_t^{+\infty} s^{\lambda-1}\rho(s)\,ds + \\ + t^{\alpha-\lambda}\alpha \int\limits_t^{+\infty} s^\lambda (\rho(s))^{1+\frac{1}{\alpha}}\,ds \text{ for } t_0 < t < +\infty, \\ \Bigg(t^\alpha \rho(t) = -h_\lambda(t) + \lambda t^{\alpha-\lambda} \int\limits_{t_\varepsilon}^{t} s^{\lambda-1}\rho(s)\,ds - \\ - t^{\alpha-\lambda}\alpha \int\limits_{t_\varepsilon}^{t} s^\lambda (\rho(s))^{1+\frac{1}{\alpha}}\,ds + t_\varepsilon^\lambda \rho(t_\varepsilon) t^{\alpha-\lambda} \text{ for } t_\varepsilon < t < +\infty \Bigg), \end{gathered} \tag{20}$$

where $h_\lambda(t)$ is the function defined by (2).

On account of (7) we have $A > 0$. By (2), (7), (8), therefore for any $0 < \varepsilon < \min\{A, p_*(\lambda)\}$ there exists $t_\varepsilon > t_0$ such that

$$A - \varepsilon < t^\alpha \rho(t) < B + \varepsilon, \quad h_\lambda(t) > p_*(\lambda) - \varepsilon \quad \text{for} \quad t_\varepsilon < t < +\infty.$$

Owing to the above arguments, we find from (20) that

$$B + \varepsilon > p_*(\lambda) - \varepsilon - \frac{\lambda(B+\varepsilon)}{\alpha - \lambda} + \frac{\alpha}{\alpha - \lambda}(A - \varepsilon)^{1+\frac{1}{\alpha}}$$

$$\left(p_*(\lambda) - \varepsilon < -(A - \varepsilon) + \frac{\lambda(B+\varepsilon)}{\lambda - \alpha} - \frac{\alpha(A-\varepsilon)^{1+\frac{1}{\alpha}}}{\lambda - \alpha}\right)$$

which implies

$$p_*(\lambda) \le B - \frac{\alpha A^{1+\frac{1}{\alpha}}}{\alpha - \lambda} + \frac{\lambda B}{\alpha - \lambda}$$

$$\left(p_*(\lambda) \le -A - \frac{\alpha A^{1+\frac{1}{\alpha}}}{\lambda - \alpha} + \frac{\lambda B}{\lambda - \alpha}\right).$$

This contradicts (18) ((19)).

Theorem 5 is proved.

1. *Agarwal R. P., Grace S. R., O'Regan D.* Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations. — Kluwer Acad. Publ., 2002.
2. *Agarwal R. P., Grace S. R., O'Regan D.* Oscillation theory for second order dynamic equations. — Taylor & Francis, 2003.
3. *Dosly O., Rehak P.* Half-linear differential equations. — North-Holland, 2005.
4. *Hardy G. H., Littlewood J. E., Polya G.* Inequalities. — Second ed. — Cambridge: Cambridge Univ. Press, 1988.
5. *Hus H. B., Yeh C. C.* Oscillation theorems for second-order half-linear differential equations // Appl. Math. Lett. — 1996. — **9**, № 6. — P. 71–77.
6. *Hong H. L., Lian W. C., Yeh C. C.* The oscillation of half-linear differential equations with an oscillatory coefficient // Math. Comput. Modelling. — 1996. — **24**, № 7. — P. 77–86.
7. *Kandelaki N., Lomtatidze A., Ugulava D.* On oscillation and nonoscillation of a second order half-linear equation // Georg. Math. J. — 2000. — **7**, № 2. — P. 329–346.
8. *Kusano T., Norio Y.* Non-oscillation theorems for a class of quasilinear differential equations second order // J. Math. Anal. and Appl. — 1995. — **198**. — P. 115–127.
9. *Kusano T., Naito Y., Ogata A.* Strong oscillation and non-oscillation of quasilinear differential equations of second order // Different. Equat. Dynam. Systems. — 1994. — **2**. — P. 1–10.
10. *Manojlović J. V.* Oscillation criteria for second-order half-linear differential equations // Math. Comput. Modelling. — 1999. — **30**. — P. 109–119.
11. *Lomtatidze A.* Oscillation and non-oscillation criteria for second-order linear differential equations // Georg. Math. J. — 1997. — **4**, № 2. — P. 129–138.
12. *Li H. J., Yeh C. C.* Non-oscillatio criteria for second-order half-linear differential equations // Appl. Math. Lett. — 1995. — **8**, № 5. — P. 63–70.

13. *Li W. T.* Interval oscillation of second-order half-linear functional differential equations // Appl. Math. Comput. — 2004. — **155**, № 2. — P. 451–468.

14. *James S. W. Wong.* A non-oscillation theorem for Emden–Fowler equations // J. Math. Anal. and Appl. — 2002. — **274**, № 2. — P. 746–754.

15. *James S. W. Wong.* Second order nonlinear forced oscillations // SIAM J. Math. Anal. — 1988. — **19**. — P. 667–675.

16. *Wang Q. R.* Oscillation and asymptiotics for second-order half-linear differential equations // Appl. Math. Comput. — 2001. — **122**. — P. 253–266.

17. *Yang X. J.* Oscillation results for second-order half-linear differential equations // Math. Comput. Modelling. — 2002. — **36**, № 4–5. — P. 503–507.

18. *Yong Zhou.* Oscillation and non-oscillation criteria for second order quasilinear difference equations // J. Math. Anal. and Appl. — 2005. — **303**. — P. 365–375.

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