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EXISTENCE AND ATTRACTIVITY RESULTS FOR NONLINEAR FIRST ORDER RANDOM DIFFERENTIAL EQUATIONS

ІСНУВАННЯ ТА АТРАКТОРНІСТЬ РОЗВ'ЯЗКІВ НЕЛІНІЙНИХ СТОХАСТИЧНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ПЕРШОГО ПОРЯДКУ

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In this paper, the existence and attractivity results are proved for nonlinear first order ordinary random differential equations. An example is indicated to demonstrate a realization of the abstract theory developed in the present paper.

Викладено результати про існування та атракторність розв'язків нелінійних стохастичних диференціальних рівнянь першого порядку. Наведено приклад реалізації абстрактної теорії.

1. Introduction. Let $\mathbb{R}$ denote the real line and $\mathbb{R}_+$ the set of nonnegative real numbers, that is, $\mathbb{R}_+ = [0, \infty) \subset \mathbb{R}$. Let $C(\mathbb{R}_+, \mathbb{R})$ denote the class of real-valued functions defined and continuous on $\mathbb{R}_+$. Given a measurable space $(\Omega, \mathcal{A})$ and, for a measurable function $x : \Omega \to C(\mathbb{R}_+, \mathbb{R})$, consider the initial value problem of nonlinear first order ordinary random differential equations (in short RDE)

$$
\begin{gathered}
x'(t, \omega) + k\, x(t, \omega) = f(t, x(t, \omega), \omega) \quad \text{a.e.} \quad t \in \mathbb{R}_+, \\
x(0, \omega) = q(\omega)
\end{gathered}
\tag{1.1}
$$

for all $\omega \in \Omega$, where $k \in \mathbb{R}_+ \setminus \{0\}$, $q : \Omega \to \mathbb{R}$ and $f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$.

By a **random solution** of the RDE (1.1) we mean a measurable function $x : \Omega \to AC(\mathbb{R}_+, \mathbb{R})$ that satisfies the equations in (1.1), where $AC(\mathbb{R}_+, \mathbb{R})$ is the space of absolutely continuous real-valued functions defined on $\mathbb{R}_+$.

The initial value problems of ordinary differential equations have been studied in the literature on bounded as well as unbounded intervals of the real line for different aspects of the solutions. See, for example, Banas and Dhage [2], Burton and Zhang [3], Burton and Furumochi [4], Dhage [5] Hu and Yan [8], and the references therein. Similarly, the initial value problem of random differential equations have also been discussed in the literature for existence theorems on bounded intervals, however, the study of such random equations has not been made on unbounded intervals of the real line for any aspects of the random solutions. Some results along these lines appear in Itoh [9], Bharucha-Reid [1] and Dhage [6].

Therefore, the nonlinear random differential equations on unbounded intervals need to be considered for existence as well as for different characterizations of the random solutions. The present paper proposes to discuss the existence and attractivity results for random differential equations (1.1) on the right half $\mathbb{R}_+$ of the real line $\mathbb{R}$. The random fixed point theory, in particular, a random version of Schauder's fixed point theorem will be employed to prove the main result of this paper. Our results generalize the stability results of Burton and Furumochi [4] in some sense and we claim that the results of this paper are new to the literature on random differential equations.

2. Auxiliary results. Let E denote a Banach space with the norm $\|\cdot\|$ and let $Q : E \to E$. Then Q is called **compact** if $Q(E)$ is a relatively compact subset of E. Q is called **totally bounded** if $Q(B)$ is totally bounded subset of E for any bounded subset B of E. Q is called **completely continuous** if it is continuous and totally bounded on E. Note that every compact operator is totally bounded, but the the converse may not be true. However, both the notions coincide on bounded sets in the Banach space E.

We further assume that the Banach space E is separable, i.e., E has a countable dense subset and let β_E be the σ-algebra of Borel subsets of E. We say a mapping $x : \Omega \to E$ is measurable if for any $B \in \beta_E$,

$$x^{-1}(B) = \{\omega \in \Omega \mid x(\omega) \in B\} \in \mathcal{A}.$$

Note that a continuous map f from a Banach space E into itself is measurable, but the converse may not be true. Let $Q : \Omega \times E \to E$ be a mapping. Then Q is called a random operator if $Q(\omega, x)$ is measurable in ω for all $x \in E$ and it is expressed as $Q(\omega)x = Q(\omega, x)$. In this case we also say that $Q(\omega)$ is a random operator on E. A random operator $Q(\omega)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $Q(\omega, x)$ is continuous (resp. compact, totally bounded and completely continuous) in x for all $\omega \in \Omega$. Details completely continuous random operators on Banach spaces appear in Itoh [9]. We employ the following random fixed point theorem in proving the main result of this paper.

Theorem 2.1 [9]. *Let X be a non-empty, closed convex bounded subset of a separable Banach space E and let $Q : \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $Q(\omega)x = x$ has a random solution, i.e., there is a measurable function $\xi : \Omega \to X$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$.*

The following theorem is often times used in the study of nonlinear discontinuous random differential equations. We also need this result in the subsequent part of this paper.

Theorem 2.2 (Carathéodory). *Let $Q : \Omega \times E \to E$ be a mapping such that $Q(\cdot, x)$ is measurable for all $x \in E$ and $Q(\omega, \cdot)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \mapsto Q(\omega, x)$ is jointly measurable.*

3. Characterizations of random solutions. We seek the random solutions of RDE (1.1) in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ of real-valued functions defined, continuous and bounded on $\mathbb{R}_+$. We equip the space $BC(\mathbb{R}_+, \mathbb{R})$ with the supremum norm $\|\cdot\|$ defined by

$$\|x\| = \sup_{t \in \mathbb{R}_+} |x(t)|.$$

It is known that the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ is separable. By $L^1(\mathbb{R}_+, \mathbb{R})$ we denote the space of Lebesgue measurable real-valued functions defined on $\mathbb{R}_+$. By $\|\cdot\|_{L^1}$ we denote the usual norm in $L^1(\mathbb{R}_+, \mathbb{R})$ defined by

$$\|x\|_{L^1} = \int_0^\infty |x(t)|\, dt.$$

In order to introduce the further concepts used in this paper, let us denote $E = BC(\mathbb{R}_+, \mathbb{R})$ and let S be a non-empty subset of E. Let $Q : \Omega \times E \to E$ be a mapping and consider the following random equation:

$$Q(\omega)\, x(t,\omega) = x(t,\omega) \tag{3.1}$$

for $t \in \mathbb{R}_+$ and $\omega \in \Omega$. A measurable function $x : \Omega \to E$ is called a random solution of the random equation (3.1) if it satisfies (3.1) on $\mathbb{R}_+$. Below we give different characterizations of the random solutions for the random equation (3.1) on $\mathbb{R}_+$.

Definition 3.1. *We say that random solutions of the random equation (3.1) are locally attractive on $\mathbb{R}_+$ if there exists a closed ball $\overline{\mathcal{B}}_r(x_0)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$ for some $x_0 \in BC(\mathbb{R}_+, \mathbb{R})$ and for some real number $r > 0$, such that for arbitrary random solutions $x = x(t,\omega)$ and $y = y(t,\omega)$ of the random equation (3.1) belonging to $\overline{\mathcal{B}}_r(x_0) \cap S$ we have that*

$$\lim_{t\to\infty} (x(t,\omega) - y(t,\omega)) = 0 \tag{3.2}$$

for all $\omega \in \Omega$, where S is a non-empty subset of $BC(\mathbb{R}_+, \mathbb{R})$. In this case when the limit (3.2) is uniform with respect to the set $\overline{\mathcal{B}}_r(x_0) \cap S$, that is, when for each $\varepsilon > 0$ there exists a $T > 0$ such that for all $t \geq T$,

$$|x(t,\omega) - y(t,\omega)| \leq \varepsilon \tag{3.3}$$

for all $\omega \in \Omega$ and for all $x, y \in \overline{\mathcal{B}}_r(x_0) \cap S$ being the random solutions of (3.1), we will say that the random solutions are uniformly locally attractive on $\mathbb{R}_+$.

Definition 3.2. *We say that random solutions of the random equation (3.1) are globally attractive on $\mathbb{R}_+$ if for arbitrary random solutions $x = x(t,\omega)$ and $y = y(t,\omega)$ of the random equation (3.1) belonging to S the condition (3.2) is satisfied. In the case when (3.2) is satisfied uniformly with respect to the set S in E, that is, for $\varepsilon > 0$ there exists a $T > 0$ such that $t \geq T$, the inequality (3.3) holds for all $x, y \in S$ being the random solutions for the random equation (3.1), we will say that the random solutions of the random equation (3.1) are uniformly globally attractive on $\mathbb{R}_+$.*

Definition 3.3. *Let $c \in \mathbb{R}$ be fixed. A line $y(t,\omega) = c$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, is called an attractor for the random solution $x : \Omega \to E$ to the random equation (3.1) if $\lim_{t\to\infty}[x(t,\omega) - -c] = 0$ for all $\omega \in \Omega$. In this case, the random solution x to the random equation (3.1) is called asymptotic to the line $y = c$ and the line is called an asymptote for the random solution x on $\mathbb{R}_+$.*

Definition 3.4. *The random solutions for the random equation (3.1) are said to be locally asymptotically attractive if there exists an a closed ball $\overline{\mathcal{B}}_r(x_0)$ in E for some $x_0 \in E$ and for some real number $r > 0$ such that for any two random solutions $x = x(t,\omega)$ and $y = y(t,\omega)$ to the random equation (3.1) belonging to $\overline{\mathcal{B}}_r(x_0) \cap S$, there is a line which is a common attractor to them on $\mathbb{R}_+$. When x and y are uniformly locally attractive and there is a line as a common attractor, we will say that the random solutions of the random equation (3.1) are uniformly locally attractive on $\mathbb{R}_+$.*

Definition 3.5. *The random solutions for the random equation (3.1) are said to be globally asymptotically attractive if for any two globally attractive solutions x and y of (3.1) there is a line which is a common attractor to them on $\mathbb{R}_+$. Furthermore, if the random solutions for the* *random equation (3.1) are uniformly globally attractive, then they are called uniformly globally asymptotically attractive on* $\mathbb{R}_+$.

We note that the global attractivity and global asymptotic attractivity implies respectively the local attractivity and local asymptotic attractivity of the random solutions. The same is also true for uniformly globally attractivity and uniformly globally asymptotic attractivity of the random solutions for the random equation (3.1). However, the reverse implication may not hold.

4. Attractivity results. We need the following definition in the sequel.

Definition 4.1. *A function* $f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ *is called random Carathéodory if*
(i) the map $\omega \mapsto f(t, x, \omega)$ *is measurable for all* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}$;
(ii) the map $(t, x) \mapsto f(t, x, \omega)$ *is jointly continuous for all* $\omega \in \Omega$.
We consider the the following set of hypotheses in what follows.
(H_1) The function $q : \Omega \to \mathbb{R}$ is measurable and bounded. Moreover,

$$\operatorname*{ess\,sup}_{\omega\in\Omega} |q(\omega)| = c_1$$

for some real number $c_1 > 0$.
(H_2) The function $f(t, x, \omega)$ is random Carathéodory.
(H_3) There exists a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, x, \omega)| \le h(t) \quad \text{a.e.} \quad t \in \mathbb{R}_+$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}$. Moreover,

$$\lim_{t\to\infty} e^{-kt} \int\limits_0^t e^{ks} h(s)\, ds = 0.$$

Remark 4.1. If the hypothesis (H_3) holds, then the function $w : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $w(t) = e^{-kt} \int_0^t e^{ks} h(s)\, ds$ is continuous and the number

$$W = \sup_{t\ge 0} w(t) = \sup_{t\ge 0} e^{-kt} \int\limits_0^t e^{ks} h(s)\, ds$$

exists. Hypothesis (H_3) has been considered in a number of papers in the literature. See for example, Banas and Dhage [2], Burton and Furrumochi [4] and the references therein.
Our main result is the following theorem.

Theorem 4.1. *Assume that the hypotheses (H_1) through (H_3) hold. Then the RDE (1.1) admits a random solution. Moreover, random solutions are uniformly globally asymptotically attractive to the zero random solution on* $\mathbb{R}_+$.

Proof. Now RDE (1.1) is equivalent to the random equation

$$x(t,\omega) = q(\omega)\,e^{-kt} + e^{-kt}\int\limits_0^t e^{ks} f(s,x(s,\omega),\omega)\,ds \tag{4.1}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

Set $E = BC(\mathbb{R}_+,\mathbb{R})$ and, for every function $\omega \mapsto x(\omega) \in E$, define a mapping Q on $\Omega \times E$ by

$$Q(\omega)x(t,\omega) = q(\omega)\,e^{-kt} + e^{-kt}\int\limits_0^t e^{ks} f(s,x(s,\omega)\omega)\,ds \tag{4.2}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$. For the sake of convenience, we write $Q(\omega)x(t,\omega) = Q(\omega)x(t)$ for omitting the double appearance of ω and we merge it into $Q(\omega)$.

Clearly, Q defines a mapping $Q : \Omega \times E \to E$. To see this, let $x \in E$ be arbitrary. Then for each $\omega \in \Omega$, the continuity of the map $t \mapsto Q(\omega)x(t)$ follows from the fact that the exponential e^{-kt} and the indefinite integral $\int_0^t f(s,x(s,\omega),\omega)\,ds$ are continuous functions of t on $\mathbb{R}_+$. Next, we show that the function $Q(\omega)x : \mathbb{R}_+ \to \mathbb{R}$ is bounded for each $\omega \in \Omega$. Now by hypotheses (H_1) and (H_2),

$$|Q(\omega)x(t)| \le |q(\omega)|\,e^{-kt} + e^{-kt}\int\limits_0^t e^{ks}|f(s,x(s,\omega),\omega)|\,ds \le$$

$$\le c_1\,e^{-kt} + e^{-kt}\int\limits_0^t e^{ks}h(s)\,ds \le c_1 + W \tag{4.3}$$

for all $\omega \in \Omega$. As a result, $Q : \Omega \times E \to E$.

Define a closed ball $\overline{\mathcal{B}}_r(0)$ in the Banach space E centered at the origin of radius $r = c_1 + W$. From (4.3),

$$\|Q(\omega)x\| \le c_1 + W$$

for all $\omega \in \Omega$ and $x \in E$. Hence $Q : \Omega \times E \to \overline{\mathcal{B}}_r(0)$, and in particular, Q defines a map $Q : \Omega \times \overline{B}_r(0) \to \overline{\mathcal{B}}_r(0)$. Now we show that Q satisfies all the conditions of Theorem 2.1 with $X = \overline{\mathcal{B}}_r(0)$.

First we show that Q is a random operator on $\Omega \times \overline{\mathcal{B}}_r(0)$. By hypothesis (H_2), the map $\omega \mapsto f(t,x,\omega)$ is measurable by the Carathéodory theorem. Since a continuous function is measurable, the map $t \mapsto e^{kt}$ is measurable and so the product $e^{kt}f(t,x(t,\omega),\omega)$ is measurable in ω for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Since the integral is a limit of the finite sum of measurable functions, we have that the function $\omega \mapsto \displaystyle\int_0^t e^{ks} f(s,x(s,\omega),\omega)\,ds$ is measurable. Similarly, the mapping

$$\omega \mapsto q(\omega)\,e^{-kt} + e^{kt}\int\limits_0^t e^{ks} f(s,x(s,\omega),\omega)\,ds$$

is measurable for all $t \in \mathbb{R}_+$. Consequently, the map $\omega \mapsto Q(\omega)x$ is measurable for all $x \in E$ and that Q is a random operator on $\Omega \times \overline{\mathcal{B}}_r(0)$.

Secondly, we show that random operator $Q(\omega)$ is continuous on $\overline{\mathcal{B}}_r(0)$. Let $\omega \in \Omega$ be fixed. Since $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} e^{-kt}\int_0^t e^{ks}h(s)\,ds = 0$, there is a real number $T > 0$ such that $w(t) < \dfrac{\varepsilon}{4}$ for all $t \geq T$. We show continuity of the random operator $Q(\omega)$ in the following two cases:

Case I: Let $t \in [0,T]$ and let $\{x_n\}$ be a sequence of points in $\overline{\mathcal{B}}_r(0)$ such that $x_n \to x$ as $n \to \infty$. Then, by the dominated convergence theorem,

$$\lim_{n\to\infty} Q(\omega)x_n(t) = \lim_{n\to\infty}\left(q(\omega)\,e^{-kt} + e^{-kt}\int_0^t e^{ks}f(s,x_n(s,\omega),\omega)\,ds\right) =$$

$$= q(\omega)e^{-kt} + \lim_{n\to\infty}\left(e^{-kt}\int_0^t e^{ks}f(s,x_n(s,\omega),\omega)\,ds\right) =$$

$$= q(\omega)e^{-kt} + \left(e^{-kt}\int_0^t \lim_{n\to\infty}\left[e^{ks}f(s,x_n(s,\omega),\omega)\right]\,ds\right) =$$

$$= q(\omega)e^{-kt} + e^{-kt}\int_0^t e^{ks}f(s,x(s,\omega),\omega)\,ds = Q(\omega)x(t)$$

for all $t \in [0,T]$ and for each fixed $\omega \in \Omega$.

Case II: Suppose that $t \geq T$. Then we have

$$|Q(\omega)x_n(t) - Q(\omega)x(t)| = \left|e^{-kt}\int_0^t e^{ks}f(s,x_n(s,\omega),\omega)\,ds - e^{-kt}\int_0^t e^{ks}f(s,x(s,\omega),\omega)\,ds\right| \leq$$

$$\leq \left|e^{-kt}\int_0^t e^{ks}f(s,x_n(s,\omega),\omega)\,ds\right| + \left|e^{-kt}\int_0^t e^{ks}f(s,x(s,\omega),\omega)\,ds\right| \leq 2w(t) < \varepsilon$$

for all $t \geq T$ and for each fixed $\omega \in \Omega$. Since ε is arbitrary, one has $\lim_{n\to\infty} Q(\omega)x_n(t) = Q(\omega)x(t)$ for all $t \geq T$ and $\omega \in \Omega$. Now combining the Case I with Case II, we conclude that $Q(\omega)$ is a pointwise continuous random operator on $\overline{\mathcal{B}}_r(0)$ into itself. Further, it is shown below that the family of functions $\{Q(\omega)x_n\}$ is an equicontinuous set in E for a fixed $\omega \in \Omega$. Hence, the above convergence is uniform on $\mathbb{R}_+$ and consequently, $Q(\omega)$ is a continuous random operator on $\overline{\mathcal{B}}_r(0)$ into itself.

Next, we show that $Q(\omega)$ is a compact random operator on $\overline{\mathcal{B}}_r(0)$. Let $\omega \in \Omega$ be fixed and consider a sequence $\{Q(\omega)x_n\}$ of points in $\overline{\mathcal{B}}_r(0)$. To finish, it is enough to show that the sequence $\{Q(\omega)x_n\}$ has a Cauchy subsequence for each $\omega \in \Omega$. Clearly, $\{Q(\omega)x_n\}$ is a uniformly bounded subset of $\overline{\mathcal{B}}_r(0)$. We show that it is an equicontinuous sequence of functions on $\mathbb{R}_+$.

Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} w(t) = 0$, there exists a real number $T > 0$ such that $w(t) < \dfrac{\varepsilon}{4}$ for all $t \geq T$. We consider the following three cases in the sequel.

Case I: Let $t_1, t_2 \in [0, T]$. Then we have

$$|Q(\omega)x_n(t_1) - Q(\omega)x_n(t_2)| =$$

$$\leq \left| q(\omega)e^{-kt_1} + e^{-kt_1}\int_0^{t_1} e^{ks} f(s, x_n(s,\omega),\omega)\,ds - \right.$$

$$\left. -q(\omega)e^{-kt_2} - e^{-kt_2}\int_0^{t_2} e^{ks} f(s, x_n(s,\omega),\omega)\,ds \right| \leq$$

$$\leq |q(\omega)|\left|e^{-kt_1} - e^{-kt_2}\right| +$$

$$+\left|e^{-kt_1}\int_0^{t_1} e^{ks} f(s, x_n(s,\omega),\omega)\,ds - e^{-kt_2}\int_0^{t_1} e^{ks} f(s, x_n(s,\omega),\omega)\,ds\right| +$$

$$+\left|e^{-kt_2}\int_0^{t_1} e^{ks} f(s, x_n(s,\omega),\omega)\,ds - e^{-kt_2}\int_0^{t_2} e^{ks} f(s, x_n(s,\omega),\omega)\,ds\right| \leq$$

$$\leq |q(\omega)|\left|e^{-kt_1} - e^{-kt_2}\right| +$$

$$+\left|e^{-kt_1} - e^{-kt_2}\right|\left|\int_0^{t_1} e^{ks} f(s, x_n(s,\omega),\omega)\,ds\right| +$$

$$+\,e^{-kt_2}\left|\int_0^{t_1} e^{ks} f(s, x_n(s,\omega),\omega)\,ds - \int_0^{t_2} e^{ks} f(s, x_n(s,\omega),\omega)\,ds\right| \leq$$

$$\leq \left[|q(\omega)| + \int_0^{T} e^{ks} h(s)\,ds\right]\left|e^{-kt_1} - e^{-kt_2}\right| +$$

$$+\,e^{-kt_2}\left|\int_{t_2}^{t_1} e^{ks}|f(s, x_n(s,\omega),\omega)|\,ds\right| \leq$$

$$\leq \left[c_1 + e^{kT}\|h\|_{L^1}\right]\left|e^{-kt_1} - e^{-kt_2}\right| + |p(t_1) - p(t_2)|$$

for all $n \in \mathbb{N}$, where $p(t) = \int_0^t e^{ks} h(s)\, ds$. Since the functions e^{-kt} and $p(t)$ are continuous on $[0, T]$, they are uniformly continuous there. Hence,

$$|Q(\omega)x_n(t_1) - Q(\omega)x_n(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $t_1, t_2 \in [0, T]$ and for all $n \in \mathbb{N}$.

Case II: If $t_1, t_2 \in [T, \infty)$, then we have

$$|Q(\omega)x_n(t_1) - Q(\omega)x_n(t_2)| \le$$

$$\le \left| e^{-kt_1} \int_0^{t_1} e^{ks} f(s, x_n(s,\omega), \omega)\, ds - e^{-kt_2} \int_0^{t_2} e^{ks} f(s, x_n(s,\omega), \omega)\, ds \right| \le$$

$$\le e^{-kt_1} \int_0^{t_1} e^{ks} |f(s, x_n(s,\omega), \omega)|\, ds + e^{-kt_2} \int_0^{t_2} e^{ks} |f(s, x_n(s,\omega), \omega)|\, ds \le$$

$$\le w(t_1) + w(t_2) \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$

Since ε is arbitrary, one has

$$|Q(\omega)x_n(t_1) - Q(\omega)x_n(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$.

Case III: If $t_1 < T < t_2$, then

$$|Q(\omega)x_n(t_1) - Q(\omega)x_n(t_2)| \le |Q(\omega)x_n(t_1) - Q(\omega)x_n(T)| + |Q(\omega)x_n(T) - Q(\omega)x_n(t_2)|.$$

As $t_1 \to t_2$, $t_1 \to T$ and $t_2 \to T$, we have

$$|Q(\omega)x_n(t_1) - Q(\omega)x_n(T)| \to 0 \quad \text{and} \quad |Q(\omega)x_n(t_2) - Q(\omega)x_n(T)| \to 0$$

as $t_1 \to t_2$ uniformly for all $n \in \mathbb{N}$. Hence,

$$|Q(\omega)x_n(t_1) - Q(\omega)x_n(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $t_1 < T$, $t_2 > T$ and for all $n \in \mathbb{N}$. Thus, in all three cases,

$$|Q(\omega)x_n(t_1) - Q(\omega)x_n(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $t_1, t_2 \in \mathbb{R}_+$ and for all $n \in \mathbb{N}$.

This shows that $\{Q(\omega)x_n\}$ is an equicontinuous sequence in $\overline{\mathcal{B}}_r(0)$. Now an application of Arzela–Ascoli theorem yields that $\{Q(\omega)x_n\}$ has a uniformly convergent subsequence on the compact subset $[0, T]$ of $\mathbb{R}_+$. Without loss of generality, call the subsequence to be the sequence itself. We show that $\{Q(\omega)x_n\}$ is Cauchy in $\overline{\mathcal{B}}_r(0)$. Now $|Q(\omega)x_n(t) - Q(\omega)x(t)| \to 0$ as $n \to \infty$ for all $t \in [0,T]$. Then for given $\varepsilon > 0$ there exits an $n_0 \in \mathbb{N}$ such that

$$\sup_{0\le p\le T} e^{-kp} \int_0^p e^{ks} |f(s, x_m(s,\omega), \omega) - f(s, x_n(s,\omega), \omega)|\, ds < \frac{\varepsilon}{2}$$

for all $m, n \ge n_0$. Therefore, if $m, n \ge n_0$, then we have

$$\|Q(\omega)x_m - Q(\omega)x_n\| = \sup_{0\le p<\infty} \left| e^{-kp} \int_0^p e^{ks} |f(s, x_m(s,\omega), \omega) - f(s, x_n(s,\omega), \omega)|\, ds \right| \le$$

$$\le \sup_{0\le p\le T} \left| e^{-kp} \int_0^p e^{ks} |f(s, x_m(s,\omega), \omega) - f(s, x_n(s,\omega), \omega)|\, ds \right| +$$

$$+ \sup_{p\ge T} e^{-kp} \int_0^p e^{ks} [|f(s, x_m(s,\omega), \omega)| + |f(s, x_n(s,\omega), \omega)|]\, ds < \varepsilon.$$

This shows that $\{Q(\omega)x_n\} \subset Q(\omega)(\overline{\mathcal{B}}_r(0)) \subset \overline{\mathcal{B}}_r(0)$ is Cauchy. Since $\overline{\mathcal{B}}_r(0)$ is complete, $\{Q(\omega)x_n\}$ converges to a point in $\overline{\mathcal{B}}_r(0)$. As $Q(\omega)(\overline{\mathcal{B}}_r(0))$ is closed $\{Q(\omega)x_n\}$ converges to a point in $Q(\omega)(\overline{\mathcal{B}}_r(0))$. Hence $Q(\omega)(\overline{\mathcal{B}}_r(0))$ is relatively compact for each $\omega \in \Omega$ and consequently Q is a continuous and compact random operator on $\Omega \times \overline{\mathcal{B}}_r(0)$ into $\overline{\mathcal{B}}_r(0)$. Now an application of Theorem 2.1 to the operator $Q(\omega)$ on $\overline{\mathcal{B}}_r(0)$ yields that Q has a fixed point in $\overline{\mathcal{B}}_r(0)$ which further implies that the RDE (1.1) has a random solution on $\mathbb{R}_+$.

Next, we show that the solutions are uniformly attractive on $\mathbb{R}_+$. Let $x, y\Omega \to \overline{\mathcal{B}}_r(0)$ be any two random solutions to the RDE (1.1) on $\mathbb{R}_+$. Then, for each $\omega \in \Omega$,

$$|x(t,\omega) - y(t,\omega)| \le \left| e^{-kt} \int_0^t e^{ks} f(s, x(s,\omega), \omega)\, ds - e^{-kt} \int_0^t e^{ks} f(s, y(s,\omega), \omega)\, ds \right| \le$$

$$\le e^{-kt} \int_0^t e^{ks} |f(s, x(s,\omega), \omega)|\, ds + e^{-kt} \int_0^t e^{ks} |f(s, y(s,\omega), \omega)|\, ds \le 2w(t) \tag{4.4}$$

for all $t \in \mathbb{R}_+$. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number $T > 0$ such that $w(t) < \dfrac{\varepsilon}{2}$ for all $t \ge T$. Therefore, $|x(t,\omega) - y(t,\omega)| \le \varepsilon$ for all $t \ge T$ and for all $\omega \in \Omega$. Hence, all random solutions of the RDE (1.1) are uniformly globally attractive on $\mathbb{R}_+$.

Finally, we prove that random solutions are asymptotically attractive to the line $y = 0$ on $\Omega \times \mathbb{R}_+$. Let $x : \Omega \to C(\mathbb{R}_+, \mathbb{R})$ be a random solution of the RDE (1.1) on $\mathbb{R}_+$. Then, for each

$\omega \in \Omega$,

$$|x(t,\omega)| \le |q(\omega)|\, e^{-kt} + e^{-kt} \int_0^t e^{ks} |f(s, x(s,\omega), \omega)|\, ds \le$$

$$\le |q(\omega)|\, e^{-kt} + e^{-kt} \int_0^t e^{ks} h(s)\, ds \le c_1 e^{-kt} + w(t)$$

for all $\omega \in \Omega$. Taking the limit superior in the above inequality as t tends to ∞ yields

$$\limsup_{t\to\infty} |x(t,\omega)| \le c_1 \limsup_{t\to\infty} e^{-kt} + \limsup_{t\to\infty} w(t) = 0$$

and so, $\lim_{t\to\infty} |x(t,\omega)| = 0$ for all $\omega \in \Omega$. Therefore, for each $\varepsilon > 0$ there exists a real number $T > 0$ such that $|x(t\omega)| < \varepsilon$ for all $t \ge T$ and $\omega \in \Omega$. Hence all random solutions of the RDE (1.1) are uniformly globally asymptotically attractive to the zero random solution on $\mathbb{R}_+$.

Example 4.1. Let $\Omega = (-\infty, 0)$. Given a function $x : \Omega \to C(\mathbb{R}_+, \mathbb{R})$, consider the RDE

$$x'(t,\omega) + x(t,\omega) = \frac{e^{-t} \sin \omega t\, x(t,\omega)}{1 + |x(t,\omega)|}, \qquad x(0,\omega) = 1 \tag{4.5}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

Here, $q(\omega) = 1$ for all $\omega \in \Omega$ and $f(t,x,\omega) = \dfrac{e^{-t} \sin \omega t\, x}{1 + |x|}$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and $\omega \in$ $\in (-\infty, 0)$. Clearly, the function f is a random L^1-Carathéodory with growth function

$$h(t) = e^{-t} \ge \left| \frac{e^{-t} \sin \omega t\, x}{1 + |x|} \right| = |f(t,x,\omega)|.$$

Again, we have

$$\lim_{t\to\infty} e^{-t} \int_0^t e^{-s} e^{s}\, ds = \lim_{t\to\infty} e^{-t} \int_0^t ds = \lim_{t\to\infty} \frac{t}{e^t} = 0.$$

Thus, both the hypotheses (H_1) and (H_2) of Theorem 4.1 are satisfied and hence the RDE (4.5) has a random solution and all random solutions are uniformly globally asymptotically attractive to the zero random solution on $\mathbb{R}_+$.

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