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WEAKLY-NONLINEAR BOUNDARY VALUE PROBLEMS FOR IMPULSIVE SYSTEMS WITH SMALL CONCENTRATED DELAYS OF THE ARGUMENT

СЛАБКО НЕЛІНІЙНІ КРАЙОВІ ЗАДАЧІ ДЛЯ ІМПУЛЬСНИХ СИСТЕМ З МАЛИМИ ЗОСЕРЕДЖЕНИМИ ЗАГАЮВАННЯМИ АРГУМЕНТУ

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We consider a nonlinear boundary value problem for an impulsive system of ordinary differential equations with concentrated delay in the general case when the number of the boundary conditions does not coincide with the order of the system sufficiently. It is supposed that the corresponding boundary value problem without delay is linear and has an r-parametric family of solutions. Then the equation for the generating amplitudes is derived, and sufficient conditions for the existence and an iteration algorithm for the construction of a solution of the initial problem are obtained in the critical case of first order if the delay is sufficiently small.

Розглядається нелінійна крайова задача для імпульсних систем звичайних диференціальних рівнянь з зосередженими загаюваннями у загальному випадку, коли кількість крайових умов не співпадає з порядком системи. Припускається, що відповідна крайова задача без загаювання є лінійною і має r-параметричну сім'ю розв'язків. У цьому випадку отримано рівнянн для породжуючих амплітуд, а також достатні умови існування та ітераційний алгоритм побудови розв'язку задачі, що розглядається в критичному випадку першого порядку, якщо загаювання є достатньо малим.

1. Introduction. Impulsive differential equations [1] with delay describe models of real processes and phenomena where both a dependence on the past and momentary disturbances are present. For instance, the size of a given population may be normally described by a delay differential equation and, at certain moments, the number of individuals can be abruptly changed. The interaction of the impulsive perturbations and the delay make difficult a qualitative investigation of such equations. In particular, the solutions are not smooth at the moments of impulse effect shifted by the delay [2].

In the present paper we consider a nonlinear boundary value problem (BVP) for an

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impulsive system of ordinary differential equations with concentrated delay in the general case when the number of the boundary conditions, m, does not coincide with the order of the system n. Suppose that the corresponding BVP without delay is linear and has an r-parametric family of solutions. Then the equation for the generating amplitudes is derived, and sufficient conditions for the existence and an iteration algorithm for the construction of a solution of the initial problem are obtained in the critical case of first order if the delay is sufficiently small. Some remarks on the critical case of second order and on the noncritical case are made as well. The results of the present paper were reported at the 24-th Summer School "Application of Mathematics in Engineering" (Sozopol, Bulgaria, 1998). A very concise version appeared in its proceedings [3].

The periodic problem is an important particular case of a BVP of Fredholm type (the number of boundary conditions, m, equals the order of the system, n). The periodic problem for an impulsive system with a small constant delay in the noncritical case (under considerably more general assumptions) as well as in the critical cases of first and second order was considered, respectively, in [4, 5] (see also the monograph [6], §8), [7] and [8]. The periodic problem for impulsive systems (without delay) as well as more general BVP's for differential systems with delay (without impulses) and with impulses (without delay) in critical cases of first and second order were considered in several papers by the first author [9 – 12] (see also the monographs [13, 14]). In all these papers, the unperturbed system is assumed to be linear. In [15], as far as we know, for the first time the critical case for a nonlinear unperturbed system is treated.

2. Statement of the problem. Preliminary assertions. Consider the boundary value problem (BVP) for the impulsive differential system with retarded argument

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f(t) + H(t, x(t), y(t)), \quad t \in [a, b],\ t \neq t_i, \\ \Delta x(t_i) &= B_i x(t_i) + a_i + I_i(x(t_i), y(t_i)), \quad i = \overline{1, p}, \\ x(t) &= \varphi(t) \text{ for } t \in [a - \varepsilon_0, a], \\ \ell x &= \alpha + \varepsilon J(x, \varepsilon), \end{aligned} \tag{1}$$

where $x \in \Omega \subset \mathbf{R}^n$, $f : \mathbf{R} \mapsto \mathbf{R}^n$, $A : \mathbf{R} \mapsto \mathbf{R}^{n \times n}$, $y = (y^1, y^2, \ldots, y^k) \in \Omega^k \subset \mathbf{R}^{nk}$, $H(t, x, y) = \sum_{j=1}^{k} H_j(t, x, y)(x - y^j)$, $H_j : \mathbf{R} \times \Omega \times \Omega^k \mapsto \mathbf{R}^{n \times n}$, Ω is a domain in $\mathbf{R}^n$, $y^j(t) = x(t - \varepsilon\omega^j(t))$, $j = \overline{1, k}$, $\omega^j : [a, b] \mapsto [0, 1]$; $\Delta x(t_i) = x(t_i + 0) - x(t_i - 0)$ are impulses at moments t_i, and $a \equiv t_0 < t_1 < t_2 < \ldots < t_p < b$; $a_i \in \mathbf{R}^n$, $B_i \in \mathbf{R}^{n \times n}$, $I_i(x, y) = \sum_{j=1}^{k} I_{ij}(x, y)(x - y^j)$, $I_{ij} : \Omega \times \Omega^k \mapsto \mathbf{R}^{n \times n} (i = \overline{1, p}, j = \overline{1, k})$, $\varepsilon \in [0, \varepsilon_0)$ is a small parameter; $\alpha = \operatorname{col}[\alpha_1, \ldots, \alpha_m] \in \mathbf{R}^m$; $\ell = \operatorname{col}[\ell_1, \ldots, \ell_m]$ and $J(x, \varepsilon)$ are, respectively, a linear and nonlinear with respect to x m-dimensional functionals; the initial function $\varphi \in C[a - \varepsilon_0, a]$, ε_0 will be specified below.

Remark 1. A functional differential equation is represented in the form

$$\dot{z}(t) = (Fz)(t), \tag{2}$$

where F is a general functional differential operator. Typical representatives of the class of equations (2) are the equations with *concentrated* deviation of the argument

$$\begin{aligned} \dot{z}(t) &= F(t, z(h_1(t)), \dots, z(h_k(t))), \quad t \in [a,b], \\ z(t) &= \psi(t) \quad \text{if} \quad t \notin [a,b], \end{aligned} \tag{3}$$

where $h_j : [a,b] \mapsto \mathbf{R}$, $j = \overline{1,k}$, are given functions, and with *distributed* deviation of the argument

$$\begin{aligned} \dot{z}(t) &= F\left(t, \int_a^b z(s)\, d_s R(t,s)\right), \\ z(t) &= \psi(t) \quad \text{if} \quad t \notin [a,b]. \end{aligned}$$

If in (3) $h_j(t) \le t, j = \overline{1,k}$, then it is a system with *concentrated delay*.

Remark 2. We could replace $H,\ I_i,\ \varepsilon J$ by nonlinearities of a more general form, vanishing for $\varepsilon = 0$.

Remark 3. In the Russian-language literature BVP's with $m \neq n (m = n)$ are usually called BVP's of *Noether* (respectively *Fredholm*) type. Here we use the second term but not the first one. Note that in the non-Russian literature *Fredholm BVP's* are such that m does not *necessarily* coincide with n; for $m = n$ we have *Fredholm BVP's of zero index*.

As usual in the theory of the impulsive differential equations, at the points of discontinuity t_i of the solution $x(t)$ we assume that $x(t_i) \equiv x(t_i - 0)$. It is clear that, in general, the derivatives $\dot{x}(t_i)$ do not exist. However, there do exist the limits $\dot{x}(t_i \pm 0)$. According to the above convention, we assume $\dot{x}(t_i) \equiv \dot{x}(t_i - 0)$. However, we assume $x(a) \equiv x(a+0)$. In general, $x(a) \neq x(a-0) = \varphi(a)$. The nonlinearity $H(t,x,y)$ is discontinuous at the points t that are solutions of the equations

$$t - \varepsilon \omega^j(t) = t_i, \quad i = \overline{0,p},\ j = \overline{1,k}. \tag{4}$$

We require the continuity of the solution $x(t)$ at such points if they are distinct from the moments of impulse effect t_i or a.

For the sake of brevity, if not stated otherwise, we shall use the notation $x_i = x(t_i), y_i = y(t_i)$ (i.e. $y_i^j = x(t_i - \varepsilon\omega^j(t_i))$).

Introduce the following conditions:

H$_1$. The components of $A(t),\ f(t)$ belong to the space $C([a,b] \setminus \{t_i\})$ of functions which are continuous or piecewise continuous, with discontinuities of the first kind at the points t_i.

H$_2$. The functions $H_j(t,x,y)$ are continuously differentiable with respect to $x,\ y$, and their components belong to $C([a,b] \setminus \{t_i\})$ with respect to t.

H$_3$. The functions $I_{ij}(x,y) \in C^1(\Omega \times \Omega^k, \mathbf{R}^{n \times n}),\ i = \overline{1,p},\ j = \overline{1,k}$.

H$_4$. The functions ω^j are Lipschitz continuous:

$$|\omega^j(t') - \omega^j(t'')| \le K|t' - t''|,\ j = \overline{1,k},\ t', t'' \in [a,b].$$

H$_5$. The matrices $E + B_i,\ i = \overline{1,p}$, are nonsingular ($E$ is the unit matrix).

H$_6$. The functional ℓ is bounded on the space $C([a,b] \setminus \{t_i\})$.

H$_7$. The functional $J(x,\varepsilon)$ is Fréchet continuously differentiable with respect to x, belongs to $C([a,b] \setminus \{t_i\})$, and is continuous with respect to ε.

We assume that

$$\varepsilon_0 = \min\left\{t_{i+1} - t_i,\, i = \overline{0, p-1},\, b - t_p,\, 1/K\right\}.$$

Then for $\varepsilon \in (0, \varepsilon_0)$ each equation (4) has a unique solution $t_i^j = t_i^j(\varepsilon)$. It obviously satisfies $t_i < t_i^j < t_i + \varepsilon$.

Together with (1) we consider the so called *generating* system,

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f(t), \quad t \in [a, b],\ t \neq t_i, \\ \Delta x(t_i) &= B_i x_i + a_i, \quad i = \overline{1, p}, \end{aligned} \tag{5}$$

$$\ell x = \alpha \tag{6}$$

obtained from (1) for $\varepsilon = 0$.

The general solution of (5) is given by

$$x(t, c) = X(t)\left(c + \int\limits_a^t X^{-1}(\tau)f(\tau)\,d\tau + \sum_{a<t_i<t} X_i^{-1}a_i\right), \tag{7}$$

$c = x(0, c) \in \mathbf{R}^n$, where $X(t)$ is the fundamental solution (i.e. $X(a) = E$) of the homogeneous system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \quad t \in [a, b],\ t \neq t_i, \\ \Delta x(t_i) &= B_i x_i, \quad i = \overline{1, p}, \end{aligned} \tag{8}$$

and $X_i \equiv X(t_i + 0) = (E + B_i)X(t_i)$ again for the sake of brevity.

If we introduce Green's function $K(t, \tau)$ for the Cauchy problem related to (5):

$$K(t, \tau) = \begin{cases} X(t)X^{-1}(\tau), & a \le \tau \le t \le b; \\ 0, & a \le t < \tau \le b, \end{cases}$$

then (7) takes the form

$$x(t, c) = X(t)c + \int\limits_a^b K(t, \tau)f(\tau)\,d\tau + \sum_{i=1}^p K(t, t_i + 0)a_i. \tag{9}$$

A solution of the form (9) satisfies the boundary condition (6) if and only if the initial condition c satisfies

$$\alpha = \ell X(\cdot)c + \ell\int\limits_a^b K(\cdot, \tau)f(\tau)\,d\tau + \sum_{i=1}^p \ell K(\cdot, t_i + 0)a_i. \tag{10}$$

Denote $Q = \ell X(\cdot)$, which is an $(m \times n)$-matrix, let Q^* be its transpose, Q^+ its unique Moore – Penrose pseudoinverse $(n \times m)$-matrix [16, 17]. Denote by $\mathcal{P} \equiv \mathcal{P}_Q$ the orthoprojector $\mathbf{R}^n \mapsto$ $\mapsto \text{Ker}\,(Q)$ and by $\mathcal{P}^* \equiv \mathcal{P}_{Q^*}$ the orthoprojector $\mathbf{R}^m \mapsto \text{Ker}\,(Q^*)$.

For $n_1 = \mathrm{Rank}\ Q \le \min\ (m, n)$, let $r = n - n_1, d = m - n_1$. Denote by $\mathcal{P}_r$ an $(n \times r)$-matrix whose columns are r linearly independent columns of $\mathcal{P}$ and, similarly, let $\mathcal{P}_d^*$ be a $(d \times m)$-matrix whose rows are d linearly independent rows of $\mathcal{P}^*$.

Then the necessary and sufficient condition for solvability of the algebraic system (10) is

$$\mathcal{P}_d^* \left(\alpha - \ell \int_a^b K(\cdot, \tau) f(\tau)\, d\tau - \sum_{i=1}^p \ell K(\cdot, t_i + 0) a_i \right) = 0. \tag{11}$$

If (11) is satisfied, system (10) has an r-parametric family of solutions

$$c = Q^+ \left(\alpha - \ell \int_a^b K(\cdot, \tau) f(\tau)\, d\tau - \sum_{i=1}^p \ell K(\cdot, t_i + 0) a_i \right) + \mathcal{P}\widetilde{c}, \quad \widetilde{c} \in \mathbf{R}^n. \tag{12}$$

If we substitute (12) into (9), we find an r-parametric family of solutions of BVP (5), (6) which can be represented in the form

$$x_0(t, c_r) = X_r(t) c_r + X(t) Q^+ \alpha + (\Gamma f)(t) + \sum_{i=1}^p \gamma_i(t) a_i, \tag{13}$$

where $X_r(t) = X(t)\mathcal{P}_r$ is an $(n \times r)$-matrix whose columns make a complete system of r linearly independent solutions of (8) satisfying $\ell x = 0$, $c_r \in \mathbf{R}^r$ is an arbitrary vector;

$$(\Gamma f)(t) = \int_a^b K(t, \tau) f(\tau) d\tau - X(t) Q^+ \ell \int_a^b K(\cdot, \tau) f(\tau)\, d\tau,$$

$$\gamma_i(t) = K(t, t_i + 0) - X(t) Q^+ \ell K(\cdot, t_i + 0), \quad i = \overline{1, p}.$$

Thus the following assertion is valid.

Lemma 1([see [11], Theorem 1). *Let conditions H_1, H_5, H_6 hold and* $\mathrm{Rank}\, Q = n_1$. *Then system (8) has just $r = n - n_1$ linearly independent solutions satisfying $\ell x = 0$. The nonhomogeneous BVP (5), (6) has solutions if and only if the nonhomogeneities $f(t)$, a_i, and α satisfy condition (11). Then BVP (5), (6) has an r-parametric family of solutions of the form (13).*

Remark 4. Suppose that the linear functional ℓ satisfies the equality

$$\ell \int_a^b K(\cdot, \tau) f(\tau)\, d\tau = \int_a^b \ell K(\cdot, \tau) f(\tau)\, d\tau$$

for any $f \in C([a, b] \setminus \{t_i\})$. Then equality (13) can be written in the form

$$x_0(t, c_r) = X_r(t) c_r + X(t) Q^+ \alpha + \int_a^b G(t, \tau) f(\tau)\, d\tau + \sum_{i=1}^p G(t, t_i + 0) a_i,$$

where $G(t,\tau)$ is the generalized Green's matrix

$$G(t,\tau) \equiv K(t,\tau) - X(t)Q^{+}\ell K(\cdot,\tau).$$

3. Main result. Here we consider the so called *critical case* when $r > 0$. For the *noncritical case* ($r = 0$) see Section 4.

3.1. Equation for the generating amplitudes. Let us find conditions for the existence of a solution $x(t,\varepsilon)$ of BVP (1) belonging to the space $C([a,b] \setminus \{t_i\})$ as a function of t, depending continuously on ε and such that, for some $c_r \in \mathbf{R}^r$, we have $x(t,0) = x_0(t,c_r)$. A necessary condition for the existence of such solutions is given by the following theorem.

Theorem 1. *Let BVP (1) satisfy conditions H_1 – H_7 and (11) have a solution $x(t,\varepsilon)$ which, for $\varepsilon = 0$ becomes a generating solution $x_0(t,c_r^*)$. Then the vector $c_r^* \in \mathbf{R}^r$ satisfies the equation*

$$\begin{aligned} F(c_r^*) \;&\equiv\; \mathcal{P}_d^* \{ J(x_0(\cdot,c_r^*),0) - \\ &-\; \ell \int\limits_a^b K(\cdot,\tau)\mathcal{H}(\tau,x_0(\tau,c_r^*))\,(A(\tau)x_0(\tau,c_r^*) + f(\tau))\,d\tau - \\ &-\; \ell K(\cdot,a)\mathbf{H}_0(x_0(a,c_r^*))(x_0(a,c_r^*) - \varphi(a)) - \\ &-\; \sum_{i=1}^{p} \ell K(\cdot,t_i+0)\,[\mathbf{J}_i(x_0(t_i,c_r^*))\,(A_i x_0(t_i,c_r^*) + f_i) + \\ &+\; \mathbf{H}_i(x_0(t_i,c_r^*))\,(B_i x_0(t_i,c_r^*) + a_i)]\} = 0, \end{aligned} \tag{14}$$

where $\mathcal{H}(\tau,x)$, $\mathbf{J}_i(x)$, $i=\overline{1,p}$, $\mathbf{H}_i(x)$, $i=\overline{0,p}$, are given below.

***Proof*.** In (1) we change the variables according to the formula

$$x(t,\varepsilon) = x_0(t,c_r^*) + z(t,\varepsilon). \tag{15}$$

This leads to the problem of finding a solution $z = z(t,\varepsilon)$ for the impulsive system of differential equations with retarded argument

$$\begin{aligned} \dot z &= A(t)z + H(t,x(t,\varepsilon),y(t,\varepsilon)), \\ \Delta z(t_i) &= B_i z_i + I_i(x_i,y_i), \quad i=\overline{1,p}, \end{aligned} \tag{16}$$

with a boundary condition $\ell z = \varepsilon J(x,\varepsilon)$, initial condition $x(t) = \varphi(t)$ for $t < a$, such that it would belong to the space $C([a,b] \setminus \{t_i\})$ as a function of t, depend continuously on ε, and that $z(t,\varepsilon) \to 0$ as $\varepsilon \to 0$.

We can formally consider (16) as a nonhomogeneous system of the form (5, 6). Then the solvability condition (11) becomes

$$\begin{aligned} &\mathcal{P}_d^* \,(\varepsilon J(x(\cdot),\varepsilon) - \ell \int\limits_a^b K(\cdot,\tau)H(\tau,x(\tau),y(\tau))\,d\tau - \\ &\qquad -\sum_{i=1}^{p} \ell K(\cdot,t_i+0)\;I_i(x_i,y_i)) = 0. \end{aligned} \tag{17}$$

For the sake of later convenience we shall denote by $\eta(\varepsilon, x)$ expressions tending to 0 as $\varepsilon \to 0$, and satisfying a Lipschitz condition with respect to x with a constant tending to 0 as $\varepsilon \to 0$. We shall sometimes write $\eta(\varepsilon)$ instead of $\eta(\varepsilon, x)$, and $x(t)$ instead of $x(t, \varepsilon)$, if this wouldn't lead to misunderstanding. Thus, for instance, we sometimes write x_i instead of $x(t_i, \varepsilon)$, etc.

Since the left-hand side of equality (17) tends to 0 as $\varepsilon \to 0$, we first divide it by ε and then study its behaviour as $\varepsilon \to 0$. First we notice that

$$(x_i - y_i^j)/\varepsilon = \left(x(t_i) - x(t_i - \varepsilon\omega^j(t_i))\right)/\varepsilon = \omega^j(t_i)\dot{x}(t_i) + \eta(\varepsilon) =$$

$$= \omega^j(t_i)(A_i x_i + f_i) + \eta(\varepsilon)$$

since the interval $(t_i - \varepsilon\omega^j(t_i), t_i)$ contains no points of discontinuity of the function $x(t, \varepsilon)$ or its derivative. Thus

$$I_i(x_i, y_i)/\varepsilon = \sum_{j=1}^{k} I_{ij}(x_i, y_i)(x_i - y_i^j)/\varepsilon = \sum_{j=1}^{k} I_{ij}(x_i, x_i, \ldots, x_i)\omega^j(t_i)(A_i x_i + f_i) + \eta(\varepsilon).$$

Let us denote

$$\mathbf{J}_i(x) = \sum_{j=1}^{k} I_{ij}(\underbrace{x, x, \ldots, x}_{k+1})\omega^j(t_i). \tag{18}$$

Then

$$I_i(x_i, y_i)/\varepsilon = \mathbf{J}_i(x_i)(A_i x_i + f_i) + \eta(\varepsilon). \tag{19}$$

We can represent the integral $\int\limits_a^b$ in (17) by a sum of integrals over intervals containing no points of discontinuity of the integrand. It is obvious that for $\tau \in (t_i, t_i + \varepsilon)$ (more precisely, for $\tau = t_i^j(\varepsilon)$), the interval $(\tau - \varepsilon\omega^j(\tau), \tau)$ contains the point of discontinuity $t_i,\ i = \overline{0, p}$, while for τ inside the remaining intervals, the interval $(\tau - \varepsilon, \tau)$ contains no such points. We denote $\Delta_1^\varepsilon = \bigcup\limits_{i=0}^{p}(t_i, t_i + \varepsilon)$, $\Delta_2^\varepsilon = [a, b] \setminus \Delta_1^\varepsilon$ and make use of the representation $\int\limits_a^b = \int\limits_{\Delta_1^\varepsilon} + \int\limits_{\Delta_2^\varepsilon}$.

We first begin with the "good" set Δ_2^ε. We have

$$\int\limits_{\Delta_2^\varepsilon} K(t, \tau)H(\tau, x(\tau), y(\tau))\, d\tau/\varepsilon =$$

$$= \int\limits_{\Delta_2^\varepsilon} K(t, \tau) \sum_{j=1}^{k} H_j(\tau, x(\tau), y(\tau)) \left(x(\tau) - x(\tau - \varepsilon\omega^j(\tau))\right) d\tau/\varepsilon =$$

$$= \int\limits_{\Delta_2^\varepsilon} K(t, \tau) \sum_{j=1}^{k} H_j(\tau, x(\tau), y(\tau))\omega^j(\tau)\dot{x}(\tau)\, d\tau + \eta(\varepsilon).$$

Denote

$$\mathcal{H}(\tau, x) = \sum_{j=1}^{k} H_j(\tau, \underbrace{x, x, \ldots, x}_{k+1})\omega^j(\tau). \tag{20}$$

Thus

$$\sum_{j=1}^{k} H_j(\tau, x(\tau), y(\tau))\omega^j(\tau) = \mathcal{H}(\tau, x(\tau)) + \eta(\varepsilon)$$

and

$$\int\limits_{\Delta_2^\varepsilon} K(t,\tau)H(\tau, x(\tau), y(\tau))\, d\tau/\varepsilon = \int\limits_{\Delta_2^\varepsilon} K(t,\tau)\mathcal{H}(\tau, x(\tau))(A(\tau)x(\tau) + f(\tau))\, d\tau + \eta(\varepsilon) =$$

$$= \int\limits_a^b K(t,\tau)\mathcal{H}(\tau, x(\tau))(A(\tau)x(\tau) + f(\tau))\, d\tau -$$

$$- \int\limits_{\Delta_1^\varepsilon} K(t,\tau)\mathcal{H}(\tau, x(\tau))(A(\tau)x(\tau) + f(\tau)) d\tau + \eta(\varepsilon) =$$

$$= \int\limits_a^b K(t,\tau)\mathcal{H}(\tau, x(\tau))(A(\tau)x(\tau) + f(\tau))\, d\tau + \eta(\varepsilon).$$

Next we estimate any of the integrals

$$\int\limits_{t_i}^{t_i+\varepsilon} K(t,\tau)H(\tau, x(\tau), y(\tau))\, d\tau/\varepsilon,$$

since $\int\limits_{\Delta_1^\varepsilon} = \sum_{i=0}^{p} \int\limits_{t_i}^{t_i+\varepsilon}$. First let $i \in \{1, \ldots, p\}$. We suppose that

$$t_i < t_i^1 < t_i^2 < \cdots < t_i^k < t_i + \varepsilon. \tag{21}$$

By definition, $t_i^j = t_i + \varepsilon\omega^j(t_i^j)$. By the Lipschitz continuity of $\omega^j(t)$ we have

$$|\omega^j(t_i^j) - \omega^j(t_i)| \le K|t_i^j - t_i| = \varepsilon K|\omega^j(t_i^j)| \le \varepsilon K,$$

thus

$$t_i^j = t_i + \varepsilon\omega^j(t_i) + \varepsilon\eta(\varepsilon). \tag{22}$$

It is easily seen that for $\varepsilon > 0$ small enough, the strict inequalities

$$\omega^1(t_i) < \omega^2(t_i) < \cdots < \omega^k(t_i) \tag{23}$$

imply (21).

Now we have

$$\int\limits_{t_i}^{t_i+\varepsilon} = \int\limits_{t_i}^{t_i^1} + \sum_{j=1}^{k-1} \int\limits_{t_i^j}^{t_i^{j+1}} + \int\limits_{t_i^k}^{t_i+\varepsilon}$$

and consider successively the integrals in the right-hand side. First

$$\int\limits_{t_i}^{t_i^1} K(t,\tau)H(\tau, x(\tau), y(\tau))\, d\tau = (t_i^1 - t_i)K(t, t_i^{1*})H(t_i^{1*}, x(t_i^{1*}), y(t_i^{1*})),$$

where $t_i < t_i^{1*} < t_i^1$. In view of (22) for $i = 1$ we have

$$(t_i^1 - t_i)/\varepsilon = \omega^1(t_i) + \eta(\varepsilon).$$

Next,

$$\begin{aligned} K(t, t_i^{1*}) &= K(t, t_i + 0) + \eta(\varepsilon), \\ x(t_i^{1*}) &= x(t_i + 0) + \eta(\varepsilon), \quad y^j(t_i^{1*}) = x_i + \eta(\varepsilon), \quad j = \overline{1,k}. \end{aligned}$$

Thus

$$\begin{aligned} &\int\limits_{t_i}^{t_i^1} K(t,\tau)H(\tau, x(\tau), y(\tau))\, d\tau/\varepsilon = \\ &\quad = \omega^1(t_i)K(t, t_i + 0) \sum_{j=1}^{k} H_j(t_i + 0, \underbrace{x(t_i + 0)}_{1}, \underbrace{x_i, \ldots, x_i}_{k})\Delta x(t_i) + \eta(\varepsilon) = \\ &\quad = \omega^1(t_i)K(t, t_i + 0) \sum_{j=1}^{k} H_j(t_i + 0, \underbrace{(E + B_i)x_i + a_i}_{1}, \underbrace{x_i}_{k})(B_i x_i + a_i) + \eta(\varepsilon). \end{aligned}$$

In the next interval we have

$$\int\limits_{t_i^1}^{t_i^2} K(t,\tau)H(\tau, x(\tau), y(\tau))\, d\tau = (t_i^2 - t_i^1)K(t, t_i^{2*})H(t_i^{2*}, x(t_i^{2*}), y(t_i^{2*})),$$

where $t_i^1 < t_i^{2*} < t_i^2$. According to (22) and for $j = 1, 2$ we have

$$(t_i^2 - t_i^1)/\varepsilon = \omega^2(t_i) - \omega^1(t_i) + \eta(\varepsilon).$$

Next,

$$\begin{aligned} K(t, t_i^{2*}) &= K(t, t_i + 0) + \eta(\varepsilon), \\ x(t_i^{2*}) &= x(t_i + 0) + \eta(\varepsilon), \quad y^1(t_i^{2*}) = x(t_i + 0) + \eta(\varepsilon), \\ y^j(t_i^{2*}) &= x_i + \eta(\varepsilon), \quad j = \overline{2,k}. \end{aligned}$$

Thus

$$\int_{t_i^1}^{t_i^2} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon = (\omega^2(t_i)-\omega^1(t_i))K(t,t_i+0)\ \times$$

$$\times\ \sum_{j=2}^{k} H_j(t_i+0,\underbrace{x(t_i+0),x(t_i+0)}_{2},\underbrace{x_i,\dots,x_i}_{k-1})\Delta x(t_i)+\eta(\varepsilon) =$$

$$= (\omega^2(t_i)-\omega^1(t_i))K(t,t_i+0)\sum_{j=2}^{k} H_j(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{2},\underbrace{x_i}_{k-1})(B_ix_i+a_i)+\eta(\varepsilon).$$

Similarly we find

$$\int_{t_i^2}^{t_i^3} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon =$$

$$= (\omega^3(t_i)-\omega^2(t_i))K(t,t_i+0)\sum_{j=3}^{k} H_j(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{3},\underbrace{x_i}_{k-2})(B_ix_i+a_i)+\eta(\varepsilon).$$

. . . .

$$\int_{t_i^{k-1}}^{t_i^k} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon =$$

$$= (\omega^k(t_i)-\omega^{k-1}(t_i))K(t,t_i+0)H_k(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{k},\underbrace{x_i}_{1})(B_ix_i+a_i)+\eta(\varepsilon),$$

$$\int_{t_i^k}^{t_i+\varepsilon} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon = \eta(\varepsilon).$$

Introduce the notation

$$\begin{aligned}\mathbf{H}_i(x_i) \ =\ & \omega^1(t_i)\sum_{j=1}^{k} H_j(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{1},\underbrace{x_i}_{k})+\\ &+(\omega^2(t_i)-\omega^1(t_i))\sum_{j=2}^{k} H_j(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{2},\underbrace{x_i}_{k-1})+\\ &+(\omega^3(t_i)-\omega^2(t_i))\sum_{j=3}^{k} H_j(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{3},\underbrace{x_i}_{k-2})\ +\ \cdots\\ &\cdots\ +\ (\omega^k(t_i)-\omega^{k-1}(t_i))H_k(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{k},\underbrace{x_i}_{1}).\end{aligned} \tag{24}$$

Now

$$\int\limits_{t_i}^{t_i+\varepsilon} K(t,\tau)H(\tau,x(\tau),y(\tau))d\tau/\varepsilon = K(t,t_i+0)\mathbf{H}_i(x_i)(B_ix_i+a_i)+\eta(\varepsilon). \tag{25}$$

Remark 5. Equality (25) involves the expression (24) which essentially depends on the assumption (21) which is implied by (23) for ε small enough. Suppose that

$$\omega^j(t_i) = \omega^{j+1}(t_i)$$

for some j. Then the difference $t_i^{j+1} - t_i^j$ can have an arbitrary sign or vanish for any ε small enough. However, in view of $(t_i^{j+1} - t_i^j)/\varepsilon = \eta(\varepsilon)$, and hence

$$\int\limits_{t_i^j}^{t_i^{j+1}} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon = \eta(\varepsilon),$$

equality (25) still holds. Thus it is valid for the ordering

$$\omega^1(t_i) \le \omega^2(t_i) \le \cdots \le \omega^k(t_i). \tag{26}$$

For any other ordering different from (26), we can replace the expression (24) by a corresponding one. Unfortunately, we were not able to write equality (25) in terms of an expression independent of the ordering (26).

Now let $i=0$. Under the assumption (26) for $i=0$, as above, we obtain (25) with $i=0$:

$$\int\limits_{a}^{a+\varepsilon} K(t,\tau)H(\tau,x(\tau),y(\tau))d\tau/\varepsilon = K(t,a)\mathbf{H}_0(x(a))(x(a)-\varphi(a))+\eta(\varepsilon),$$

where

$$\begin{aligned}
\mathbf{H}_0(x) \;=\; & \omega^1(a)\sum_{j=1}^{k} H_j(a,\underbrace{x}_{1},\underbrace{\varphi(a)}_{k}) \;+ \\
& + (\omega^2(a)-\omega^1(a))\sum_{j=2}^{k} H_j(a,\underbrace{x}_{2},\underbrace{\varphi(a)}_{k-1}) \;+ \\
& + (\omega^3(a)-\omega^2(a))\sum_{j=3}^{k} H_j(a,\underbrace{x}_{3},\underbrace{\varphi(a)}_{k-2}) \;+\; \cdots \\
& \cdots \;+\; (\omega^k(a)-\omega^{k-1}(a))H_k(a,\underbrace{x}_{k},\underbrace{\varphi(a)}_{1}).
\end{aligned}$$

Summing up equalities (23), $i=\overline{0,p}$, we find

$$\int\limits_{\Delta_1^\varepsilon} K(t,\tau)H(\tau,x(\tau),y(\tau))d\tau/\varepsilon =$$

$$= K(t,a)\mathbf{H}_0(x(a))(x(a)-\varphi(a)) + \sum_{i=1}^{p} K(t,t_i+0)\mathbf{H}_i(x_i)(B_i x_i + a_i) + \eta(\varepsilon)$$

and

$$\int\limits_a^b K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon \ =$$

$$= \int\limits_a^b K(t,\tau)\mathcal{H}(\tau,x(\tau))(A(\tau)x(\tau)+f(\tau))\,d\tau + K(t,a)\mathbf{H}_0(x(a))(x(a)-\varphi(a)) \ +$$

$$+ \sum_{i=1}^{p} K(t,t_i+0)\mathbf{H}_i(x_i)(B_i x_i + a_i) + \eta(\varepsilon). \tag{27}$$

Thus, dividing equality (17) by ε, in view of equalities (19) and (27), we have

$$\mathcal{P}_d^* \{J(x(\cdot),0) \ - \ \ell \int\limits_a^b K(\cdot,\tau)\mathcal{H}(\tau,x(\tau))(A(\tau)x(\tau)+f(\tau))d\tau \ -$$

$$- \ell K(\cdot,a)\mathbf{H}_0(x(a))(x(a)-\varphi(a)) \ -$$

$$- \sum_{i=1}^{p} \ell K(\cdot,t_i+0) \ (\mathbf{J}_i(x_i)(A_i x_i + f_i) + \mathbf{H}_i(x_i)(B_i x_i + a_i)) + \eta(\varepsilon)\} \ = \ 0. \tag{28}$$

Now we easily see that (14) is obtained from (28) by passing to the limit as $\varepsilon \to 0$. The theorem is proved.

Equation (14) can be called *equation for the generating amplitudes* (see, for instance, [18] or a number of works of the first author [9 – 14]) for the BVP for the impulsive system with concentrated delay (1).

3.2. Reduction of the problem to an operator system in a suitable function space. Now suppose that c_r^* is a solution of equation (14). Then the solution $z(t,\varepsilon)$ of system (16) such that $z(t,0)\equiv 0$ can be represented in the form

$$z(t,\varepsilon) = X_r(t)c + \varepsilon z^{(1)}(t,\varepsilon), \tag{29}$$

where the unknown constant vector $c = c(\varepsilon) \in \mathbf{R}^r$ must satisfy an equation derived below from (28), while the unknown vector-valued function $z^{(1)}(t,\varepsilon)$ can be represented as

$$z^{(1)}(t,\varepsilon) = X(t)Q^+ J(x(\cdot),\varepsilon) + (\Gamma H)(t)/\varepsilon + \sum_{i=1}^{p} \gamma_i(t) I_i(x_i,y_i)/\varepsilon, \tag{30}$$

where

$$(\Gamma H)(t) = \int\limits_a^b K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau - X(t)Q^+\ell \int\limits_a^b K(\cdot,\tau)H(\tau,x(\tau),y(\tau))\,d\tau.$$

In view of the above considerations we can write the solvability condition (17) in the form

$$\mathcal{P}_d^*\left\{ J(x_0(\cdot, c_r^*) + z(\cdot, \varepsilon), \varepsilon) - \ell \int_a^b K(\cdot, \tau) h(\tau, x_0(\tau, c_r^*) + z(\tau, \varepsilon))\, d\tau - \right.$$

$$\left. - \sum_{i=0}^{p} \ell K(\cdot, t_i + 0) \mathcal{J}_i(x_0(t_i, c_r^*) + z(t_i, \varepsilon)) + \eta(\varepsilon, x) \right\} = 0, \qquad (31)$$

where $t_0 = a = t_0 + 0$, $h(\tau, x) \equiv \mathcal{H}(\tau, x)(A(\tau)x + f(\tau))$, $\mathcal{J}_0(x) \equiv \mathbf{H}_0(x - \varphi(a))$, $\mathcal{J}_i(x) \equiv \equiv \mathbf{J}_i(x)(A_i x + f_i) + \mathbf{H}_i(x)(B_i x + a_i)$, and the quantity $\eta(\varepsilon, x)$ tends to 0 as $\varepsilon \to 0$.

Similarly, equality (30) can be represented in the form

$$z^{(1)}(t, \varepsilon) = X(t) Q^+ J(x_0(\cdot, c_r^*) + z(\cdot, \varepsilon), \varepsilon) +$$

$$+ \left(\Gamma h(\cdot, x) \Big|_{x = x_0(\cdot, c_r^*) + z(\cdot, \varepsilon)} \right)(t) +$$

$$+ \sum_{i=0}^{p} \gamma_i(t) \mathcal{J}_i(x_0(t_i, c_r^*) + z(t_i, \varepsilon)) + \eta(\varepsilon, x(t, \varepsilon)), \qquad (32)$$

where $\gamma_0(t) \equiv K(t, a)$.

We expand the left-hand side of (31) about the point $z = 0$. We have

$$h(\tau, x_0(\tau, c_r^*) + z) = h_0(\tau) + h_1(\tau) z + h_2(\tau, z),$$

where

$$h_0(\tau) = h(\tau, x_0(\tau, c_r^*)), h_1(\tau) = \frac{\partial}{\partial x} h(\tau, x) \Big|_{x = x_0(\tau, c_r^*)},$$

$h_2(\tau, z)$ is such that

$$h_2(\tau, 0) = 0, \quad \frac{\partial}{\partial z} h_2(\tau, 0) = 0.$$

Analogously we represent

$$\mathcal{J}_i(x_0(t_i, c_r^*) + z) = \mathcal{J}_{0i} + \mathcal{J}_{1i} z + \mathcal{J}_{2i}(z), \quad i = \overline{0, p},$$

where $\mathcal{J}_{0i}$, $\mathcal{J}_{1i}$ are represented in a similar way, while $\mathcal{J}_{2i}(z)$ is such that

$$\mathcal{J}_{2i}(0) = 0, \quad \frac{\partial}{\partial z} \mathcal{J}_{2i}(0) = 0,$$

and

$$J(x_0(\cdot, c_r^*) + z, \varepsilon) = J_0 + J_1 z + J_2(z, \varepsilon).$$

Now by virtue of the assumption $F(c_r^*) = 0$, equality (31) takes the form

$$\mathcal{P}_d^*\left\{ J_1 z(\cdot, \varepsilon) + J_2(z(\cdot, \varepsilon), \varepsilon) - \ell \int_a^b K(\cdot, \tau) \left(h_1(\tau) z(\tau, \varepsilon) + h_2(\tau, z(\tau, \varepsilon)) \right) d\tau - \right.$$

$$\left. - \sum_{i=0}^{p} \ell K(\cdot, t_i + 0) \left(\mathcal{J}_{1i} z(t_i, \varepsilon) + \mathcal{J}_{2i}(z(t_i, \varepsilon)) \right) + \eta(\varepsilon, x) \right\} = 0. \qquad (33)$$

In view of the representation (29) let us denote

$$\mathcal{B}_0 = \mathcal{P}_d^* \left(J_1 X_r(\cdot) - \ell \int_a^b K(\cdot,\tau) h_1(\tau) X_r(\tau) d\tau - \sum_{i=0}^{p} \ell K(\cdot, t_i + 0) \mathcal{J}_{1i} X_r(t_i) \right),$$

which is a $(d \times r)$-matrix. Then we have

$$\begin{aligned} \mathcal{B}_0 c \;=\; -\mathcal{P}_d^* \Bigg\{ & \varepsilon J_1 z^{(1)}(\cdot,\varepsilon) + J_2(z(\cdot,\varepsilon),\varepsilon) \;- \\ & - \ell \int_a^b K(\cdot,\tau) \left(\varepsilon h_1(\tau) z^{(1)}(\tau,\varepsilon) + h_2(\tau, z(\tau,\varepsilon)) \right) d\tau \;- \\ & - \sum_{i=0}^{p} \ell K(\cdot, t_i + 0) \left(\varepsilon \mathcal{J}_{1i} z^{(1)}(t_i,\varepsilon) + \mathcal{J}_{2i}(z(t_i,\varepsilon)) \right) + \eta(\varepsilon, x) \Bigg\}. \end{aligned} \tag{34}$$

In the representation (32) we use the same expansions and obtain

$$\begin{aligned} z^{(1)}(t,\varepsilon) = & X(t) Q^+ (J_0 + J_1(X_r(\cdot)c + \varepsilon z^{(1)}(\cdot,\varepsilon)) + J_2(z(\cdot,\varepsilon),\varepsilon)) \;+ \\ & + (\Gamma h_0)(t) + \left(\Gamma \left[h_1(\cdot)(X_r(\cdot)c + \varepsilon z^{(1)}(\cdot,\varepsilon)) \right] \right)(t) + (\Gamma h_2(\cdot, z(\cdot,\varepsilon)))(t) \;+ \\ & + \sum_{i=0}^{p} \gamma_i(t) \Big(\mathcal{J}_{0i} + \mathcal{J}_{1i}(X_r(t_i)c + \varepsilon z^{(1)}(t_i,\varepsilon)) + \mathcal{J}_{2i}(z(t_i,\varepsilon)) \Big) + \eta(\varepsilon, x(t,\varepsilon)). \end{aligned} \tag{35}$$

Thus we have reduced problem (1) to the equivalent operator system (15), (29), (34), (35).

3.3. Critical case of first order. Denote by $\mathcal{P}_0 \equiv \mathcal{P}_{\mathcal{B}_0}$ and $\mathcal{P}_0^* \equiv \mathcal{P}_{\mathcal{B}_0^*}$, respectively, the orthoprojectors $\mathcal{P}_0 : \mathbf{R}^r \mapsto \mathrm{Ker}\ (\mathcal{B}_0)$ and $\mathcal{P}_0^* : \mathbf{R}^d \mapsto \mathrm{Ker}\ (\mathcal{B}_0^*)$. Suppose that $\mathrm{Rank}\mathcal{B}_0 = r$, i.e. $\mathcal{P}_0 = 0$. This is the so called *critical case of first order*. In this case the inequality $d \geq r$ necessarily holds, which implies $m \geq n$. Then equation (34) can be solved with respect to c if and only if

$$\begin{aligned} \mathcal{P}_0^* \mathcal{P}_d^* \Bigg\{ & \varepsilon J_1 z^{(1)}(\cdot,\varepsilon) \;+\; J_2(z(\cdot,\varepsilon),\varepsilon) \;-\; \ell \int_a^b K(\cdot,\tau) \left(\varepsilon h_1(\tau) z^{(1)}(\tau,\varepsilon) + h_2(\tau, z(\tau,\varepsilon)) \right) d\tau \;- \\ & - \sum_{i=0}^{p} \ell K(\cdot, t_i + 0) \left(\varepsilon \mathcal{J}_{1i} z^{(1)}(t_i,\varepsilon) + \mathcal{J}_{2i}(z(t_i,\varepsilon)) \right) + \eta(\varepsilon, x) \Bigg\} = 0. \end{aligned} \tag{36}$$

If $\mathcal{P}_0^* \mathcal{P}_d^* = 0$, then condition (36) is always fulfilled and equation (34) can be uniquely solved with respect to c:

$$\begin{aligned} c \;=\; -\; \mathcal{B}_0^+ \mathcal{P}_d^* \Bigg\{ & \varepsilon J_1 z^{(1)}(\cdot,\varepsilon) + J_2(z(\cdot,\varepsilon),\varepsilon) \;- \\ & - \; \ell \int_a^b K(\cdot,\tau) \left(\varepsilon h_1(\tau) z^{(1)}(\tau,\varepsilon) + h_2(\tau, z(\tau,\varepsilon)) \right) d\tau \;- \\ & - \; \sum_{i=0}^{p} \ell K(\cdot, t_i + 0) \left(\varepsilon \mathcal{J}_{1i} z^{(1)}(t_i,\varepsilon) + \mathcal{J}_{2i}(z(t_i,\varepsilon)) \right) + \eta(\varepsilon, x) \Bigg\}, \end{aligned} \tag{37}$$

where $\mathcal{B}_0^+$ is an $(r \times d)$-matrix which Moore – Penrose pseudoinverse to $\mathcal{B}_0$. Thus we obtain [18] an operator system (15), (29), (37), (35) to which a convergent simple iteration method can be applied.

Theorem 2. *For BVP (1) let conditions H_1 – H_7 and (11) hold and* $\operatorname{Rank} Q = n_1$, $r = n - n_1 > 0$. *Then for any root $c_r = c_r^* \in \mathbf{R}^r$ of equation (14) such that $\mathcal{P}_0 = 0$, $\mathcal{P}_0^* \mathcal{P}_d^* = 0$ there exists a constant $\varepsilon_* \in (0, \varepsilon_0)$ such that for $\varepsilon \in [0, \varepsilon_*]$ BVP (1) has a unique solution $x(t, \varepsilon) \in C([a, b] \backslash \{t_i\})$ as a function of t, depending continuously on ε, and such that $x(t, 0) = x_0(t, c_r^*)$. This solution is determined by means of a convergent, for $\varepsilon \in [0, \varepsilon_*]$, simple iteration method*

$$\begin{aligned}
c_\nu &= -\mathcal{B}_0^+ \mathcal{P}_d^* \Bigg\{ \varepsilon J_1 z_\nu^{(1)}(\cdot, \varepsilon) + J_2(z_\nu(\cdot, \varepsilon), \varepsilon) \; - \\
&- \; \ell \int_a^b K(\cdot, \tau) \left(\varepsilon h_1(\tau) z_\nu^{(1)}(\tau, \varepsilon) + h_2(\tau, z_\nu(\tau, \varepsilon)) \right) d\tau \; - \\
&- \; \sum_{i=0}^{p} \ell K(\cdot, t_i + 0) \left(\varepsilon \mathcal{J}_{1i} z_\nu^{(1)}(t_i, \varepsilon) + \mathcal{J}_{2i}(z_\nu(t_i, \varepsilon)) \right) + \eta(\varepsilon, x_\nu) \Bigg\}, \\
z_{\nu+1}^{(1)}(t, \varepsilon) &= X(t) Q^+ \left(J_0 + J_1(X_r(\cdot) c_\nu + \varepsilon z_\nu^{(1)}(\cdot, \varepsilon) \right) + J_2(z_\nu(\cdot, \varepsilon), \varepsilon)) \; + \\
&+ \; (\Gamma h_0)(t) + \left(\Gamma \left[h_1(\cdot)(X_r(\cdot) c_\nu + \varepsilon z_\nu^{(1)}(\cdot, \varepsilon)) \right] \right)(t) + (\Gamma h_2(\cdot, z_\nu(\cdot, \varepsilon)))(t) \; + \\
&+ \; \sum_{i=0}^{p} \gamma_i(t) \Big(\mathcal{J}_{0i}. + \mathcal{J}_{1i}(X_r(t_i) c_\nu + \varepsilon z_\nu^{(1)}(t_i, \varepsilon)) + \mathcal{J}_{2i}(z_\nu(t_i, \varepsilon)) \Big) + \eta(\varepsilon, x_\nu(t, \varepsilon)), \\
z_{\nu+1}(t, \varepsilon) &= X_r(t) c_\nu + \varepsilon z_{\nu+1}^{(1)}(t, \varepsilon), \qquad (38) \\
x_\nu(t, \varepsilon) &= \begin{cases} x_0(t, c_r^*) + z_\nu(t, \varepsilon), & t \in [a, b]; \\ \varphi(t), & t \in [a - \varepsilon_*, a), \end{cases} \\
\nu &= 0, 1, 2, \ldots; \quad z_0(t, \varepsilon) = z_0^{(1)}(t, \varepsilon) = 0.
\end{aligned}$$

The convergence of the method can be proved by using Lyapunov's majorants technique as in [13, 18].

4. Notes and comments. ***4.1. Critical case of first order for boundary value problem s of Fredholm type.*** Let $m = n$ (the number of the boundary conditions equals the order of the system). Now the condition $\mathcal{P}_0 = 0$ implies $\mathcal{P}_0^* = 0$, thus the condition $\mathcal{P}_0^* \mathcal{P}_d^* = 0$ is automatically fulfilled. Moreover, $\mathcal{P}_0 = 0$ implies $\det \mathcal{B}_0 \neq 0$, and in equality (37) and in the first equality of the iteration procedure (38), we write $\mathcal{B}_0^{-1}$ instead of $\mathcal{B}_0^+$. It is easy to see [13, 12, 18] that this condition is equivalent to the simplicity of the root $c_r = c_r^*$ of the equation for the generating amplitudes:

$$F(c_r^*) = 0, \qquad \det \left(\frac{\partial F(c_r)}{\partial c_r} \right) \Bigg|_{\, c_r = c_r^*} \neq 0.$$

In particular, the periodic BVP with impulse effect for system (1) is of Fredholm type. It was considered by the first author in [12] (see also the monographs [13, 14]). The existence of a periodic solution of an impulsive system with a small constant delay ($\omega^j(t) \equiv 1$ in system

(1)) was proved in [7] for the critical case of first order. Moreover, under some simplifying assumptions the result was extended to the case of a *nonlinear* generating system [15].

4.2. Critical case of second order. Now suppose that $\mathcal{P}_0 \neq 0$. Then the operator system (15), (29), (34), (35) does not belong to the class of systems to which the simple iteration method is applicable. Introducing an additional variable, system (15), (29), (34), (35) under condition (36) is regularized (reduced to an operator system of higher dimension to which the simple iteration method can be applied). Below we just sketch this regularization.

If condition (36) is satisfied, then from (34) we determine

$$c = c^{(0)} + c^{(1)},$$

where $c^{(0)}$ is given by the right-hand side of equality (37), $c^{(1)}$ is an arbitrary constant vector in $\mathrm{Ker}\,(\mathcal{B}_0)$, $c^{(1)} = \mathcal{P}_0 c$, $c^{(0)} = (\mathrm{Id} - \mathcal{P}_0)c$. Then equality (35) can be written in the form

$$z^{(1)}(t,\varepsilon) = G_1(t)c^{(1)} + z^{(2)}(t,\varepsilon), \tag{39}$$

where

$$G_1(t) \equiv X(t)Q^+ J_1 X_r(\cdot) + (\Gamma\,[h_1(\cdot)X_r(\cdot)])\,(t) + \sum_{i=0}^{p} \gamma_i(t)\mathcal{J}_{1i}X_r(t_i),$$

and $z^{(2)}(t,\varepsilon)$ is given by the right-hand side of equality (35), with c being replaced by $c^{(0)}$.

If $\mathcal{P}_0^*\mathcal{P}_d^* = 0$, then condition (36) is always fulfilled and BVP (1) has an $(r - \mathrm{Rank}\,\mathcal{B}_0)$-parametric family of solutions. If this is not the case, further computations need more precise expansions with respect to the “small parameter” ε, which require the existence of a piecewise continuous second derivative of the solution x, respective piecewise continuous differentiability of the known functions in system (1) with respect to t, the existence of some continuous second derivatives of the functions H_j, I_{ij} and continuous differentiability of $J(x,\varepsilon)$ with respect to ε.

Thus the solvability condition (36) is represented in the form

$$\begin{aligned}\mathcal{P}_0^*\mathcal{P}_d^* \Big\{ &\varepsilon J_1 z^{(1)}(\cdot,\varepsilon) + \varepsilon J_2' z(\cdot,\varepsilon) + J_2''(z(\cdot,\varepsilon),\varepsilon) \; - \\ &- \ell \int_a^b K(\cdot,\tau)\Big(\varepsilon h_1(\tau) z^{(1)}(\tau,\varepsilon) + \varepsilon h_2'(\tau) z(\tau,\varepsilon) + h_2''(\tau, z(\tau,\varepsilon))\Big) d\tau \; - \\ &- \sum_{i=0}^{p} \ell K(\cdot, t_i+0)\Big(\varepsilon \mathcal{J}_{1i} z^{(1)}(t_i,\varepsilon) + \varepsilon \mathcal{J}_{2i}' z(t_i,\varepsilon) + \mathcal{J}_{2i}''(z(t_i,\varepsilon))\Big) + \eta(\varepsilon,x)\Big\} = 0.\end{aligned} \tag{40}$$

In view of (29) and (39) we obtain the system

$$\begin{aligned}\varepsilon\mathcal{B}_1 c^{(1)} = -\mathcal{P}_0^*\mathcal{P}_d^* \Big\{ &\varepsilon J_1 z^{(2)}(\cdot,\varepsilon) + \varepsilon J_2' \Big(X_r(\cdot)c^{(0)} + \varepsilon z^{(1)}(\cdot,\varepsilon)\Big) + J_2''(z(\cdot,\varepsilon),\varepsilon) \; - \\ -\ell \int_a^b K(\cdot,\tau) \Big[&\varepsilon h_1(\tau) z^{(2)}(\tau,\varepsilon) + \varepsilon h_2'(\tau) \Big(X_r(\tau)c^{(0)} + \varepsilon z^{(1)}(\tau,\varepsilon)\Big) + h_2''(\tau, z(\tau,\varepsilon))\Big] d\tau \; - \\ &- \sum_{i=0}^{p} \ell K(\cdot, t_i+0)\Big[\varepsilon \mathcal{J}_{1i} z^{(2)}(t_i,\varepsilon) + \varepsilon \mathcal{J}_{2i}' \Big(X_r(t_i)c^{(0)} + \varepsilon z^{(1)}(t_i,\varepsilon)\Big) + \\ &+ \mathcal{J}_{2i}''(z(t_i,\varepsilon))\Big] + \eta(\varepsilon,x)\Big\},\end{aligned} \tag{41}$$

where

$$\mathcal{B}_1 = \mathcal{P}_0^*\mathcal{P}_d^*\Big\{ J_1G_1(\cdot) + J_2'X_r(\cdot) - \ell\int\limits_a^b K(\cdot,\tau)\left(h_1(\tau)G_1(\tau) + h_2'(\tau)X_r(\tau)\right)d\tau \; -$$

$$- \sum_{i=0}^{p} \ell K(\cdot, t_i+0)\left(\mathcal{J}_{1i}G_1(t_i) + \mathcal{J}_{2i}'X_r(t_i)\right)\Big\}\mathcal{P}_0$$

is a $(d\times r)$-matrix.

Denote by $\mathcal{P}_1 \equiv \mathcal{P}_{\mathcal{B}_1}$ and $\mathcal{P}_1^* \equiv \mathcal{P}_{\mathcal{B}_1^*}$, respectively, the orthoprojectors $\mathcal{P}_1 : \mathbf{R}^r \mapsto \mathrm{Ker}\ (\mathcal{B}_1)$ and $\mathcal{P}_1^* : \mathbf{R}^d \mapsto \mathrm{Ker}\ (\mathcal{B}_1^*)$. Then system (41) is solvable with respect to $\varepsilon c^{(1)} \in \mathrm{Ker}(\mathcal{B}_0)$ if and only if

$$\mathcal{P}_1^*\mathcal{P}_0^*\mathcal{P}_d^*\Big\{\varepsilon J_1 z^{(2)}(\cdot,\varepsilon) + \varepsilon J_2'\left(X_r(\cdot)c^{(0)} + \varepsilon z^{(1)}(\cdot,\varepsilon)\right) + J_2''(z(\cdot,\varepsilon),\varepsilon) \; -$$

$$-\ell\int\limits_a^b K(\cdot,\tau)\left[\varepsilon h_1(\tau)z^{(2)}(\tau,\varepsilon) + \varepsilon h_2'(\tau)\left(X_r(\tau)c^{(0)} + \varepsilon z^{(1)}(\tau,\varepsilon)\right) + h_2''(\tau, z(\tau,\varepsilon))\right]d\tau \; -$$

$$- \sum_{i=0}^{p} \ell K(\cdot, t_i+0)\left[\varepsilon \mathcal{J}_{1i} z^{(2)}(t_i,\varepsilon) + \varepsilon\mathcal{J}_{2i}'\left(X_r(t_i)c^{(0)} + \varepsilon z^{(1)}(t_i,\varepsilon)\right) + \right.$$

$$\left. + s\ \mathcal{J}_{2i}''(z(t_i,\varepsilon))\right] + \eta(\varepsilon,x)\Big\} = 0, \tag{42}$$

and

$$\varepsilon c^{(1)} = -\mathcal{B}_1^+\mathcal{P}_0^*\mathcal{P}_d^*\Big\{\varepsilon J_1 z^{(2)}(\cdot,\varepsilon) + \varepsilon J_2'\left(X_r(\cdot)c^{(0)} + \varepsilon z^{(1)}(\cdot,\varepsilon)\right) + J_2''(z(\cdot,\varepsilon),\varepsilon) \; -$$

$$-\ell\int\limits_a^b K(\cdot,\tau)\left[\varepsilon h_1(\tau)z^{(2)}(\tau,\varepsilon) + \varepsilon h_2'(\tau)\left(X_r(\tau)c^{(0)} + \varepsilon z^{(1)}(\tau,\varepsilon)\right) + h_2''(\tau, z(\tau,\varepsilon))\right]d\tau \; -$$

$$- \sum_{i=0}^{p} \ell K(\cdot, t_i+0)\left[\varepsilon \mathcal{J}_{1i} z^{(2)}(t_i,\varepsilon) + \varepsilon\mathcal{J}_{2i}'\left(X_r(t_i)c^{(0)} + \varepsilon z^{(1)}(t_i,\varepsilon)\right) + \right.$$

$$\left. + \mathcal{J}_{2i}''(z(t_i,\varepsilon))\right] + \eta(\varepsilon,x)\Big\} + c^{(2)},$$

where $\mathcal{B}_1^+$ is the matrix Moore – Penrose pseudoinverse to $\mathcal{B}_1, c^{(2)}$ is an arbitrary vector in $\mathrm{Ker}(\mathcal{B}_0)\cap\mathrm{Ker}(\mathcal{B}_1)$.

Suppose that $\mathrm{Ker}(\mathcal{B}_0)\cap\mathrm{Ker}(\mathcal{B}_1) = 0$. Then (41) has a unique solution. A sufficient condition for (42) is $\mathcal{P}_1^*\mathcal{P}_0^*\mathcal{P}_d^* = 0$, i.e.

$$\mathrm{Ker}(\mathcal{B}_0^*)\cap\mathrm{Ker}(\mathcal{B}_1^*)\cap\mathrm{Ker}(Q^*) = 0.$$

Thus, under the conditions

$$\mathcal{P}_0 \neq 0, \mathcal{P}_0\mathcal{P}_1 = 0, \mathcal{P}_1^*\mathcal{P}_0^*\mathcal{P}_d^* = 0$$

system (15), (29), (34), (35) is reduced to a system to which a simple iteration method can be applied. As stated above, this requires additional smoothness assumptions and cumber some computations. For the case of a periodic problem for an impulsive system with a small constant delay ($\omega^j(t) \equiv 1$), these computations were carried out in details in [8].

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