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STUDYING THE STABILITY OF EQUILIBRIUM SOLUTIONS IN THE PLANAR CIRCULAR RESTRICTED FOUR-BODY PROBLEM

ВИВЧЕННЯ СТІЙКОСТІ РІВНОВАЖНИХ РОЗВ'ЯЗКІВ У ПЛАНАРНІЙ КРУГОВІЙ ЗРІЗАНІЙ ПРОБЛЕМІ ЧОТИРЬОХ ТІЛ

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The Newtonian circular restricted four-body problem is considered. We have obtained nonlinear algebraic equations, determining equilibrium solutions in the rotating frame, and found six possible equilibrium configurations of the system. Studying the stability of equilibrium solutions, we have proved that the radial equilibrium solutions are unstable while the bisector equilibrium solutions are stable in Liapunov's sense if the mass parameter satisfies $\mu \in (0, \mu_0)$, where μ_0 is0 a sufficiently small number, and $\mu \neq \mu_j,\ j = 1, 2, 3$. We have also proved that for $\mu = \mu_1$ and $\mu = \mu_3$ the resonance conditions of the third and the fourth orders, respectively, are fulfilled and for these values of μ the bisector equilibrium are unstable and stable in Liapunov's sense, respectively. All symbolic and numerical calculations are done with the computer algebra system Mathematica.

Розглядається кругова зрізана проблема ньютонівської динаміки для чотирьох тіл. Отримано нелінійні алгебраїчні рівняння, що визначають рівноважні розв'язки відносно обертаючої системи відліку, та знайдено шість рівноважних конфігурацій системи. При вивченні стійкості рівноважних розв'язків доведено, що радіальні рівноважні розв'язки є нестійкими, проте бісекторні рівноважні розв'язки є стійкими за Ляпуновим, якщо параметр маси $\mu \in (0, \mu_0)$, де μ_0 — досить мале число, і $\mu \neq \mu_j,\ j = 1, 2, 3$. Також доведено, що для $\mu = \mu_1$ та $\mu = \mu_3$ умови резонансу відповідно третього та четвертого порядків виконано, і для цих значень μ бісекторні рівноважні розв'язки є відповідно нестійкими та стійкими за Ляпуновим. Усі символьні та числові обчислення виконано за допомогою системи комп'ютерної алгебри „Математика".

Introduction. In the fifties-sixties of the last century one of the authors of the present paper had a happy occasion to attend lectures on celestial mechanics given by the outstanding scientist, bright representative of the world famous Ukrainian Mathematical School, academician Yuri Mitropolskii. Those lectures were devoted to analytical and qualitative investigations of the famous problem in astronomy and classic dynamics, namely, the Newtonian problem of three and more bodies. It was noted that Gauss, together with Laplace, was one of the first who demonstrated efficiency of the averaging methods in developing the analytical theories for planets motion and investigated the general problem of integrability of differential equations in the Newtonian three-body problem on the basis of these methods. It followed from the lectures that Gauss proposed a method of constructing a general integral for the averaged systems of differential equations in the restricted three-body problem long before "the theorems on cyclic coordinates"appeared (in the planar case of the problem a single averaging in one fast variable is used while in the spacial case double averaging is necessary).

General integrals for the mentioned differential equations are expressed via complex implicit functions of phase variables and here a problem, being topical for all times, arises, — how to construct a general solution of differential equations in celestial mechanics or how to obtain phase variables in the form of explicit functions of time? This means that it is necessary to develop constructive methods for obtaining coordinates as explicit functions of time if we are investigating a motion of some concrete object in asymptotically large time intervals.

Remarkable lectures of Yu. Mitropolskii convinced us that constructive methods of nonlinear analysis were and will be in the future used for solving the problems of celestial mechanics and astrodynamics. And the present paper may be considered as an illustration of this statement.

It is well-known that for the two-body problem, first stated and solved by Newton, a complete and general solution exist. Using this solution it is natural to study the motion of a third particle of infinitesimal mass under the influence of the gravitational attraction of two massive particles moving in circular or elliptic orbits about their center of mass. This model is a special case of the three-body problem and is known as the restricted three-body problem [1]. It is very interesting from the theoretical point of view and is widely used in astronomy and cosmic dynamics. Many eminent mathematicians have investigated it in the last three centuries and a great progress in this field has been achieved [2]. This problem stimulated development of new qualitative, analytical and asymptotic methods for studying nonlinear Hamiltonian systems [3 – 5]. Nevertheless, the development of the stability theory of Hamiltonian systems is not completed yet and investigations in this field are very topical.

In [6 – 8] a new class of the exact particular solutions in the planar Newtonian many-body problem was found. On this basis two new dynamical models were proposed which are known as Newtonian restricted many-body problems [9, 10]. They seem to be not integrable in general. Hence, similarly to the case of the restricted three-body problem, one can start their studying from seeking exact particular solutions of the equations of motion and investigating their stability. In the simplest case of four interacting particles it was shown [11] that there exist six equilibrium solutions but only two of them are stable in linear approximation. In the present paper the stability problem is resolved in a strict nonlinear formulation. It should be emphasized that this problem is very complicated and it can be solved only on the basis of the KAM-theory [12 – 14]. Besides, very cumbersome symbolic calculations are involved, which can be reasonably done only with a modern computer software. Here all symbolic and numeric calculations are done with the computer algebra system *Mathematica* [15].

Equations of motion and their equilibrium solutions. Let two particles P_1, P_2 having equal masses m move uniformly in a circular orbit about their common center of mass, where the third particle P_0 of mass m_0 rests. The orbit is situated in the Oxy plane of the barycentric inertial frame of reference and its center is in the origin. The particles attract each other according to the Newtonian law of gravitation and form symmetric with respect to the origin configuration at any instant of time (see Fig. 1). The corresponding solution of the three-body problem is well-known [1] and angular velocity of the particles is given by

$$\frac{d\nu}{dt} \equiv \omega_0 = \left(\frac{Gm_0(4+\mu)}{4r_0^3}\right)^{1/2}, \quad (1)$$

where G is the gravity constant, ν is the polar angle, r_0 is the radius of the circle and the mass parameter $\mu = m/m_0$.

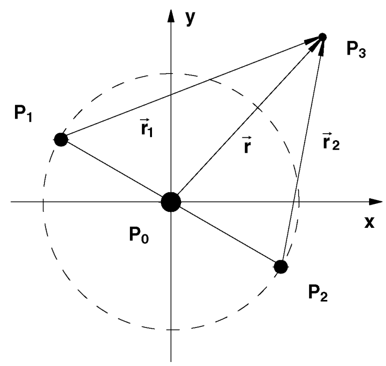

Fig. 1. Geometrical configuration of the system.

Let us consider the motion of the particle P_3 of infinitesimal mass m_1 in the gravitational field generated by the particles P_0, P_1, P_2. Denoting their polar coordinates by ρ, φ, we can write the Lagrangian of the system in the form

$$L = \frac{m_1}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + Gm_1\left(\frac{m_0}{r} + \frac{m}{r_1} + \frac{m}{r_2}\right), \quad (2)$$

where

$$r = \rho, \;\; r_j = \sqrt{\rho^2 + \rho_j^2 - 2\rho\rho_j\cos(\varphi - \varphi_j)}, \quad j = 1, 2,$$

are the distances between the particle P_3 and the particles P_0, P_1, P_2, respectively, and the dot denotes the derivative d/dt.

Obviously, it is convenient to analyze the motion of the particle P_3 in the frame of reference rotating about Oz axis with the angular velocity ω_0. Taking into account (1), (2), using the polar angle $\nu = \omega_0 t$ as a new independent variable and doing some standard transformations, we can easily obtain the Hamiltonian of the system,

$$H = \frac{1}{2}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} - 2p_\varphi\right) -$$

$$- \frac{4}{4+\mu}\left(\frac{1}{\rho} + \frac{\mu}{\sqrt{1+\rho^2-2\rho\cos\varphi}} + \frac{\mu}{\sqrt{1+\rho^2+2\rho\cos\varphi}}\right), \quad (3)$$

where p_ρ, p_φ are momenta canonically conjugated to the coordinates ρ, φ. Note that in the rotating frame, the particles $P_0,\ P_1,\ P_2$ rest in the points $(0,\ 0),\ (-1,\ 0),\ (1,\ 0)$, respectively.

With the Hamiltonian (3) the equations of motion of the particle P_3 can be written as

$$\frac{d\rho}{d\nu} = p_\rho\,,\ \ \frac{d\varphi}{d\nu} = \frac{p_\varphi}{\rho^2} - 1,$$

$$\frac{dp_\rho}{d\nu} = \frac{p_\varphi^2}{\rho^3} - \frac{4}{4+\mu}\left(\frac{1}{\rho^2} + \frac{\mu(\rho-\cos\varphi)}{(1+\rho^2-2\rho\cos\varphi)^{3/2}} + \frac{\mu(\rho+\cos\varphi)}{(1+\rho^2+2\rho\cos\varphi)^{3/2}}\right), \tag{4}$$

$$\frac{dp_\varphi}{d\nu} = \frac{4\mu\rho}{4+\mu}\left(\frac{1}{(1+\rho^2+2\rho\cos\varphi)^{3/2}} - \frac{1}{(1+\rho^2-2\rho\cos\varphi)^{3/2}}\right)\sin\varphi.$$

The equilibrium solutions of system (4) are determined from the condition $\rho = R = \mathrm{const},\ \varphi = \beta = \mathrm{const},\ p_\rho = \mathrm{const},\ p_\varphi = \mathrm{const}$. In this case equations (4) take the form

$$p_\rho = 0\,,\ \ p_\varphi = R^2,$$

$$R - \frac{4}{(4+\mu)R^2} - \frac{4\mu}{4+\mu}\left(\frac{R-\cos\beta}{(1+R^2-2R\cos\beta)^{3/2}} + \frac{R+\cos\beta}{(1+R^2+2R\cos\beta)^{3/2}}\right) = 0, \tag{5}$$

$$\left(\frac{1}{(1+R^2+2R\cos\beta)^{3/2}} - \frac{1}{(1+R^2-2R\cos\beta)^{3/2}}\right)\sin\beta = 0.$$

One can readily see that the last equation of (5) is satisfied for any R if $\beta = 0,\ \pi/2,\ \pi,\ 3\pi/2$. In terms of [10], we'll call the equilibrium positions located on the rays $\beta = 0$ and $\beta = \pi$ the radial equilibrium solutions. The corresponding values of R are obtained as roots of the equation

$$\frac{(4+\mu)R}{4} = \frac{1}{R^2} + \frac{\mu}{(1+R)^2} - \frac{\mu(1-R)}{|1-R|^3}. \tag{6}$$

The equilibrium positions on the rays $\beta = \pi/2$ and $\beta = 3\pi/2$ will be called the bisector equilibrium solutions. They are determined as roots of the equation

$$\frac{(4+\mu)R}{4} = \frac{1}{R^2} + \frac{2R\mu}{(1+R)^{3/2}}. \tag{7}$$

Equations (6), (7) are nonlinear and can not be solved analytically. But with the system *Mathematica* we can find their roots numerically with arbitrary precision for any value of the parameter μ. In the case of $\mu = 0$ equation (6) has only one root $R = 1$. But for any $\mu > 0$ there is one root in the interval $0 < R_1 < 1$ and another one $R_2 > 1$ (points $N_1,\ N_3$ and $N_2,\ N_4$ in Fig. 2). They tend to the limits $R_1 = 0$ and $R_2 = 2,39681$, respectively, as $\mu \to \infty$.

Equation (7) also has one root $R = 1$ (points $S_1,\ S_2$ in Fig. 2) if $\mu = 0$ which tends to the limit $R = \sqrt{3}$ as $\mu \to \infty$. Note that this limit corresponds to the famous Lagrange triangular solution for the three-body problem [1, 2].

Thus, there exist six equilibrium positions of the particle P_3 in the rotating frame. They determine circular trajectories of this particle in the barycentric inertial frame of reference and, hence, there exist six equilibrium solutions in the restricted four-body problem.

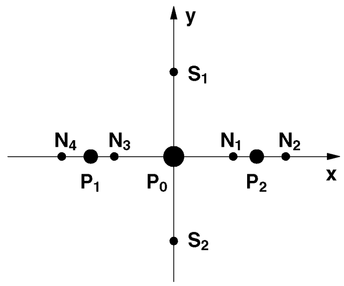

Fig. 2. Equilibrium positions of the particle P_3 in the Oxy plane.

Stability analysis in linear approximation. To study the stability of equilibrium solutions let us make the following canonical transformation:

$$\rho \to R + u,\ \varphi \to \beta + \frac{\gamma}{R},\ P_\rho \to v,\ P_\varphi \to R(R + \omega). \tag{8}$$

Then considering the functions $u(\nu)$, $\gamma(\nu)$, $v(\nu)$, $\omega(\nu)$ as small perturbations of the equilibrium solutions, we can expand the Hamiltonian (3) in the Taylor series in powers of u, γ, v and ω and neglect all terms of the third and higher orders. The first term in the obtained expansion depends only on the independent variable ν and it may be omitted because it does not influence the equations of motion. The the first order term in perturbations is equal to zero as a consequence of the equations for equilibrium solutions (5) – (7). So we obtain the Hamiltonian in the form

$$H_2 = \frac{1}{2}\left(v^2 + \omega^2 - 4u\omega + au^2 + b\gamma^2\right), \tag{9}$$

where

$$a = 1 + \frac{8\mu}{4+\mu}\left(\frac{2}{(R^2+1)^{3/2}} - \frac{1}{(1-R)^3} - \frac{1}{(R+1)^3}\right), \qquad b = \frac{8\mu(3+R^2)}{(4+\mu)(1-R^2)^3}, \tag{10}$$

$$a = 1 + \frac{8\mu}{4+\mu}\left(\frac{2}{(R^2+1)^{3/2}} - \frac{1}{(R-1)^3} - \frac{1}{(R+1)^3}\right), \qquad b = \frac{8\mu(1+3R^2)}{(4+\mu)R(R^2-1)^3} \tag{11}$$

for the radial equilibrium solutions $N_1,\ N_3$ and $N_2,\ N_4$, respectively, and

$$a = 1 + \frac{24\mu}{(4+\mu)(1+R^2)^{5/2}}, \qquad b = -\frac{24\mu}{(4+\mu)(1+R^2)^{5/2}} \tag{12}$$

in the case of bisector equilibrium solutions $S_1,\ S_2$. With the Hamiltonian (9) we obtain the equations of the disturbed motion in the form

$$\begin{aligned} \frac{du}{d\nu} &= \frac{\partial H_2}{\partial v} = v, \qquad & \frac{dv}{d\nu} &= -\frac{\partial H_2}{\partial u} = 2\omega - au, \\ \frac{d\gamma}{d\nu} &= \frac{\partial H_2}{\partial \omega} = \omega - 2u, \qquad & \frac{d\omega}{d\nu} &= -\frac{\partial H_2}{\partial \gamma} = -b\gamma. \end{aligned} \tag{13}$$

One can readily see that we have obtained a system of four linear differential equations of the first order with constant coefficients. Behavior of its solutions is determined by the corresponding characteristic exponents which can easily be found and may be written in the form

$$\lambda = \pm i\sigma_{1,2}, \tag{14}$$

where i is the imaginary unit ($i^2 = -1$) and

$$\sigma_{1,2} = \frac{1}{\sqrt{2}}\left(a + b \pm \sqrt{a^2 + 16b - 2ab + b^2}\right)^{1/2}. \tag{15}$$

Using the system *Mathematica*, we have investigated characteristic exponents (14) as functions of the parameter μ numerically. It turned out that for all radial equilibrium solutions there is one characteristic exponent (14) with a positive real part for any μ from the interval $0 \le \mu < \infty$. So, according to the Liapunov's theorem on linearized stability [16], we can conclude that radial equilibrium solutions of the circular restricted problem of four bodies are unstable. The same is true for the bisector equilibrium solutions if the parameter μ is large enough. But our calculations have shown that there is the value $\mu_0 = 0,0853217$ such that for

$$0 < \mu < \mu_0 \tag{16}$$

characteristic exponents (14) are different pure imaginary numbers. Note that for such values of the parameter μ, expression (15) takes the form

$$\sigma_{1,2} = \frac{1}{\sqrt{2}}\left(1 \pm \sqrt{1 + 12b + 4b^2}\right)^{1/2}, \tag{17}$$

where the parameter b is defined in (12) and the imaginary parts σ_1, σ_2 of the characteristic exponents satisfy the following inequalities:

$$0 < \sigma_2 < \frac{1}{\sqrt{2}} < \sigma_1 < 1. \tag{18}$$

Thus, the bisector equilibrium solutions are stable in linear approximation if the parameter μ belongs to the interval (16).

Normalizing the third order term in the Hamiltonian. It is well-known that the stability problem for a Hamiltonian system of differential equations belongs to the critical case, in Liapunov's sense [16], and it can be resolved only in a strict nonlinear formulation. The most general approach for studying such systems is the Poincare method of normal forms that was used successfully in solving many problems of nonlinear mechanics [3]. According to this method we have to construct the Birkhoff canonical transformation [17] that reduces the Hamiltonian function to some simplest form when the equations of motion can be solved. In a neighborhood of the bisector equilibrium, solutions the Hamiltonian (3) can be expanded in a Taylor series in powers of the perturbations u, γ, v and ω and can be represented in the form

$$H = H_2 + H_3 + H_4 + \ldots, \tag{19}$$

where the quadratic part H_2 is defined in (9) and

$$H_3 = -\frac{6+2b+(6+7b)R^2}{6R(1+R^2)}u^3 + \frac{3u^2\omega}{R} - \frac{u\omega^2}{R} - \frac{(2-3R^2)b}{2R(1+R^2)}u\gamma^2, \tag{20}$$

$$H_4 = \frac{36(1+R^2)^2+b(8+21R^2+48R^4)}{24R^2(1+R^2)^2}u^4 + \frac{b}{24}\left(\frac{4}{R^2} - \frac{35}{(1+R^2)^2}\right)\gamma^4 -$$

$$- \frac{4u^3\omega}{R^2} + \frac{3u^2\omega^2}{2R^2} - \frac{(2-21R^2+12R^4)b}{4R^2(1+R^2)^2}u^2\gamma^2. \tag{21}$$

Thus, we have to normalize successively the terms $H_2, H_3, \ldots$ in the expansion (19).

First of all, let us normalize the quadratic part H_2 of the Hamiltonian. Using the algorithm proposed in [2], we construct a canonical transformation of the form

$$\begin{aligned} u &= 2c_1p_1 + 2c_2p_2, \qquad \gamma = -\frac{c_1\sigma_1(b+\sigma_2^2)}{b}q_1 + \frac{c_2\sigma_2(b+\sigma_1^2)}{b}q_2, \\ v &= -2c_1\sigma_1q_1 + 2c_2\sigma_2q_2, \qquad \omega = c_1(b+\sigma_2^2)p_1 + c_2(b+\sigma_1^2)p_2, \end{aligned} \tag{22}$$

where

$$c_1 = \left(\frac{\sigma_1}{(\sigma_1^2-\sigma_2^2)(3-b+\sigma_1^2)}\right)^{1/2}, \qquad c_2 = \left(\frac{\sigma_2}{(\sigma_1^2-\sigma_2^2)(3-b+\sigma_2^2)}\right)^{1/2}.$$

Then the quadratic part H_2 of the Hamiltonian (3) takes a normal form,

$$H_2 = \frac{1}{2}\left(\sigma_1(p_1^2+q_1^2) - \sigma_2(p_2^2+q_2^2)\right). \tag{23}$$

Here p_1, q_1 and p_2, q_2 are two pairs of canonically conjugated variables.

Substituting (22) into (20) we can rewrite the third order term H_3 in the form

$$H_3 = \sum_{i+j+k+l=3} h_{ijkl}^{(3)} q_1^i q_2^j p_1^k p_2^l, \tag{24}$$

where all non-zero terms $h^{(3)}_{ijkl}$ are given by

$$h^{(3)}_{0030} = -\frac{2c_1^3}{3R}\left(\frac{2b(R^2-4)}{1+R^2} - 3 + 3b^2 + 6(2-b)\sigma_1^2 + 3\sigma_1^4\right),$$

$$h^{(3)}_{0003} = -\frac{2c_2^3}{3R}\left(\frac{2b(R^2-4)}{1+R^2} - 3 + 3b^2 + 6(2-b)\sigma_2^2 + 3\sigma_2^4\right),$$

$$h^{(3)}_{0012} = \frac{2c_1c_2^2}{R}\left(\frac{3+8b-3b^2+3R^2-2bR^2-3b^2R^2}{1+R^2} + 4(b-2)\sigma_2^2 - \sigma_2^4 + 2\sigma_1^2(b-2-\sigma_2^2)\right),$$

$$h^{(3)}_{0021} = \frac{2c_1^2c_2}{R}\left(\frac{3+8b-3b^2+3R^2-2bR^2-3b^2R^2}{1+R^2} + 2(b-2)\sigma_2^2 - \sigma_1^4 + 2\sigma_1^2(2b-4-\sigma_2^2)\right),$$

$$h^{(3)}_{2010} = \frac{c_1^3(3R^2-2)\sigma_1^2(b+\sigma_2^2)^2}{bR(1+R^2)}, \qquad h^{(3)}_{2001} = \frac{c_1^2c_2(3R^2-2)\sigma_1^2(b+\sigma_2^2)^2}{bR(1+R^2)}, \tag{25}$$

$$h^{(3)}_{0210} = \frac{c_1c_2^2(3R^2-2)\sigma_2^2(b+\sigma_1^2)^2}{bR(1+R^2)}, \qquad h^{(3)}_{0201} = \frac{c_2^3(3R^2-2)\sigma_2^2(b+\sigma_1^2)^2}{bR(1+R^2)},$$

$$h^{(3)}_{1110} = \frac{2(2-3R^2)c_1^2c_2\sigma_1\sigma_2(b+\sigma_1^2)(b+\sigma_2^2)}{bR(1+R^2)},$$

$$h^{(3)}_{1101} = \frac{2(2-3R^2)c_1c_2^2\sigma_1\sigma_2(b+\sigma_1^2)(b+\sigma_2^2)}{bR(1+R^2)}.$$

The second step in normalizing the Hamiltonian is to find a canonical transformation such that the third order term H_3 would be cancelled. Following Birkhoff [17], we can try to find the corresponding generating function in the form

$$S(\tilde{p}_1,\tilde{p}_2,q_1,q_2) = q_1\tilde{p}_1 + q_2\tilde{p}_2 + \sum_{i+j+k+l=3} s^{(3)}_{ijkl} q_1^i q_2^j p_1^k p_2^l. \tag{26}$$

Then the new momenta $\tilde{p}_1, \tilde{p}_2$ and the coordinates $\tilde{q}_1, \tilde{q}_2$ are determined by the following relationships:

$$\tilde{q}_1 = \frac{\partial S}{\partial \tilde{p}_1},\ \tilde{q}_2 = \frac{\partial S}{\partial \tilde{p}_2},\ p_1 = \frac{\partial S}{\partial q_1},\ p_2 = \frac{\partial S}{\partial q_2}. \tag{27}$$

These relationships are just equations with respect to the old canonical variables q_1, q_2, p_1, p_2 which are analytic functions in neighborhood of the point $\tilde{q}_1 = \tilde{q}_2 = \tilde{p}_1 = \tilde{p}_2 = 0$ if $\tilde{q}_1$, $\tilde{q}_2$, $\tilde{p}_1$ and $\tilde{p}_2$ are sufficiently small. Hence, substituting (26) into (27) and taking into account the terms up to the second order in $\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2$, we can rewrite it in the form

$$\begin{aligned}
q_1 &= \tilde{q}_1 - s^{(3)}_{0012}\tilde{p}_2^2 - 2s^{(3)}_{0021}\tilde{p}_1\tilde{p}_2 - 3s^{(3)}_{0030}\tilde{p}_1^2 - s^{(3)}_{0111}\tilde{p}_2\tilde{q}_2 - 2s^{(3)}_{0120}\tilde{p}_1\tilde{q}_2 - s^{(3)}_{0210}\tilde{q}_2^2 - \\
&\quad - s^{(3)}_{1011}\tilde{p}_2\tilde{q}_1 - 2s^{(3)}_{1020}\tilde{p}_1\tilde{q}_1 - s^{(3)}_{1110}\tilde{q}_1\tilde{q}_2 - s^{(3)}_{2010}\tilde{q}_1^2, \\
q_2 &= \tilde{q}_2 - 3s^{(3)}_{0003}\tilde{p}_2^2 - 2s^{(3)}_{0012}\tilde{p}_1\tilde{p}_2 - s^{(3)}_{0021}\tilde{p}_1^2 - 2s^{(3)}_{0102}\tilde{p}_2\tilde{q}_2 - s^{(3)}_{0111}\tilde{p}_1\tilde{q}_2 - s^{(3)}_{0201}\tilde{q}_2^2 - \\
&\quad - 2s^{(3)}_{1002}\tilde{p}_2\tilde{q}_1 - s^{(3)}_{1011}\tilde{p}_1\tilde{q}_1 - s^{(3)}_{1101}\tilde{q}_1\tilde{q}_2 - s^{(3)}_{2001}\tilde{q}_1^2, \\
p_1 &= \tilde{p}_1 + s^{(3)}_{1002}\tilde{p}_2^2 + s^{(3)}_{1011}\tilde{p}_1\tilde{p}_2 + s^{(3)}_{1020}\tilde{p}_1^2 + s^{(3)}_{1101}\tilde{p}_2\tilde{q}_2 + s^{(3)}_{1110}\tilde{p}_1\tilde{q}_2 + s^{(3)}_{1200}\tilde{q}_2^2 + \\
&\quad + 2s^{(3)}_{2001}\tilde{p}_2\tilde{q}_1 + 2s^{(3)}_{2010}\tilde{p}_1\tilde{q}_1 + 2s^{(3)}_{2100}\tilde{q}_1\tilde{q}_2 + 3s^{(3)}_{3000}\tilde{q}_1^2, \\
p_2 &= \tilde{p}_2 + s^{(3)}_{0102}\tilde{p}_2^2 + s^{(3)}_{0111}\tilde{p}_1\tilde{p}_2 + s^{(3)}_{0120}\tilde{p}_1^2 + 2s^{(3)}_{0201}\tilde{p}_2\tilde{q}_2 + 2s^{(3)}_{0210}\tilde{p}_1\tilde{q}_2 + 3s^{(3)}_{0300}\tilde{q}_2^2 + \\
&\quad + s^{(3)}_{1101}\tilde{p}_2\tilde{q}_1 + s^{(3)}_{1110}\tilde{p}_1\tilde{q}_1 + 2s^{(3)}_{1200}\tilde{q}_1\tilde{q}_2 + s^{(3)}_{2100}\tilde{q}_1^2.
\end{aligned} \tag{28}$$

Now we can substitute (28) into (19) and expand the Hamiltonian H in a Taylor series in powers of $\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2$. One can readily check that the second order term $\tilde{H}_2$ in this expansion will have the form (23) while the third order term $\tilde{H}_3$ will be a sum of twenty terms of the form

$$\tilde{h}^{(3)}_{ijkl}\tilde{q}_1^i\tilde{q}_2^j\tilde{p}_1^k\tilde{p}_2^l, \quad i+j+k+l = 3.$$

Coefficients $\tilde{h}^{(3)}_{ijkl}$ can be divided into three independent groups. The first group is

$$\begin{aligned}
\tilde{h}^{(3)}_{0012} &= h^{(3)}_{0012} + s^{(3)}_{1002}\sigma_1 - s^{(3)}_{0111}\sigma_2, \\
\tilde{h}^{(3)}_{0210} &= h^{(3)}_{0210} + s^{(3)}_{1200}\sigma_1 + s^{(3)}_{0111}\sigma_2, \\
\tilde{h}^{(3)}_{1101} &= h^{(3)}_{1101} - s^{(3)}_{0111}\sigma_1 + 2s^{(3)}_{1002}\sigma_2 - 2s^{(3)}_{1200}\sigma_2
\end{aligned} \tag{29}$$

and corresponds to the coefficients of $\tilde{p}_1\tilde{p}_2^2$, $\tilde{p}_1\tilde{q}_2^2$, $\tilde{q}_1\tilde{q}_2\tilde{p}_2$ in the expression for $\tilde{H}_3$.

Coefficients of $\tilde{q}_2\tilde{p}_1\tilde{p}_2$, $\tilde{q}_1\tilde{p}_2^2$, $\tilde{q}_1\tilde{q}_2^2$ in $\tilde{H}_3$ form the second group and are given by

$$\begin{aligned}
\tilde{h}^{(3)}_{0111} &= s^{(3)}_{1101}\sigma_1 + 2s^{(3)}_{0012}\sigma_2 - 2s^{(3)}_{0210}\sigma_2, \\
\tilde{h}^{(3)}_{1002} &= -s^{(3)}_{0012}\sigma_1 - s^{(3)}_{1101}\sigma_2, \\
\tilde{h}^{(3)}_{1200} &= -s^{(3)}_{0210}\sigma_1 + s^{(3)}_{1101}\sigma_2.
\end{aligned} \tag{30}$$

The rest fourteen coefficients $\tilde{h}^{(3)}_{ijkl}$ form the third group

$$\tilde{h}^{(3)}_{0003} = h^{(3)}_{0003} - s^{(3)}_{0102}\sigma_2, \quad \tilde{h}^{(3)}_{0021} = h^{(3)}_{0021} + s^{(3)}_{1011}\sigma_1 - s^{(3)}_{0120}\sigma_2,$$

$$\tilde{h}^{(3)}_{0030} = h^{(3)}_{0030} + s^{(3)}_{1020}\sigma_1, \quad \tilde{h}^{(3)}_{0102} = 3s^{(3)}_{0003}\sigma_2 - 2s^{(3)}_{0201}\sigma_2,$$

$$\tilde{h}^{(3)}_{0120} = s^{(3)}_{1110}\sigma_1 + s^{(3)}_{0021}\sigma_2, \quad \tilde{h}^{(3)}_{0201} = h^{(3)}_{0201} + 2s^{(3)}_{0102}\sigma_2 - 3s^{(3)}_{0300}\sigma_2,$$

$$\tilde{h}^{(3)}_{0300} = s^{(3)}_{0201}\sigma_2, \quad \tilde{h}^{(3)}_{1011} = -2s^{(3)}_{0021}\sigma_1 + 2s^{(3)}_{2001}\sigma_1 - s^{(3)}_{1110}\sigma_2, \tag{31}$$

$$\tilde{h}^{(3)}_{1020} = -3s^{(3)}_{0030}\sigma_1 + 2s^{(3)}_{2010}\sigma_1, \quad \tilde{h}^{(3)}_{1110} = h^{(3)}_{1110} - 2s^{(3)}_{0120}\sigma_1 + 2s^{(3)}_{2100}\sigma_1 + s^{(3)}_{1011}\sigma_2,$$

$$\tilde{h}^{(3)}_{2001} = h^{(3)}_{2001} - s^{(3)}_{1011}\sigma_1 - s^{(3)}_{2100}\sigma_2, \quad \tilde{h}^{(3)}_{2010} = h^{(3)}_{2010} - 2s^{(3)}_{1020}\sigma_1 + 3s^{(3)}_{3000}\sigma_1,$$

$$\tilde{h}^{(3)}_{2100} = -s^{(3)}_{1110}\sigma_1 + s^{(3)}_{2001}\sigma_2, \quad \tilde{h}^{(3)}_{3000} = -s^{(3)}_{2010}\sigma_1.$$

We can consider relationships (29) – (31) as the equations determining unknown coefficients $s^{(3)}_{ijkl}$ of canonical transformation (28). Recall that we would like to find coefficients $s^{(3)}_{ijkl}$ such that the third order term $\tilde{H}_3$ cancells or all the coefficients $\tilde{h}^{(3)}_{ijkl}$ are equal to zero. One can readily see that the system of three equations (29) determines coefficients $s^{(3)}_{0111}$, $s^{(3)}_{1002}$, $s^{(3)}_{1200}$. The determinant of this system is equal to $\sigma_1(4\sigma_2^2 - \sigma_1^2)$. Hence, if $\sigma_1 \neq 0$ and the condition

$$\sigma_1 \pm 2\sigma_2 \neq 0 \tag{32}$$

is fulfilled, system (29) has a solution. In this case, we can set $\tilde{h}^{(3)}_{ijkl} = 0$ in (29) and find

$$s^{(3)}_{1002} = \frac{1}{\sigma_1^3 - 4\sigma_1\sigma_2^2}(h^{(3)}_{1101}\sigma_1\sigma_2 - h^{(3)}_{0012}\sigma_1^2 + 2\left(h^{(3)}_{0012} + h^{(3)}_{0210})\sigma_2^2\right),$$

$$s^{(3)}_{0111} = \frac{1}{\sigma_1^2 - 4\sigma_2^2}\left(h^{(3)}_{1101}\sigma_1 + 2(h^{(3)}_{0210} - h^{(3)}_{0012})\sigma_2\right), \tag{33}$$

$$s^{(3)}_{1200} = -\frac{1}{\sigma_1^3 - 4\sigma_1\sigma_2^2}\left(h^{(3)}_{1101}\sigma_1\sigma_2 + h^{(3)}_{0210}\sigma_1^2 - 2(h^{(3)}_{0012} + h^{(3)}_{0210})\sigma_2^2\right).$$

The determinant of system (30) for the coefficients $s^{(3)}_{0012}, s^{(3)}_{0210}, s^{(3)}_{1101}$ is equal to $\sigma_1(\sigma_1^2 - 4\sigma_2^2)$. Again, supposing that conditions (32) are fulfilled and setting $\tilde{h}^{(3)}_{ijkl} = 0$ in (30), we obtain

$$s^{(3)}_{0012} = s^{(3)}_{0210} = s^{(3)}_{1101} = 0. \tag{34}$$

The determinant of system (31) is also easily found and equal to $81\sigma_1^4\sigma_2^6(4\sigma_1^2 - \sigma_2^2)^2$. If $\sigma_1 \neq 0,\ \sigma_2 \neq 0$ and

$$2\sigma_1 \pm \sigma_2 \neq 0, \tag{35}$$

we can set $\tilde{h}^{(3)}_{ijkl} = 0$ in (31) and find

$$s^{(3)}_{0300} = \frac{1}{3\sigma_2}(2h^{(3)}_{0003} + h^{(3)}_{0201}), \;\; s^{(3)}_{0102} = \frac{1}{\sigma_2}h^{(3)}_{0003}, \;\; s^{(3)}_{1020} = -\frac{1}{\sigma_1}h^{(3)}_{0030},$$

$$s^{(3)}_{0120} = \frac{1}{4\sigma_1^2\sigma_2 - \sigma_2^3}(h^{(3)}_{1110}\sigma_1\sigma_2 - h^{(3)}_{0021}\sigma_2^2 + 2(h^{(3)}_{0021} + h^{(3)}_{2001})\sigma_1^2),$$

$$s^{(3)}_{1011} = \frac{1}{4\sigma_1^2 - \sigma_2^2}(h^{(3)}_{1110}\sigma_2 - 2(h^{(3)}_{0021} - h^{(3)}_{2001})\sigma_1),$$

$$s^{(3)}_{2100} = -\frac{1}{4\sigma_1^2\sigma_2 - \sigma_2^3}(h^{(3)}_{1110}\sigma_1\sigma_2 + h^{(3)}_{2001}\sigma_2^2 - 2(h^{(3)}_{0021} + h^{(3)}_{2001})\sigma_1^2), \tag{36}$$

$$s^{(3)}_{3000} = -\frac{1}{3\sigma_1}(2h^{(3)}_{0030} + h^{(3)}_{2010}), \; s^{(3)}_{1110} = s^{(3)}_{2010} = 0,$$

$$s^{(3)}_{0003} = s^{(3)}_{0030} = s^{(3)}_{0021} = s^{(3)}_{0201} = s^{(3)}_{2001} = 0.$$

It should be emphasized that the coefficients $s^{(3)}_{ijkl}$ have form (33), (34), (36) only if the conditions (32), (35) are fulfilled. Note that these inequalities mean the absence of the third order resonances in the system [3]. In this case there exists canonical transformation (28) with coefficients (33), (34), (36) such that the third order term $\tilde{H}_3$ in the Hamiltonian (19) is cancelled.

Analyzing expressions (17), one can easily find that there is only one value of the parameter μ in the interval (16), namely, $\mu_1 = 0,0529422$, where the condition $\sigma_1 - 2\sigma_2 = 0$ is fulfilled. Hence, with canonical transformation (28) the third order term in the expansion (19) of the Hamiltonian (3) is cancelled for any value of the parameter μ in the interval (16), except for $\mu = \mu_1$.

Now let us consider the case $\mu = \mu_1$ when the third order resonance takes place. The determinant of system (31) is not equal to zero and, hence, we again obtain solution (36) if all coefficients $\tilde{h}^{(3)}_{ijkl} = 0$. On the contrary, determinants of the systems (29), (30) become equal to zero. Nevertheless, system (30) has the same solution (34) for $\tilde{h}^{(3)}_{ijkl} = 0$ because it doesn't contain the coefficients $h^{(3)}_{ijkl}$. But we can not set $\tilde{h}^{(3)}_{ijkl} = 0$ in system (29) because it will not have solutions at all. It means that in the case of the third order resonance we can not cancel all terms in $\tilde{H}_3$ with canonical transformation (28). We can try only to reduce $\tilde{H}_3$ to a form such that theorem of Markeev [2], for example, on the stability of Hamiltonian system under the third order resonance would be applied. Setting

$$\tilde{h}^{(3)}_{0012} = -\tilde{h}^{(3)}_{0210} = -\frac{1}{2}\tilde{h}^{(3)}_{1101} = \frac{B}{2\sqrt{2}}$$

in system (29), we obtain its solution in the form

$$s^{(3)}_{1002} = -\frac{1}{2\sigma_2}\left(h^{(3)}_{0012} + h^{(3)}_{0210} + 2\sigma_2 s^{(3)}_{1200}\right),$$

$$s^{(3)}_{0111} = -\frac{1}{4\sigma_2}\left(h^{(3)}_{0012} + 3h^{(3)}_{0210} - h^{(3)}_{1101} + 8\sigma_2 s^{(3)}_{1200}\right),$$

$$B = \frac{1}{\sqrt{2}}\left(h^{(3)}_{0012} - h^{(3)}_{0210} - h^{(3)}_{1101}\right),$$

where $s^{(3)}_{1200}$ is an arbitrary constant. Then the Hamiltonian (19) is reduced to the form

$$\tilde{H} = \sigma_2(\tilde{p}_1^2 + \tilde{q}_1^2) - \frac{1}{2}\sigma_2(\tilde{p}_2^2 + \tilde{q}_2^2) + \frac{B}{2\sqrt{2}}(\tilde{p}_1\tilde{p}_2^2 - \tilde{p}_1\tilde{q}_2^2 - 2\tilde{p}_2\tilde{q}_1\tilde{q}_2) + \tilde{H}_4 + \ldots. \quad (37)$$

And using the standard canonical transformation

$$\tilde{q}_1 = \sqrt{2\tau_1}\sin\varphi_1,\ \tilde{p}_1 = \sqrt{2\tau_1}\cos\varphi_1,$$
$$\tilde{q}_2 = \sqrt{2\tau_2}\sin\varphi_2,\ \tilde{p}_2 = \sqrt{2\tau_2}\cos\varphi_2, \quad (38)$$

we rewrite (37) as

$$H = 2\sigma_2\tau_1 - \sigma_2\tau_2 + B\tau_2\sqrt{\tau_1}\cos(\varphi_1 + 2\varphi_2) + H_4^*(\varphi_1, \varphi_2, \tau_1, \tau_2) + \ldots, \quad (39)$$

where $B = -0,365822 \neq 0$ for $\mu = \mu_1$. Now on the basis of theorem of Markeev [2] we can make the following conclusion.

Theorem 1. *The bisector equilibrium solutions of the circular restricted problem of four bodies are unstable for the parameter $\mu = \mu_1$ in the interval (16) when the third order resonance takes place.*

Normalizing the fourth order term in the Hamiltonian. Let us suppose further that $\mu \in \in (0, \mu_0)$ but $\mu \neq \mu_1$. Substituting successively (22) and (28) with coefficients (33), (34), (36) into (19) and expanding it in the Taylor series in powers of $\tilde{q}_1$, $\tilde{q}_2$, $\tilde{p}_1$, $\tilde{p}_2$, we obtain a new Hamiltonian in the form

$$\tilde{H} = \tilde{H}_2 + \tilde{H}_4 + \ldots, \quad (40)$$

where the second order term,

$$\tilde{H}_2 = \frac{1}{2}\left(\sigma_1(\tilde{p}_1^2 + \tilde{q}_1^2) - \sigma_2(\tilde{p}_2^2 + \tilde{q}_2^2)\right), \quad (41)$$

is normalized, the third order term $\tilde{H}_3$ is absent and the fourth order term $\tilde{H}_4$ may be written as

$$\tilde{H}_4 = \sum_{i+j+k+l=4} \tilde{h}^{(4)}_{ijkl}\tilde{q}_1^i\tilde{q}_2^j\tilde{p}_1^k\tilde{p}_2^l. \quad (42)$$

The sum (42) contains 19 non-zero terms but the expressions for the coefficients $\tilde{h}^{(4)}_{ijkl}$ are quite cumbersome and we do not write them here. Again we can try to find the function

$$S(p_1^*, p_2^*, \tilde{q}_1, \tilde{q}_2) = \tilde{q}_1 p_1^* + \tilde{q}_2 p_2^* + \sum_{i+j+k+l=4} s^{(4)}_{ijkl} \tilde{q}_1^i \tilde{q}_2^j p_1^{*k} p_2^{*l}, \tag{43}$$

generating a canonical transformation that reduces the fourth order term $\tilde{H}_4$ to its simplest form. Then new momenta p_1^*, p_2^* and coordinates q_1^*, q_2^* are determined by

$$q_1^* = \frac{\partial S}{\partial p_1^*},\ q_2^* = \frac{\partial S}{\partial p_2^*},\ \tilde{p}_1 = \frac{\partial S}{\partial \tilde{q}_1},\ \tilde{p}_2 = \frac{\partial S}{\partial \tilde{q}_2}. \tag{44}$$

Resolving (44) with respect to the old canonical variables $\tilde{q}_1$, $\tilde{q}_2$, $\tilde{p}_1$, $\tilde{p}_2$ in a neighborhood of the point $q_1^* = q_2^* = p_1^* = p_2^* = 0$ and substituting the solution into (40), we expand the Hamiltonian $\tilde{H}$ in a Taylor series in powers of q_1^*, q_2^*, p_1^*, p_2^*. Obviously, the second order term H_2^* in this expansion again has the form (41), the third order term H_3^* is absent and the fourth order term H_4^* is a sum of 35 terms of the form

$$h^{*(4)}_{ijkl} \tilde{q}_1^i \tilde{q}_2^j \tilde{p}_1^k \tilde{p}_2^l, \quad i+j+k+l = 4.$$

The coefficients $h^{*(4)}_{ijkl}$ are again divided into three independent groups. The first group corresponds to the coefficients of $p_1^* p_2^{*3}$, $q_2^{*2} p_1^* p_2^*$, $q_1^* q_2^* p_2^{*2}$, $q_1^* q_2^{*3}$ in the expression for H_4^* and is given by

$$\begin{aligned} h^{*(4)}_{0013} &= \tilde{h}^{(4)}_{0013} + s^{(4)}_{1003}\sigma_1 - s^{(4)}_{0112}\sigma_2, \\ h^{*(4)}_{0211} &= \tilde{h}^{(4)}_{0211} + s^{(4)}_{1201}\sigma_1 + 2s^{(4)}_{0112}\sigma_2 - 3s^{(4)}_{0310}\sigma_2, \\ h^{*(4)}_{1102} &= \tilde{h}^{(4)}_{1102} - s^{(4)}_{0112}\sigma_1 + 3s^{(4)}_{1003}\sigma_2 - 2s^{(4)}_{1201}\sigma_2, \\ h^{*(4)}_{1300} &= \tilde{h}^{(4)}_{1300} - s^{(4)}_{0310}\sigma_1 + s^{(4)}_{1201}\sigma_2. \end{aligned} \tag{45}$$

Relationships (45) form a system of equations determining the coefficients $s^{(4)}_{1003}$, $s^{(4)}_{0112}$, $s^{(4)}_{1201}$, $s^{(4)}_{0310}$ in the expansion (43). The determinant of this system is equal to $(\sigma_1^2 - \sigma_2^2)(\sigma_1^2 - 9\sigma_2^2)$ and is not equal to zero if $\sigma_1 \neq \sigma_2$ and

$$\sigma_1 \pm 3\sigma_2 \neq 0. \tag{46}$$

Hence, if conditions (46) are fulfilled, which means the absence of the fourth order resonances, system (45) will have a nontrivial solution in the case of $h^{*(4)}_{ijkl} = 0$ and the corresponding terms in H_4^* can be cancelled.

The second group consists of the coefficients of $q_2^* p_1^* p_2^{*2}$, $p_1^* q_2^{*3}$, $q_1^* p_2^{*3}$, $q_1^* q_2^{*2} p_2^*$ in H_4^* and can be written as

$$h^{*(4)}_{0112} = s^{(4)}_{1102}\sigma_1 + 3s^{(4)}_{0013}\sigma_2 - 2s^{(4)}_{0211}\sigma_2,$$

$$h^{*(4)}_{0310} = s^{(4)}_{1300}\sigma_1 + s^{(4)}_{0211}\sigma_2,$$

$$h^{*(4)}_{1003} = -s^{(4)}_{0013}\sigma_1 - s^{(4)}_{1102}\sigma_2,$$

$$h^{*(4)}_{1201} = -s^{(4)}_{0211}\sigma_1 + 2s^{(4)}_{1102}\sigma_2 - 3s^{(4)}_{1300}\sigma_2.$$

This system is similar to equations (30) and has a trivial solution,

$$s^{(4)}_{1102} = s^{(4)}_{0013} = s^{(4)}_{0211} = s^{(4)}_{1300} = 0,$$

in the case $h^{*(4)}_{ijkl} = 0$ despite its determinant becoming equal to zero for $\sigma_1 \pm 3\sigma_2 = 0$.

The other twenty seven coefficients $h^{*(4)}_{ijkl}$ in the expression H^*_4 form the third group but we do not write it here. This group is just a system of equations determining the rest twenty seven coefficients $s^{(4)}_{ijkl}$ in the expansion (43). It should be emphasized that the determinant of this system is equal to zero and it will not have any solutions in the case $h^{*(4)}_{ijkl} = 0$. Thus, the fourth order term H^*_4 can not be cancelled entirely in the expansion (40). Nevertheless, analyzing the system, we have shown that if, in addition to inequalities (32), (35), (46), the conditions

$$\sigma_1 \neq 0,\ \sigma_2 \neq 0,\ \sigma_1 \pm \sigma_2 \neq 0,\ 3\sigma_1 \pm \sigma_2 \neq 0, \tag{47}$$

are fulfilled, the fourth order term $\tilde{H}_4$ is reduced to the form

$$7H^*_4 = \frac{1}{4}\left(c_{20}(p^{*2}_1 + q^{*2}_1)^2 + c_{11}(p^{*2}_1 + q^{*2}_1)(p^{*2}_2 + q^{*2}_2) + c_{02}(p^{*2}_2 + q^{*2}_2)^2\right). \tag{48}$$

Then, using the standard canonical transformation (38), we rewrite the Hamiltonian (40) as

$$H = \sigma_1\tau_1 - \sigma_2\tau_2 + c_{20}\tau_1^2 + c_{11}\tau_1\tau_2 + c_{02}\tau_2^2 + H^*_5(\varphi_1, \varphi_2, \tau_1, \tau_2) + \ldots. \tag{49}$$

Now the Arnold – Moser theorem [13, 14] can be applied; it states that in the case of absence of the resonances up to the fourth order, stability of the bisector equilibrium solutions depends on the value of the parameter

$$f = c_{20}\sigma_2^2 + c_{11}\sigma_1\sigma_2 + c_{02}\sigma_1^2.$$

Since the expessions for the coefficients $c_{20},\ c_{11},\ c_{02}$ are very complicated we do not write them here. But using the system *Mathematica*, we can easily plot the curve $f = f(\mu)$ in the $O\mu f$ plane (see Fig. 3). Now we see that there is only one point $\mu_2 = 0,0502039$ in the interval (16) where $f = 0$. There are also two values of μ, namely, $\mu_1 = 0,0529423$ and $\mu_3 = 0,0291011$, corresponding to the cases of resonance of the third and the fourth orders, respectively. Hence, on the basis of the Arnold – Moser theorem we can make the following conclusion.

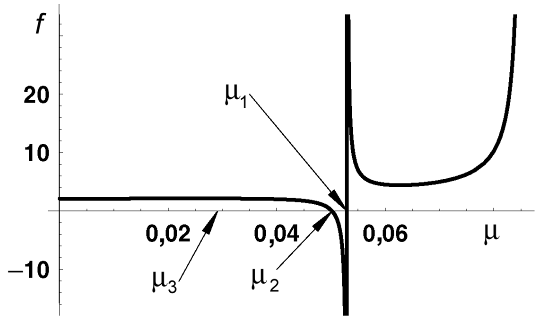

Fig. 3. Parameter f as a function of μ.

Theorem 2. *The bisector equilibrium solutions of the circular restricted problem of four bodies are stable in Liapunov's sense for any values of the parameter μ from the interval (16), except for three points, namely, μ_1, μ_2 and μ_3.*

Let us consider the case $\mu = \mu_3$ when the fourth order resonance takes place, i.e., the condition $\sigma_1 - 3\sigma_2 = 0$ is fulfilled. Now the determinant of system (45) becomes equal to zero and has not any solutions for $h^{*(4)}_{ijkl} = 0$. It means that we can not cancel the terms

$$h^{*(4)}_{0013} p^*_1 p^{*3}_2, \; h^{*(4)}_{0211} q^{*2}_2 p^*_1 p^*_2, \; h^{*(4)}_{1102} q^*_1 q^*_2 p^{*2}_2, \; h^{*(4)}_{1300} q^*_1 q^{*3}_2$$

in the expression for H^*_4. But we can try to reduce H^*_4 to form such a that the theorem of Markeev [2] on the stability of a Hamiltonian system under the fourth order resonance would be applied. Setting

$$h^{*(4)}_{0013} = -\frac{1}{3} h^{*(4)}_{0211} = -\frac{1}{3} h^{*(4)}_{1102} = h^{*(4)}_{1300} = \frac{B}{4}$$

in system (45), we obtain its solution in the form

$$s^{(4)}_{1003} = -\frac{1}{24\sigma_2}(9\tilde{h}^{(4)}_{0013} + 3\tilde{h}^{(4)}_{0211} - \tilde{h}^{(4)}_{1102} - 3\tilde{h}^{(4)}_{1300} + 8\sigma_2 s^{(4)}_{1201}),$$

$$s^{(4)}_{0112} = -\frac{1}{4\sigma_2}(\tilde{h}^{(4)}_{0013} + \tilde{h}^{(4)}_{0211} - \tilde{h}^{(4)}_{1102} - \tilde{h}^{(4)}_{1300} + 4\sigma_2 s^{(4)}_{1201}),$$

$$s^{(4)}_{0310} = -\frac{1}{24\sigma_2}(\tilde{h}^{(4)}_{0013} - \tilde{h}^{(4)}_{0211} - \tilde{h}^{(4)}_{1102} - 7\tilde{h}^{(4)}_{1300} - 8\sigma_2 s^{(4)}_{1201}),$$

$$B = \frac{1}{2}(\tilde{h}^{(4)}_{0013} - \tilde{h}^{(4)}_{0211} - \tilde{h}^{(4)}_{1102} + \tilde{h}^{(4)}_{1300}),$$

where $s^{(4)}_{1201}$ is an arbitrary constant. Then the Hamiltonian (40) is reduced to the form

$$H^* = \frac{3\sigma_2}{2}\left(p_1^{*2} + q_1^{*2}\right) - \frac{\sigma_2}{2}\left(p_2^{*2} + q_2^{*2}\right) +$$

$$+ \frac{1}{4}\left(c_{20}(p_1^{*2} + q_1^{*2})^2 + c_{11}(p_1^{*2} + q_1^{*2})(p_2^{*2} + q_2^{*2}) + c_{02}(p_2^{*2} + q_2^{*2})^2\right) +$$

$$+ \frac{B}{4}\left(p_1^* p_2^{*3} - 3q_2^{*2} p_1^* p_2^* - 3q_1^* q_2^* p_2^{*2} + q_1^* q_2^{*3}\right). \quad (50)$$

At last, making the transformation (38), we rewrite the Hamiltonian (50) as

$$H = 3\sigma_2\tau_1 - \sigma_2\tau_2 + c_{20}\tau_1^2 + c_{11}\tau_1\tau_2 + c_{02}\tau_2^2 +$$

$$+ B\tau_2\sqrt{\tau_1\tau_2}\cos(\varphi_1 + 3\varphi_2) + H_5^*(\varphi_1, \varphi_2, \tau_1, \tau_2) + \ldots . \quad (51)$$

According to Markeev's theorem [2], stability of the equilibrium solutions under the fourth order resonance depends on the values of $c_{20} + 3c_{11} + 9c_{02}$ and $3\sqrt{3}B$. Our calculations show that, for $\mu = \mu_3$,

$$c_{20} + 3c_{11} + 9c_{02} = 21,4802, \;\; 3\sqrt{3}B = 8,99408$$

and, hence,

$$c_{20} + 3c_{11} + 9c_{02} > 3\sqrt{3}B.$$

Thus, on the basis of Markeev's theorem [2] we can make the following conclusion.

Theorem 3. *The bisector equilibrium solutions of the circular restricted problem of four bodies are stable in Liapunov's sense for the parameter $\mu = \mu_3$ from the interval (16) when the fourth order resonance takes place.*

Conclusion. In the present paper we have studied the stability of equilibrium solutions in the Newtonian circular restricted four-body problem in the strict nonlinear formulation. We have proved that the radial equilibrium solutions are unstable while the bisector equilibrium solutions are stable in Liapunov's sense if the mass parameter belongs to the interval (16), except for two points $\mu = \mu_j,\ j = 1, 2$. We have showed that for $\mu = \mu_1$ the bisector equilibrium solutions become unstable because of the third order resonance. Nevertheless, for $\mu = \mu_3$, when the fourth order resonance takes place, the bisector equilibrium solutions are stable in Liapunov's sense.

In the case $\mu = \mu_2$, the parameter f becomes equal zero, which means that the third and the fourth order terms in expansion (40) are absent. Hence, in order to conclude on the stability of the bisector equilibrium solutions for $\mu = \mu_2$, a further analysis of higher order terms in the Hamiltonian expansion is required.

All symbolic and numerical calculations in the present paper are done with the computer algebra system *Mathematica*.

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