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PERIODIC SOLUTIONS OF IMPULSIVE SYSTEMS WITH A SMALL DELAY IN THE CRITICAL CASE OF SECOND ORDER

ПЕРІОДИЧНІ РОЗВ'ЯЗКИ ІМПУЛЬСНИХ СИСТЕМ ІЗ МАЛИМ ЗАПІЗНЕННЯМ У КРИТИЧНОМУ ВИПАДКУ ДРУГОГО ПОРЯДКУ

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We consider an impulsive differential-difference system such that the corresponding system without delay is linear and has an r-parametric family of ω-periodic solutions. For this case, an equation for the generating amplitudes is derived, and sufficient conditions are obtained for the existence of ω-periodic solutions of the initial system in the critical case of the second order if the delay is sufficiently small.

Розглядається нелінійна періодична імпульсна диференціальна система із запізненням у припущенні, що відповідна система без запізнення є лінійною і має r-параметричну сім'ю періодичних розв'язків. Побудовано рівняння для породжуючих амплітуд такої задачі, що дає необхідну умову існування розв'язків. Одержані достатні умови існування періодичних розв'язків вихідної нелінійної системи у критичному випадку другого порядку при досить малому аргументу, що запізнюється.

1. Introduction. Impulsive differential equations [1] with delay describe models of real processes and phenomena where both dependence on the past and momentary disturbances are present. For instance, the size of a given population may be normally described by a delay differential equation and, at certain moments, the number of individuals can be abruptly changed. The interaction of the impulsive perturbations and the delay makes the qualitative investigation of such equations difficult. In particular, the solutions are not smooth at the moments of impulse effect shifted by the delay [2].

A classical problem of the qualitative theory of differential equations is the existence of periodic solutions. Numerous references on this matter concerning differential equations with delay and impulsive differential equations can be found in [3]. A traditional approach to this problem is the investigation of the linearized system (also called *system in variations*) with

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respect to a periodic solution of the unperturbed system satisfying certain nondegeneracy assumptions.

In [3], for an impulsive system with small delay, it is proved that if the corresponding system without delay has an isolated ω-periodic solution, then in any sufficiently small neighbourhood of this orbit, the system considered also has a unique ω-periodic solution. In an earlier version of that paper, this result was proved under considerably more restrictive assumptions with the use of the contraction mapping principle (see [4], §8). Moreover, this result was extended to the case of a neutral impulsive system with small delay [5].

In the present paper, we consider an impulsive differential-difference system such that the corresponding system without delay is linear and has an r-parametric family of ω-periodic solutions. In the critical case of the first order, in the paper [6], sufficient conditions were obtained for the existence of ω-periodic solutions of the initial system if the delay is sufficiently small. Here, we consider the critical case of the second order. Necessary and sufficient conditions are obtained for the existence of ω-periodic solutions of the initial system for a sufficiently small delay. The periodic problem for impulsive systems (without delay) as well as more general boundary-value problems for differential systems with delay (and without impulses) and with impulses (and without delay) in critical cases of the first and second order were considered in several papers by Boichuk (see the monographs [7, 8]).

2. Statement of the problem. Preliminary assertions. Consider the impulsive system of functional differential equations

$$\begin{aligned}
&\dot{x}(t) = A(t)x(t) + f(t) + H(t, x(t), x(t-h)), \quad t \neq t_i, \quad t \neq t_i + h,\\
&\Delta x(t_i) = B_i x(t_i) + a_i + I_i(x(t_i, x(t_i - h)), \qquad i \in \mathbf{Z},\\
&\Delta x(t_i + h) = 0 \ \text{ if } \ h > 0,
\end{aligned} \tag{1}$$

where $x \in \Omega \subset \mathbf{R}^n$, $f : \mathbf{R} \to \mathbf{R}^n$, $A : \mathbf{R} \to \mathbf{R}^{n \times n}$, $I_i(t,x)\bar{x} = g(t,x,\bar{x})\ (x - \bar{x})$, $g : \mathbf{R} \times \Omega \times \Omega \to$ $\to \mathbf{R}^{n\times n}$, Ω is a domain in $\mathbf{R}^n$; $\Delta x(t_i) = x(t_i + 0) - x(t_i - 0)$ are impulses at moments t_i $\{t_i\}_{i \in \mathbf{Z}}$ is a strictly increasing sequence such that $\lim\limits_{i \to \pm\infty} t_i = \pm\infty$, $a_i \in \mathbf{R}^n$, $B_i \in \mathbf{R}^{n\times n}$, $I_i(x, \bar{x}) =$ $= J_i(x,\bar{x})(x - \bar{x})$, $J_i : \Omega \times \Omega \to \mathbf{R}^{n \times n} (i \in \mathbf{Z})$, and $h \geq 0$ is the delay.

As usual in the theory of impulsive differential equations, we assume that $x(t_i) \equiv x(t_i - 0)$ at the points of discontinuity t_i of the solution $x(t)$. It is clear that, in general, the derivatives $\dot{x}(t_i)$, $\dot{x}(t_i + h)$ do not exist. However, there exist the limits $\dot{x}(t_i \pm 0)$, $\dot{x}(t_i + h \pm 0)$. According to the above convention, we assume that $\dot{x}(t_i) \equiv \dot{x}(t_i - 0)$, $\dot{x}(t_i + h) \equiv \dot{x}(t_i + h - 0)$. Then the differential equation in (1) is valid everywhere. For the sake of brevity, unless otherwise stated, we use the following notation:

$$\bar{x}(t) = x(t-h), \quad x_i = x(t_i).$$

Introduce the following conditions:

H1. The components of $A(t)$, $f(t)$ belong to the space $\widetilde{C}_\omega\{t_i\}$ of all ω-periodic functions, continuous or piecewise continuous with discontinuities of the first kind at the points t_i, $i \in \mathbf{Z}$.

H2. The function $g(t,x,\bar{x})$ is continuously differentiable with respect to x, $\bar{x}$, and its components belong to $\widetilde{C}_\omega\{t_i\}$ as functions of t.

H3. The functions $J_i(x,\bar{x}) \in C^1(\Omega \times \Omega, \mathbf{R}^{n\times n})$, $i \in \mathbf{Z}$.

H4. There exists a positive integer m such that $t_{i+m} = t_i + \omega$, $B_{i+m} = B_i$, $a_{i+m} = a_i$, $J_{i+m}(x,\bar{x}) = J_i(x,\bar{x})$, $i \in \mathbf{Z}$, $x, \bar{x} \in \Omega$.

H5. The matrices $E + B_i$, $i \in \mathbf{Z}$, are nonsingular (E is the unit matrix).

Suppose, for the sake of definiteness, that

$$0 < t_1 < t_2 < \cdots < t_m < \omega.$$

Let $h_1 > 0$ be so small that, for any $h \in [0, h_1]$, we have

$$h < t_1, \quad t_i + h < t_{i+1}, \quad i = \overline{1, m-1}, \quad t_m + h < \omega.$$

Together with (1), we consider the so called *generating* system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f(t), \\ \Delta x(t_i) &= B_i x_i + a_i, \quad i \in \mathbf{Z}, \end{aligned} \tag{2}$$

obtained from (1) for $h = 0$.

Let $X(t)$ be the fundamental solution (i.e., $X(0) = E$) of the homogeneous system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \\ \Delta x(t_i) &= B_i x_i, \quad i \in \mathbf{Z}. \end{aligned} \tag{3}$$

Denote $Q = E - X(\omega)$, let Q^* be its transpose, and let Q^+ be its Moore – Penrose pseudoinverse [10]. If the matrix Q is nonsingular, we have to do with the so called *noncritical* case considered under more general assumptions in [3]. Instead we consider here the *critical* case where

H6. Rank $Q = n_1 < n$.

If we denote $r = n - n_1$, then the homogeneous impulsive system (3) has an r-parametric family of ω-periodic solutions. Denote by $\mathcal{P} = \mathcal{P}_Q$ the orthoprojector $\mathbf{R}^n \to \operatorname{Ker}(Q)$ and by $\mathcal{P}^* = \mathcal{P}_{Q^*}$ the orthoprojector $\mathbf{R}^n \to \operatorname{Ker}(Q^*)$.

Then the nonhomogeneous system (2) has ω-periodic solutions if and only if

$$\mathcal{P}^* X(\omega)\Big(\int\limits_0^\omega X^{-1}(\tau) f(\tau)\, d\tau + \sum_{i=1}^m X_i^{-1} a_i \Big) = 0. \tag{4}$$

Here, again for the sake of brevity, we have denoted

$$X_i \equiv X(t_i + 0) = (E + B_i) X(t_i)$$

instead of X_i^+.

Since rank $\mathcal{P}^* = n - \operatorname{rank} Q^* = n - n_1 = r$, condition (4) consists of r linearly independent scalar equalities. Denote by $\mathcal{P}_r^* = \mathcal{P}_{Q_r^*}$ an $(r \times n)$-matrix whose rows are r linearly independent rows of $\mathcal{P}^*$. Then (4) takes the form

$$\mathcal{P}_r^* X(\omega)\Big(\int\limits_0^\omega X^{-1}(\tau) f(\tau)\, d\tau + \sum_{i=1}^m X_i^{-1} a_i \Big) = 0. \tag{5}$$

If condition (5) is satisfied, then system (2) has an r-parametric family of ω-periodic solutions

$$x_0(t, c_r) = X_r(t) c_r + \int\limits_0^\omega G(t, \tau) f(\tau)\, d\tau + \sum_{i=1}^m G(t, t_i) a_i, \tag{6}$$

where $X_r(t)$ is an $(n \times r)$ matrix whose columns are a complete system of r linearly independent ω-periodic solutions of (3), $c_r \in \mathbf{R}^r$ is an arbitrary vector, and $G(t,\tau)$ is the generalized Green's function

$$G(t,\tau) = \begin{cases} X(t)(E + Q^+ X(\omega))X^{-1}(\tau), & 0 \le \tau \le t \le \omega, \\ X(t)Q^+ X(\omega)X^{-1}(\tau), & 0 \le t < \tau \le \omega. \end{cases}$$

3. Main result. ***3.1. Preliminaries. Equation for the generating amplitudes.*** Let us find conditions for the existence of ω-periodic solutions $x(t,h)$ of system (1) depending continuously on h and such that, for some $c_r \in \mathbf{R}^r$, we have $x(t,0) = x_0(t,c_r)$. A necessary condition for the existence of such solutions is given by the following statement:

Theorem 1. *Let system (1) satisfying conditions H1–H6 and (5) have an ω-periodic solution $x(t,h)$ which, for $h = 0$, turns into a generating solution $x_0(t,c_r^*)$. Then the vector $c_r^* \in \mathbf{R}^r$ satisfies the equation*

$$\begin{aligned} F(c_r^*) \;\equiv\; & \mathcal{P}_r^* X(\omega)\Big\{ \int_0^\omega X^{-1}(\tau) g(\tau, x_0(\tau,c_r^*), x_0(\tau,c_r^*))(A(\tau)x_0(\tau,c_r^*) + f(\tau))\, d\tau + \\ + \; & \sum_{i=1}^m X_i^{-1}[J_i(x_0(t_i,c_r^*), x_0(t_i,c_r^*))(A_i x_0(t_i,c_r^*) + f_i) + \\ + \; & g(t_i+0, (E+B_i)x_0(t_i,c_r^*) + a_i, x_0(t_i,c_r^*))(B_i x_0(t_i,c_r^*) + a_i)]\Big\} = 0. \end{aligned} \tag{7}$$

This theorem was proved in [6] where the so called critical case of the first order was considered. It will be proved here once again while reducing system (1) to an equivalent operator system. Here, we need more precise expansions with respect to the „small parameter" h, which require the existence of a piecewise continuous second derivative of the solution x, respectively piecewise continuous differentiability with respect to t of the known functions in system (1), and the existence of some second derivatives ensured by conditions H7, H8 introduced below.

Proof. In (1), we change the variables according to the formula

$$x(t,h) = x_0(t,c_r^*) + y(t,h) \tag{8}$$

and are led to the problem of finding ω-periodic solutions $y = y(t,h)$ of the impulsive system of functional differential equations

$$\begin{aligned} &\dot y(t) = A(t)y(t) + H(t, x(t,h), x(t-h,h)), \\ &\Delta y(t_i) = B_i y_i + I_i(x(t_i,h), x(t_i-h,h)), \quad i \in \mathbf{Z}, \end{aligned} \tag{9}$$

such that $y(t,h) \to 0$ as $h \to 0$.

We can formally consider (9) as a nonhomogeneous system of the form (2). Then the solvability condition (5) becomes

$$\begin{aligned} &\mathcal{P}_r^* X(\omega)\Big(\int_0^\omega X^{-1}(\tau) H(\tau, x(\tau,h), x(\tau-h,h)) d\tau + \\ &\qquad + \sum_{i=1}^m X_i^{-1} I_i(x(t_i,h), x(t_i-h,h))\Big) = 0. \end{aligned} \tag{10}$$

For the sake of later convenience, we denote by $\varepsilon(h, x)$ expressions tending to 0 as $h \to 0$, and satisfying the Lipschitz condition with respect to x with a constant tending to 0 as $h \to 0$. We shall sometimes write $\varepsilon(h)$ instead of $\varepsilon(h, x)$ and $x(t)$ instead of $x(t, h)$ if this does not lead to misunderstanding. Thus, for instance, we may write x_i instead of $x(t_i, h)$, etc.

Since the left-hand side of equality (10) tends to 0 as $h \to 0$, we first divide it by h and then study its behaviour as $h \to 0$. Unlike [6], we shall also need the terms linear in h. This is why we introduce condition H7 which is not necessary just to derive equation (7), or to consider the critical case of the first order.

H7. Conditions H1, H2 still hold if the functions $A(t)$, $f(t)$, and $g(t, x, \bar{x})$ are replaced by $\dot{A}(t)$, $\dot{f}(t)$, and $\frac{\partial g}{\partial t}(t, x, \bar{x})$, respectively.

First, we note that

$$\begin{aligned}(x_i - \bar{x}_i)/h &= \dot{x}_i - h\ddot{x}_i/2 + h\varepsilon(h) = \\ &= A_i x_i + f_i + g(t_i, x_i, \bar{x}_i)(x_i - \bar{x}_i) - (\dot{A}_i x_i + A_i \dot{x}_i + \dot{f}_i)h/2 + h\varepsilon(h)\end{aligned}$$

since the interval $(t_i - h, t_i)$ contains no points of discontinuity of the function $x(t, h)$ or its derivative.

This equality implies that

$$\begin{aligned}&(E - hg(t_i, x_i, \bar{x}_i))(x_i - \bar{x}_i)/h = \\ &= A_i x_i + f_i - (\dot{A}_i x_i + \dot{f}_i + A_i(A_i x_i + f_i))h/2 + h\varepsilon(h).\end{aligned}$$

If h is small enough, we have

$$\begin{aligned}&(x_i - \bar{x}_i)/h = (E - hg(t_i, x_i, \bar{x}_i))^{-1}[A_i x_i + f_i - \\ &-(\dot{A}_i x_i + \dot{f}_i + A_i(A_i x_i + f_i))h/2] + h\varepsilon(h) = \\ &= A_i x_i + f_i + h[g(t_i(x_i, \bar{x}_i)(A_i x_i + f_i) - (\dot{A}_i x_i + \dot{f}_i + A_i(A_i x_i + f_i))/2] + h\varepsilon(h).\end{aligned}$$

Thus,

$$\begin{aligned}&I_i(x(t_i, h), x(t_i - h, h))/h = J_i(x_i, \bar{x}_i)(x_i - \bar{x}_i)/h = \\ &\qquad = J_i(x_i, \bar{x}_i) \times (A_i x_i + f_i) + h\mathbf{J}_i(x_i) + \varepsilon(h),\end{aligned} \tag{11}$$

where

$$\begin{aligned}\mathbf{J}_i(x_i) &= [J(x_i, \bar{x}_i)(g(t_i, (x_i, \bar{x}_i) - A_i/2) - \\ &- \frac{\partial}{\partial \bar{x}} J_i(x_i, \bar{x}_i)(A_i x_i + f_i)](A_i x_i + f_i) - J_i(x_i, \bar{x}_i)(\dot{A}_i x_i + \dot{f}_i)/2.\end{aligned} \tag{12}$$

We can represent the integral $\int_0^\omega$ in (10) by a sum of integrals over intervals containing no points of discontinuity of the integrand. It is obvious that, for $\tau \in (t_i, t_i + h)$, the interval $(\tau - h, \tau)$ contains the point of discontinuity t_i, while for τ inside the remaining intervals, the interval $(\tau - h, \tau)$ contains no such points. We denote $\Delta_1^h = \bigcup_{i=1}^{m} (t_i, t_i + h)$, $\Delta_2^h = [0, \omega] \setminus \Delta_1^h$ and use the representation $\int_0^\omega = \int_{\Delta_1^h} + \int_{\Delta_2^h}$.

We first begin with the „bad" set Δ_1^h. For any $i \in \{1,\dots, m\}$, we can find a constant $\theta_i \in (0,1)$ such that

$$h^{-1} \int\limits_{t_i}^{t_i+h} X^{-1}(\tau)H(\tau, x(\tau,h), x(\tau-h,h))\,d\tau =$$

$$= X^{-1}(t_i+\theta_i h)H(t_i+\theta_i h, x(t_i+\theta_i h, h), x(t_i-(1-\theta_i)h,h)) =$$

$$= X_i^{-1}(g(t_i+0,(E+B_i)x_i+a_i,x_i)(B_ix_i+a_i)+h\mathbf{G}_i(x_i))+\varepsilon(h), \tag{13}$$

where

$$\mathbf{G}_i(x_i) = \Biggl\{ \biggl[\theta_i(\partial_t - A_i^+)g(t,x,\bar{x}) + \partial_x g(t,x,\bar{x})(J_i(x_i,\bar{x}_i)(A_ix_i+f_i)+\theta_i(A_i^+(B_ix_i+a_i)+f_i^+))-$$

$$-(1-\theta_i)\partial_{\bar{x}}g(t,x,\bar{x})(A_ix_i+f_i)\biggr] \left| \begin{array}{l} t=t_i+0 \\ x=(E+B_i)x_i+a_i \\ \bar{x}=x_i \end{array} \right. \Biggr\} (B_ix_i+a_i)+$$

$$+g(t_i+0,(E+B_i)x_i+a_i,x_i)(J_i(x_i,\bar{x}_i)(A_ix_i+f_i)+$$

$$+\theta_i(A_i^+(B_ix_i+a_i)+f_i^+)+(1-\theta_i)(A_ix_i+f_i)). \tag{14}$$

Recall that $A_i^+ \equiv A(t_i+0)$, $\dot{X}_i^+ \equiv \dot{X}(t_i+0)$, etc. (but $X_i \equiv X(t_i+0)$).

On the other hand, for τ in the „good" set Δ_2^h, we have as above (with $x \equiv x(\tau,h)$, $\bar{x} \equiv \equiv x(\tau-h,h)$, $A \equiv A(\tau)$, etc.):

$$(x-\bar{x})/h = Ax+f+h[g(\tau,x,\bar{x})(Ax+f)-(\dot{A}x+\dot{f}+A(Ax+f))/2]+h\varepsilon(h).$$

Thus,

$$h^{-1}\int_{\Delta_2^h} X^{-1}(\tau)H(\tau,x(\tau,h),x(\tau-h,h))\,d\tau =$$

$$= \int\limits_{\Delta_2^h} X^{-1}(\tau)(g(\tau,x,\bar{x}) - h\partial_{\bar{x}}g(\tau,x,\bar{x})(Ax+f)+h\varepsilon(h))\times$$

$$\times\{Ax+f+h[g(\tau,x,\bar{x})(Ax+f)-(\dot{A}x+\dot{f}+A(Ax+f))/2]+h\varepsilon(h)\}\,d\tau =$$

$$= \int\limits_0^\omega X^{-1}(\tau)(g(\tau,x(\tau,h),x(\tau,h))(A(\tau)x(\tau,h)+f(\tau))+h\boldsymbol{\mathcal{H}}(\tau,x(\tau,h)))\,d\tau-$$

$$-\int\limits_{\Delta_1^h} X^{-1}(\tau)g(\tau,x(\tau,h),x(\tau,h))(A(\tau)x(\tau,h)+f(\tau))\,d\tau + h\varepsilon(h), \tag{15}$$

where

$$\mathcal{H}(\tau,x)\equiv g(\tau,x,\bar{x})[g(\tau,x,\bar{x})(A(\tau)x+f(\tau))-(\dot{A}(\tau)x+\dot{f}(\tau)+$$
$$+A(\tau)(A(\tau)x+f(\tau)))/2]-\partial_{\bar{x}}g(\tau,x,\bar{x})(A(\tau)x+f(\tau))^2. \tag{16}$$

Further,

$$\int\limits_{\Delta_1^h} X^{-1}(\tau)g(\tau,x(\tau,h),x(\tau,h))(A(\tau)x(\tau,h)+f(\tau))\,d\tau=h\sum_{i=1}^{m}X_i^{-1}\mathbf{g}_i(x_i)+h\varepsilon(h),$$

where

$$\mathbf{g}_i(x_i)=g(t_i+0,(E+B_i)x_i+a_i,(E+B_i)x_i+a_i)(A_i^{+}((E+B_i)x_i+a_i)+f_i^{+}).$$

In view of (11), (13), and (15), we can write the solvability condition (10) in the form

$$\mathcal{P}_r^{*}X(\omega)\Big\{\int\limits_0^{\omega}[g(\tau,x_0(\tau,c_r^{*})+y(\tau,h),x_0(\tau,c_r^{*})+y(\tau,h))\times$$
$$\times[A(\tau)(x_0(\tau,c_r^{*})+y(\tau,h))+f(\tau)]+h\mathcal{H}(\tau,x_0(\tau,c_r^{*})+y(\tau,h))]\,d\tau+$$
$$+\sum_{i=1}^{m}X_i^{-1}[J_i(x_0(t_i,c_r^{*})+y(t_i,h),x_0(t_i,c_r^{*})+y(t_i,h))\times$$
$$\times(A_i(x_0(t_i,c_r^{*})+y(t_i,h))+f_i)+h\mathbf{J}_i(x_0(t_i,c_r^{*})+y(t_i,h))+$$
$$+g(t_i+0,(E+B_i)(x_0(t_i,c_r^{*})+y(t_i,h))+a_i,x_0(t_i,c_r^{*})+y(t_i,h))\times$$
$$\times(B_i(x_0(t_i,c_r^{*})+y(t_i,h))+a_i)+h(\mathbf{G}_i(x_0(t_i,c_r^{*})+y(t_i,h))-$$
$$-\mathbf{g}_i(x_0(t_i,c_r^{*})+y(t_i,h)))]+h\varepsilon(h,x)\Big\}\;=\;0. \tag{17}$$

We now easily see that (7) is obtained from (17) by passing to the limit as $h\to 0$.

Equation (7) can be called an *equation for the generating amplitudes* (see, for instance, [9] or numerous works of the first author, e.g. [7, 8]) of the problem of finding ω-periodic solutions of the impulsive system with delay (1).

3.2. Reduction of the problem to an operator system in a suitable function space. Now suppose that c_r^{*} is a solution of equation (7). Then the ω–periodic solution $y(t,h)$ of system (9) such that $y(t,0)\equiv 0$ can be represented in the form

$$y(t,h)=X_r(t)c+hy^{(1)}(t,h), \tag{18}$$

where the unknown constant vector $c=c(h)\in\mathbf{R}^r$ must satisfy an equation derived below from (17), while the unknown ω-periodic vector-valued function $y^{(1)}(t,h)$ can be represented as

$$y^{(1)}(t,h)=\Big(\int\limits_0^{\omega}G(t,\tau)H(\tau,x_0(\tau,c_r^{*})+y(\tau,h),x_0(\tau-h,c_r^{*})+y(\tau-h,h))\,d\tau+$$
$$+\sum_{i=1}^{m}G(t,t_i)I_i(x_0(t_i,c_r^{*})+y(t_i,h),x_0(t_i-h,c_r^{*})+y(t_i-h,h))\Big)/h.$$

By arguments similar to those above, we find

$$y^{(1)}(t,h) = \int\limits_0^\omega G(t,\tau) g(\tau, x_0(\tau, c_r^*) + y(\tau,h), x_0(\tau, c_r^*) + y(\tau,h)) \times$$

$$\times [A(\tau)(x_0(\tau, c_r^*) + y(\tau,h)) + f(\tau)]\, d\tau +$$

$$+ \sum_{i=1}^m G(t,t_i)[J_i(x_0(t_i, c_r^*) + y(t_i,h), x_0(t_i, c_r^*) + y(t_i,h)) \times$$

$$\times (A_i(x_0(t_i, c_r^*) + y(t_i,h)) + f_i) +$$

$$+ g(t_i + 0, (E + B_i)(x_0(t_i, c_r^*) + y(t_i,h)) + a_i, x_0(t_i, c_r^*) + y(t_i,h)) \times$$

$$\times (B_i(x_0(t_i, c_r^*) + y(t_i,h)) + a_i)] + \varepsilon(h, x(t,h)). \tag{19}$$

We expand the left-hand side of (17) about the point $y = 0$. We have

$$g(\tau, x_0(\tau, c_r^*) + y, x_0(\tau, c_r^*) + y)(A(\tau)(x_0(\tau, c_r^*) + y) + f(\tau)) = g_0(\tau) + g_1(\tau) y + g_2(\tau, y),$$

where $g_2(\tau, y)$ is such that

$$g_2(\tau, 0) = 0, \ \frac{\partial}{\partial y} g_2(\tau, 0) = 0.$$

Analogously, we have

$$J_i(x_0(t_i, c_r^*) + y, x_0(t_i, c_r^*) + y)(A_i(x_0(t_i, c_r^*) + y) + f_i) +$$

$$+ g(t_i + 0, (E + B_i)(x_0(t_i, c_r^*) + y) + a_i, x_0(t_i, c_r^*) + y)(B_i(x_0(t_i, c_r^*) + y) + a_i) =$$

$$= J_{0i} + J_{1i}(y) + J_{2i}(y),$$

where $J_{2i}(y)$ is such that

$$J_{2i}(0) = 0, \ \frac{\partial}{\partial y} J_{2i}(0) = 0.$$

By virtue of the assumption $F(c_r^*) = 0$, equality (17) now takes the form

$$\mathcal{P}_r^* X(\omega) \Big\{ \int\limits_0^\omega X_{-1}(\tau)(g_1(\tau) y(\tau,h) + g_2(\tau, y(\tau,h)) + h \boldsymbol{\mathcal{H}}(\tau, x_0(\tau, c_r^*) + y(\tau,h)))\, d\tau +$$

$$+ \sum_{i=1}^m X_i^{-1} [J_{1i} y(t_i,h) + J_{2i}(y(t_i,h)) + h \boldsymbol{\mathcal{G}}_i(x_0(t_i, c_r^*) + y(t_i,h))] +$$

$$+ h \varepsilon_1(h,x) \Big\} = 0, \tag{20}$$

where $\boldsymbol{\mathcal{G}}_i(x_i) \equiv \mathbf{J}_i(x_i) + \mathbf{G}_i(x_i) - \mathbf{g}_i(x_i), \ i = \overline{1,m}.$

In view of representation (18), denote

$$\mathcal{B}_0 = \mathcal{P}_r^* X(\omega)\Big(\int_0^\omega X^{-1}(\tau) g_1(\tau) X_r(\tau)\, d\tau \;+\; \sum_{i=1}^m X_i^{-1} J_{1i} X_r(t_i)\Big),$$

which is an $(r \times r)$ matrix. Then

$$\mathcal{B}_0 c = -\mathcal{P}_r^* X(\omega)\Big\{\int_0^\omega X^{-1}(\tau)(h g_1(\tau) y^{(1)}(\tau,h) + g_2(\tau, y(\tau,h)) +$$

$$+ h\mathcal{H}(\tau, x_0(\tau, c_r^*) + y(\tau,h)))\, d\tau + \sum_{i=1}^m X_i^{-1}[h J_{1i} y^{(1)}(t_i,h) +$$

$$+ J_{2i}(y(t_i,h)) + h\mathcal{G}_i(x_0(t_i,c_r^*) + y(t_i,h))] + h\varepsilon_1(h,x)\Big\}. \tag{21}$$

In representation (19), we use the same expansions and obtain

$$y^{(1)}(t,h) = \int_0^\omega G(t,\tau)[g_0(\tau) + g_1(\tau)[X_r(\tau)c + h y^{(1)}(\tau,h)] + g_2(\tau, y(\tau,h))]\, d\tau +$$

$$+ \sum_{i=1}^m G(t,t_i)[J_{0i} + J_{1i}[X_r(t_i)c + h y^{(1)}(t_i,h)] + \; J_{2i}(y(t_i,h))] +$$

$$+ \varepsilon_2(h, x(t,h)). \tag{22}$$

Thus, we have reduced problem (1) to the equivalent operator system (8), (18), (21), (22).

Suppose that $\det \mathcal{B}_0 \neq 0$. It is easy to see [7, 9] that this condition is equivalent to the simplicity of the root $c_r = c_r^*$ of the equation for the generating amplitudes:

$$F(c_r^*) = 0, \ \det \left(\frac{\partial F(c_r)}{\partial c_r}\right)\bigg|_{c_r = c_r^*} \neq 0.$$

This is the so called *critical case of the first order.* Then equation (21) can be solved with respect to c and we obtain [9] a Fredholm operator system of the second type to which a convergent simple iteration method can be applied.

Theorem 2 [6]. *For system (1), let conditions H1–H6 and (5) hold. Then for any simple* ($\det \mathcal{B}_0 \neq 0$) *root* $c_r = c_r^* \in \mathbf{R}^r$ *of equation (7), there exists a constant* $h_0 > 0$ *such that, for* $h \in [0, h_0]$, *system (1) has a unique* ω*-periodic solution* $x(t,h)$ *depending continuously on* h *and such that* $x(t,0) = x_0(t, c_r^*)$. *This solution is determined by a simple iteration method convergent for* $h \in [0, h_0]$.

3.3. Critical case of the second order. Now suppose that $\det \mathcal{B}_0 = 0$. Denote by $\mathcal{B}_0^*$ the matrix transpose to $\mathcal{B}_0$, by $\mathcal{B}_0^+$ its Moore – Penrose pseudoinverse, and by $\mathcal{P}_{\mathcal{B}_0} \neq 0$ and $\mathcal{P}_{\mathcal{B}_0^*}$ the orthoprojectors of $\mathbf{R}^r$ onto Ker $(\mathcal{B}_0)$ and Ker $(\mathcal{B}_0^*)$, respectively.

Equation (21) is solvable with respect to $c \in \mathbf{R}^r$ if and only if its right-hand side belongs to the orthocomplement of Ker $(\mathcal{B}_0^*)$, i.e.,

$$\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int_0^\omega X^{-1}(\tau)(hg_1(\tau)y^{(1)}(\tau,h)+g_2(\tau,y(\tau,h))+$$

$$+h\mathcal{H}(\tau,x_0(\tau,c_r^*)+y(\tau,h)))\,d\tau+\sum_{i=1}^m X_i^{-1}[hJ_{1i}y^{(1)}(t_i,h)+$$

$$+J_{2i}(y(t_i,h))+h\mathcal{G}_i(x_0(t_i,c_r^*)+y(t_i,h))]+h\varepsilon_1(h,x)\Big\}=0. \tag{23}$$

If this equality is satisfied, then from (21), we determine

$$c=-\mathcal{B}_0^+\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int_0^\omega X^{-1}(\tau)(hg_1(\tau)y^{(1)}(\tau,h)+g_2(\tau,y(\tau,h))+$$

$$+h\mathcal{H}(\tau,x_0(\tau,c_r^*)+y(\tau,h)))\,d\tau+$$

$$+\sum_{i=1}^m X_i^{-1}[hJ_{1i}y^{(1)}(t_i,h)+J_{2i}(y(t_i,h))+h\mathcal{G}_i(x_0(t_i,c_r^*)+y(t_i,h))]+$$

$$+h\varepsilon_1(h,x)\Big\}\;+\;c^{(1)}\;\equiv\;c^{(0)}+c^{(1)},$$

where $c^{(1)}$ is an arbitrary constant vector in $\operatorname{Ker}(\mathcal{B}_0)$, $c^{(1)}=\mathcal{P}_{\mathcal{B}_0}c$, $c^{(0)}=(\mathrm{Id}-\mathcal{P}_{\mathcal{B}_0})c$. Then equality (22) can be rewritten in the form

$$y^{(1)}(t,h)=G_1(t)c^{(1)}+y^{(2)}(t,h), \tag{24}$$

where

$$G_1(t)=\int_0^\omega G(t,\tau)g_1(\tau)X_r(\tau)\,d\tau\;+\;\sum_{i=1}^m G(t,t_i)J_{1i}X_r(t_i)$$

and

$$y^{(2)}(t,h)=\int_0^\omega G(t,\tau)[g_0(\tau)+g_1(\tau)[X_r(\tau)c^{(0)}+hy^{(1)}(\tau,h)]+g_2(\tau,y(\tau,h))]\,d\tau+$$

$$+\sum_{i=1}^m G(t,t_i)[J_{0i}+J_{1i}[X_r(t_i)c^{(0)}+hy^{(1)}(t_i,h)]+\;J_{2i}(y(t_i,h))]+$$

$$+\,\varepsilon_2(h,x(t,h)). \tag{25}$$

Now let us linearize the solvability condition (23) with respect to y. We use the expansion

$$\mathcal{H}(\tau,x_0(\tau,c_r^*)+y)=\mathcal{H}(\tau,x_0(\tau,c_r^*))+\mathcal{H}_1(\tau)y+\mathcal{H}_2(\tau,y),$$

where

$$\mathcal{H}_1(\tau)=\partial_x\mathcal{H}(\tau,x)\Big|_{x\,=\,x_0(\tau,c_r^*)},\qquad \mathcal{H}_2(\tau,0)=0,\;\partial_y\mathcal{H}_2(\tau,0)=0,$$

and, analogously,

$$\mathcal{G}_i(x_0(t_i,c_r^*)+y_i)=\mathcal{G}_i(x_0(t_i,c_r^*))+\mathcal{G}_{1i}y_i+\mathcal{G}_{2i}(y_i),$$

$$\mathcal{G}_{1i}=\partial_x\mathcal{G}_i(x)\Big|_{x\,=\,x_0(t_i,c_r^*)},\qquad \mathcal{G}_{2i}(0)=0,\ \partial_y\mathcal{G}_{2i}(0)=0.$$

In view of equalities (12), (14), (16), these expansions require the existence of the continuous derivatives $\partial_{x\bar{x}}g(t,x,\bar{x})$, $\partial_{\bar{x}\bar{x}}g(t,x,\bar{x})$, $\partial_{x\bar{x}}J_i(x,\bar{x})$, $\partial_{\bar{x}\bar{x}}J_i(x,\bar{x})$ ensured by the following condition H8:

H8. Conditions H2, H3 still hold if the functions g and J_i are replaced by the partial derivatives $\partial_{\bar{x}}g$ and $\partial_{\bar{x}}J_i$, respectively.

Let us denote

$$\gamma(h)\equiv\int\limits_0^{\omega}X^{-1}(\tau)\mathcal{H}(\tau,x_0(\tau,c_r^*))\,d\tau\ +\sum_{i=1}^{m}X_i^{-1}\mathcal{G}_i(x_0(t_i,c_r^*))+\ \varepsilon_1(h,x_0),$$

$$\tilde{\varepsilon}_1(h,y)\equiv\varepsilon_1(h,x_0+y)-\varepsilon_1(h,x_0).$$

Thus, $\tilde{\varepsilon}_1(h,0)=\tilde{\varepsilon}_1(0,y)=0$, while the quantity $\varepsilon_1(h,x_0)$ depends just on the generating solution $x_0(t,c_r^*)$, and so does $\gamma(h)$.

We can now rewrite equality (23) in the form

$$\begin{aligned}&\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int_0^{\omega}X^{-1}(\tau)(hg_1(\tau)y^{(1)}(\tau,h)+g_2(\tau,y(\tau,h))+\\&+h\mathcal{H}_1(\tau)y(\tau,h)+h\mathcal{H}_2(\tau,y(\tau,h)))\,d\tau+\\&+\sum_{i=1}^{m}X_i^{-1}[hJ_{1i}y^{(1)}(t_i,h)+J_{2i}(y(t_i,h))+h(\mathcal{G}_{1i}y(t_i,h)+\mathcal{G}_{2i}(y(t_i,h)))]+\\&+h\gamma(h)\ +\ h\tilde{\varepsilon}_1(h,y)\Big\}\ =\ 0.\end{aligned}\tag{26}$$

We substitute (24) into (26) to obtain a system with respect to $c^{(1)}=\mathcal{P}_{\mathcal{B}_0}c\in\mathrm{Ker}(\mathcal{B}_0)$:

$$\begin{aligned}&h\mathcal{B}_1c^{(1)}=-\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int\limits_0^{\omega}X^{-1}(\tau)(hg_1(\tau)y^{(2)}(\tau,h)+g_2(\tau,y(\tau,h))+\\&+h\mathcal{H}_1(\tau)(X_r(\tau)c^{(0)}+hy^{(1)}(\tau,h))+h\mathcal{H}_2(\tau,y(\tau,h)))\,d\tau+\\&+\sum_{i=1}^{m}X_i^{-1}[hJ_{1i}y^{(2)}(t_i,h)+J_{2i}(y(t_i,h))+\\&+h(\mathcal{G}_{1i}(X_r(t_i)c^{(0)}+hy^{(1)}(t_i,h))+\mathcal{G}_{2i}(y(t_i,h)))]+h\gamma(h)+h\tilde{\varepsilon}_1(h,y)\Big\},\end{aligned}\tag{27}$$

where

$$\mathcal{B}_1 = \mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int\limits_0^\omega X^{-1}(\tau)(g_1(\tau)G_1(\tau) + \boldsymbol{\mathcal{H}}_1(\tau)X_r(\tau))\,d\tau +$$

$$+\sum_{i=1}^m X_i^{-1}(J_{1i}G_1(t_i) + \boldsymbol{\mathcal{G}}_{1i}X_r(t_i))\Big\}\mathcal{P}_{\mathcal{B}_0}$$

is an $(r \times r)$ matrix.

As above, denote by $\mathcal{B}_1^*$ the matrix transpose to $\mathcal{B}_1$, by $\mathcal{B}_1^+$ its Moore – Penrose pseudoinverse, and by $\mathcal{P}_{\mathcal{B}_1}$ and $\mathcal{P}_{\mathcal{B}_1^*}$ the orthoprojectors of $\mathbf{R}^r$ onto $\mathrm{Ker}(\mathcal{B}_1)$ and $\mathrm{Ker}\,(\mathcal{B}_1^*)$, respectively. System (27) is solvable with respect to $hc^{(1)} \in \mathrm{Ker}\,(\mathcal{B}_0)$ if and only if its right-hand side belongs to the orthocomplement of $\mathrm{Ker}\,(\mathcal{B}_1^*)$, i.e.,

$$\mathcal{P}_{\mathcal{B}_1^*}\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int\limits_0^\omega X^{-1}(\tau)(hg_1(\tau)y^{(2)}(\tau,h) + g_2(\tau,y(\tau,h))+$$

$$+h\boldsymbol{\mathcal{H}}_1(\tau)(X_r(\tau)c^{(0)} + hy^{(1)}(\tau,h)) + h\boldsymbol{\mathcal{H}}_2(\tau,y(\tau,h)))\,d\tau+$$

$$+\sum_{i=1}^m X_i^{-1}[hJ_{1i}y^{(2)}(t_i,h) + J_{2i}(y(t_i,h))+$$

$$+h(\boldsymbol{\mathcal{G}}_{1i}(X_r(t_i)c^{(0)} + hy^{(1)}(t_i,h)) + \boldsymbol{\mathcal{G}}_{2i}(y(t_i,h)))] + h\gamma(h) + h\tilde{\varepsilon}_1(h,y)\Big\} \;=\; 0.$$

Since no additional constraints are imposed on the solution sought $y(t,h)$, the above condition is fulfilled if $\mathcal{P}_{\mathcal{B}_1^*}\mathcal{P}_{\mathcal{B}_0^*} \;=\; 0$, i.e., $\mathrm{Ker}(\mathcal{B}_1^*) \cap \mathrm{Ker}(\mathcal{B}_0^*) \;=\; \{0\}$. It is easy to see that this condition is equivalent to $\mathcal{P}_{\mathcal{B}_0}\mathcal{P}_{\mathcal{B}_1} \;=\; 0$.

Thus, if

$$\mathcal{P}_{\mathcal{B}_0} \neq 0, \quad \mathcal{P}_{\mathcal{B}_0}\mathcal{P}_{\mathcal{B}_1} \;=\; 0, \tag{28}$$

then system (27) is uniquely solvable with respect to $hc^{(1)} \in \mathrm{Ker}(\mathcal{B}_0)$, and the operator system (8), (18), (21), (22) is reduced to the operator system (8),

$$y(t,h) = X_r(t)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)} + hG_1(t)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + hy^{(2)}(t,h),$$

$$c^{(0)} \;=\; -\mathcal{B}_0^+\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int\limits_0^\omega X^{-1}(\tau)(hg_1(\tau)(G_1(\tau)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(\tau,h))+$$

$$+\;g_2(\tau,y(\tau,h)) \;+\; h\boldsymbol{\mathcal{H}}(\tau,x_0(\tau,c_r^*) + y(\tau,h)))\,d\tau+$$

$$+\sum_{i=1}^m X_i^{-1}[hJ_{1i}(G_1(t_i)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(t_i,h) + J_{2i}(y(t_i,h))+$$

$$+\,h\boldsymbol{\mathcal{G}}_i(x_0(t_i,c_r^*) + y(t_i,h))] + h\varepsilon_1(h,x)\Big\}, \tag{29}$$

$$hc^{(1)} = -\mathcal{B}_1^+\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int_0^\omega X^{-1}(\tau)(hg_1(\tau)y^{(2)}(\tau,h) + g_2(\tau, y(\tau,h))+$$

$$+h\mathcal{H}_1(\tau)(X_r(\tau)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)} + hy^{(1)}(\tau,h)) + h\mathcal{H}_2(\tau, y(\tau,h)))\,d\tau+$$

$$+\sum_{i=1}^m X_i^{-1}[hJ_{1i}y^{(2)}(t_i,h) + J_{2i}(y(t_i,h)) + h(\mathcal{G}_{1i}(X_r(t_i)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)}+$$

$$+hy^{(1)}(t_i,h)) + \mathcal{G}_{2i}(y(t_i,h)))] + h\gamma(h) + h\tilde{\varepsilon}_1(h,y)\Big\}, \tag{30}$$

$$y^{(2)}(t,h) = \int_0^\omega G(t,\tau)[g_0(\tau) + g_1(\tau)[X_r(\tau)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)}+$$

$$+h(G_1(\tau)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(\tau,h))] + g_2(\tau, y(\tau,h))]\,d\tau+$$

$$+\sum_{i=1}^m G(t,t_i)[J_{0i} + J_{1i}[X_r(t_i)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)}+$$

$$+h(G_1(t_i)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(t_i,h)] + J_{2i}(y(t_i,h))] + \varepsilon_2(h, x(t,h)).$$

To this system, a convergent simple iteration method can be applied [9] starting with, say,

$$y_k(t,h) = y_k^{(2)}(t,h) = 0, \; k = 0,1, \tag{31}$$

and the following assertion is valid:

Theorem 3. *For system (1), let conditions H1 - H8 and (5) hold. Let $c_r = c_r^* \in \mathbf{R}^r$ be a root of equation (7) such that (28) is satisfied as well as the condition*

$$\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int_0^\omega X^{-1}(\tau)\mathcal{H}(\tau, x_0(\tau, c_r^*))\,d\tau+$$

$$+\sum_{i=1}^m X_i^{-1}\mathcal{G}_i(x_0(t_i, c_r^*)) + \varepsilon_1(h, x_0)\Big\} = 0. \tag{32}$$

Then there exists a constant $h_0 > 0$ such that, for $h \in [0, h_0]$, system (1) has a unique ω-periodic solution $x(t,h)$ such that $x(t,0) = x_0(t, c_r^*)$. This solution can be determined by a simple iteration method convergent for $h \in [0, h_0]$.

Condition (32) is obtained from the solvability condition (26) by virtue of (31). It is a necessary and sufficient condition for finding

$$c_0^{(0)} = -h\mathcal{B}_0^+\mathcal{P}_{Q_r^*}X(\omega)\Big\{\int_0^\omega X^{-1}(\tau)\mathcal{H}(\tau, x_0(\tau, c_r^*))\,d\tau+$$

$$+\sum_{i=1}^m X_i^{-1}\mathcal{G}_i(x_0(t_i, c_r^*)) + \varepsilon_1(h, x_0)\Big\}$$

from (21). The finding of the subsequent iterations of $c^{(0)}$ from (29) is enabled by the choice of the corresponding iterations of $c^{(1)}$ from (30).

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