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STABILITY OF EXACT SOLUTIONS OF THE CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION WITH PERIODIC POTENTIAL

СТАБІЛЬНІСТЬ ТОЧНИХ РОЗВ'ЯЗКІВ РІВНЯННЯ ШРЕДІНГЕРА З НЕЛІНІЙНІСТЮ ТРЕТЬОГО ТА П'ЯТОГО ПОРЯДКІВ І ПЕРІОДИЧНИМ ПОТЕНЦІАЛОМ

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The nonlinear Schrödinger equation with attractive quintic nonlinearity in periodic potential in 1D, modelling a dilute gas Bose – Einstein condensate in a lattice potential, is considered and one family of exact stationary solutions is discussed. Some of these solutions have neither an analog in the linear Schrödinger equation nor in the integrable nonlinear Schrödinger equation. Their stability is examined analytically and numerically.

Розглянуто нелінійне рівняння Шредінгера з притягуючою нелінійністю п'ятого порядку в одновимірному періодичному потенціалі, яке моделює розряджений газовий конденсат Бозе – Ейнштейна в решітчатому потенціалі, а також деяку сім'ю точних стаціонарних розв'язків. Деякі з цих розв'язків не мають аналогів серед розв'язків ні лінійного рівняння Шредінгера, ні інтегровного нелінійного рівняння Шредінгера. Досліджено стабільність таких розв'язків аналітичними та чисельними методами.

I. Introduction. It is well-known that a collapse phenomenon is observed in the Bose – Einstein condensates (BECs) with attractive interaction if the number of atoms N exceeds a critical value N_c, as in the case of atomic condensates with Li [1, 2]. In this case, experiments with attractive two-body interaction have been performed [3, 4] with results consistent with the limitation in the number of atoms and with the growth and collapse scenario. The nonlinear Schrödinger (NLS) equation with cubic nonlinearity used to describe the BECs has stable solutions in the one-dimensional (1D) case when the dispersion and nonlinearity effects can effectively balance each other. In two and three dimensions, the focusing nonlinearity overcomes the dispersion and a blow-up phenomenon occurs [5].

A few mechanisms, as the dispersion [6, 7] and nonlinearity management methods [5, 6, 8 – 12] have been suggested for the arrest of collapse. Based on the variational approach, method of moments, and numerical simulations, the analysis showed that the nonlinearity management method is effective in suppressing collapse in the 1D and 2D NLS equations with focusing cubic nonlinearity. The arrest of collapse by using a strong cubic nonlinearity management scheme in the 1D cubic-quintic nonlinear Schrödinger equations is of practical interest since it appears in many branches of physics such as BEC. Here, it models the condensate with two- and three-body interactions [13, 14]. In BEC, the variation of the atomic scattering length by the Feshbach resonance technique leads to the oscillations of the mean-field cubic nonlinearity [15], but it also induces variations of the quintic nonlinearity because the three-body interaction is dependent on the scattering length [16].

In this paper we consider a quasi-one-dimensional BEC with two- and three-body interactions in a periodic potential. The governing equation is given by the NLS equation with a periodic and with cubic and quintic terms

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x) + \lambda_2 |\psi|^2 + \lambda_3 |\psi|^4\right)\psi, \tag{1}$$

where $\psi(x,t)$ represents the macroscopic wave function of the condensate and $V(x)$ is an external macroscopic potential. In this equation we assume dimensionless variables: the unit of energy is $\hbar\omega/2$, the unit of length is $\sqrt{\hbar/(m\omega)}$, and the unit of time is $1/\omega$. The parameters λ_2 and λ_3 of the two- and three-body interactions in general can be complex quantities. In the case of complex parameters, the imaginary parts of λ_2 and λ_3 describe the effects of inelastic two- and three-body collisions on the dynamics of BEC's, respectively. In this work, we do not consider dissipative terms, and such cubic and quintic parameters are real. In general, the parameter of the two-body interaction λ_2 is proportional to the two-body scattering length a_s, and is given by $\lambda_2 = 8\pi a_s$ [17].

In the absence of the three-body interaction parameter ($\lambda_3 = 0$), exact solutions of Eq. (1) have been constructed with the experimentally generated potential $V(x) = V_0 \sin^2(x)$ and their stability was investigated in [18, 21]. In fact, a potential more general than sinusoidal potential was considered: $V(x) = -V_0\,\mathrm{sn}^2(x,k)$, where $\mathrm{sn}(x,k)$ denotes the Jacobian elliptic sine function with elliptic modulus $0 \le k \le 1$ [19]. For a potential of this form, some exact solutions were also found and their stability has been analyzed in [20], taking $\lambda_2 = -1$. Currently, no experiments are being performed where a BEC with two- and three-body interaction is trapped in 1D periodic potential. Although motivated by the developments in BECs, in this paper we consider Eq. (1) with attractive three-body interaction ($\lambda_3 < 0$). Thus we consider

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x) + \lambda_2 |\psi|^2 - |\psi|^4\right)\psi. \tag{2}$$

As in BECs with two-body interaction [21–23], the proper choices for the potential allow for the construction of a large class of exact solutions. The external potential considered in this work is a generalization of the sinusoidal, standing light wave potential [24]:

$$V(x) = A_0 + B_0\,\mathrm{sn}^2(lx,k). \tag{3}$$

For $B_0 = 0$, Eq. (2) becomes a cubic-quintic nonlinear Schrödinger equation and hence is integrable [25, 26]. In the limit as $k \to 1-0$, $V(x)$ becomes an array of well-separated hyperbolic secant potential barriers or wells, while in the limit as $k \to +0$ it becomes purely sinusoidal. Because $\mathrm{sn}(lx,k)$ is a periodic function with period

$$4K(k)l = \frac{4}{l}\frac{\int_0^{\pi/2} dz}{\sqrt{1-k^2\sin^2 z}},$$

the external potential $V(x)$ is also a periodic function with period $2K(k)$. This period approaches infinity as $k \to 1-0$. Thus, as $k \to 1-0$, the potential (3) is a periodic lattice of separated peaks or troughs. Hence, by changing the parameter k, various interesting regimes of BECs are considered. This is the reason for considering potential of the form (3). The parameters A_0, B_0 and l are introduced to facilitate the construction of exact solutions and, for simplicity, we take $l > 0$. Although this exact expression for the potential is necessary to allow the construction of exact solutions, it is its qualitative features, that is, its periodicity and amplitude, that are the most important.

The paper is outlined as follows. In the next section we derive and consider various properties and limit of explicit solutions of Eq. (2) with potential (3). In Section III we develop the analytic framework for linear stability properties of solutions of Section II. In section IV, the results of the numerical simulations are discussed. A brief summary of the results concludes the paper in Section V.

II. Stationary solutions. Equation (2) with $V(x) = \text{const}$ is an integrable cubic-quintic nonlinear Schrödinger equation of which many explicit solutions are known [25–27]. If $V(x) \neq \neq \text{const}$ the cubic-quintic NLS equation is not integrable, and only small classes of explicit solutions can be obtained. A judicious choice of the potential $V(x)$ allows for the cancellation of the nonlinear terms in Eq. (2) so that exact solutions can be constructed. Of course, one can always find a suitable potential $V(x)$ by solving Eq. (2) for $V(x)$, given a certain $\psi(x,t)$. This results in a time-dependent potential and hence is not of interest. In this section, we give a dictionary of the families of exact solutions we were able to construct. These families are built as the families of exact solutions found for the two-body interaction case, discussed in [18, 20].

For the exact solutions, the density of the condensate $|\psi(x,t)|^2$ is a linear function of either $\mathrm{dn}(x,k)$ or $\mathrm{sn}(lx,k)$, or $\mathrm{cn}(lx,k)$, where $\mathrm{sn}(lx,k)$ and $\mathrm{cn}(lx,k)$ are the Jacobian elliptic sine and cosine functions, respectively. These solutions are given by $\psi(x,t) = r(x)\exp\left(i\theta(x) - \omega t\right)$, where $r(x)$ and $\theta(x)$ are two real functions to be determined and ω is the chemical potential of the condensate. Inserting this ansatz into Eq. (2) yields

$$r^3 r'' + 2\omega r^4 - 2Vr^4 - 2\lambda_2 r^6 + 2r^8 - C^2 = 0. \tag{4}$$

The parameter C is defined via the relation $\theta'(x) = C/r^2(x)$, which expresses conservation of angular momentum. Null angular momentum solutions, which constitute an important special case, satisfy $C = 0$ and we choose $\theta(x) = 0$. First we discuss the solution with $\mathrm{dn}(lx,k)$. The quantities associated with this solution will be denoted with the subscript 1. The quantities associated with the $\mathrm{sn}(lx,k)$ and $\mathrm{cn}(lx,k)$ solutions will be subscripted by 2 and 3, respectively. In order to find some restrictions on the domain of the parameters of the solutions, we will use the fact that both $\mathrm{sn}(lx,k)$ and $\mathrm{cn}(lx,k)$ have zero average as functions of x and lie in $[-1,1]$, while $\mathrm{dn}(lx,k)$ has nonzero average, and its range is $\left[\sqrt{1-k^2}, 1\right]$.

To find the $\mathrm{dn}(lx,k)$ solutions, we set

$$r_1^2(x) = A_1 + B_1\,\mathrm{dn}(lx,k). \tag{5}$$

Inserting Eq. (5) into Eq. (4) yields

$$A_1 = \frac{3k^2l^2 - 8B_0}{8\left(k^2l^2 - 2B_0\right)}\lambda_2, \quad B_1^2 = \frac{3k^2l^2 - 8B_0}{8k^2},$$

$$\omega_1 = \frac{32B_0^2 + 9k^4l^4 - 36k^2l^2B_0}{32\left(k^2l^2 - 2B_0\right)^2}\lambda_2^2 + \frac{B_0 + k^2A_0}{k^2} + \frac{l^2\left(k^2 - 2\right)}{8}, \tag{6}$$

$$C_1^2 = \left(\lambda_2^2 - \frac{24\left(k^2l^2 - 2B_0\right)^2\left(l^2k^2 - 4B_0\right)}{k^2\left(3k^2l^2 - 8B_0\right)^2}\right) \times$$

$$\times \left(\lambda_2^2 - \frac{8\left(k^2l^2 - 2B_0\right)^2\left(1 - k^2\right)}{k^2\left(3k^2l^2 - 8B_0\right)}\right)\frac{k^2l^2\left(3k^2l^2 - 8B_0\right)^2}{512\left(k^2l^2 - 2B_0\right)^3}.$$

The freedom in choosing the potential gives five free parameters, λ_2, A_0, B_0, k, and l. The requirements that $r_1^2(x)$, B_1^2, and C_1^2 be nonnegative imposes conditions on the domain of these parameters. The condition on the sign of $r_1^2(x)$ and B_1^2 gives $\frac{3k^2l^2}{8} > B_0$ and

$$\lambda_2 \geq -\frac{4\left(k^2l^2 - 2B_0\right)\sqrt{1 - k^2}}{k\sqrt{2\left(3k^2l^2 - 8B_0\right)}}$$

for $B_1 > 0$, and $\frac{3k^2l^2}{8} > B_0$ and

$$\lambda_2 \geq \frac{4\left(k^2l^2 - 2B_0\right)}{k\sqrt{2\left(3k^2l^2 - 8B_0\right)}}$$

for $B_1 < 0$. Solving the inequality $C_1^2 \geq 0$, one obtains the supplementary conditions on the domain of parameters λ_2, A_0, B_0, k, and l for the validity of solutions (5), (6).

The $\operatorname{sn}(lx, k)$ solutions are found by substituting

$$r_2^2(x) = A_2 + B_2 \operatorname{sn}(lx, k) \tag{7}$$

in Eq. (4). Equating different powers of $\operatorname{sn}(lx, k)$ imposes the following constraints on the parameters:

$$A_2 = \frac{3k^2l^2 - 8B_0}{8\left(k^2l^2 - 2B_0\right)}\lambda_2, \quad B_2^2 = \frac{8B_0 - 3k^2l^2}{8},$$

$$\omega_2 = A_0 + \frac{32B_0^2 - 36k^2l^2B_0 + 9k^4l^4}{32\left(k^2l^2 - 2B_0\right)^2}\lambda_2^2 + \frac{l^2\left(1 + k^2\right)}{8}, \tag{8}$$

$$C_2^2 = \frac{l^2\left(8B_0 - 3k^2l^2\right)^3}{2048\left(k^2l^2 - 2B_0\right)^4}\left(\frac{8\left(k^2l^2 - 2B_0\right)^3}{8B_0 - 3k^2l^2} - \lambda_2^2\right)\left(k^2\lambda_2^2 - \frac{8\left(k^2l^2 - 2B_0\right)^3}{8B_0 - 3k^2l^2}\right).$$

The freedom in choosing the potential gives five free parameters, λ_2, A_0, B_0, k, and l. The requirements that $r_2^2(x)$, B_2^2, and C_2^2 be nonnegative imposes conditions on the domain of these parameters. For the positivity of both $r_2^2(x)$ and B_2^2, we obtain that either $B_0 > \frac{k^2l^2}{2}$ and $\lambda_2 \geq \frac{4\left(2B_0 - k^2l^2\right)}{\sqrt{2\left(8B_0 - 3k^2l^2\right)}} > 0$ or $\frac{3k^2l^2}{8} < B_0 < \frac{k^2l^2}{2}$ and $\lambda_2 \leq \frac{4\left(2B_0 - k^2l^2\right)}{\sqrt{2\left(8B_0 - 3k^2l^2\right)}} < 0$. The complementary conditions on the values of the parameters λ_2, A_0, B_0, k, and l for the validity of solutions (7), (8) are obtained by solving the inequality $C_2^2 \geq 0$.

For the $\operatorname{cn}(lx, k)$ solutions, we substitute

$$r_3^2(x) = A_3 + B_3 \operatorname{cn}(lx, k) \tag{9}$$

into Eq. (4) and obtain

$$A_3 = \frac{3k^2l^2 - 8B_0}{8\left(k^2l^2 - 2B_0\right)}\lambda_2, \quad B_3^2 = \frac{3k^2l^2 - 8B_0}{8},$$

$$\omega_3 = \frac{32B_0^2 - 36k^2l^2B_0 + 9k^4l^4}{32\left(k^2l^2 - 2B_0\right)^2}\lambda_2^2 + \frac{l^2\left(1 - 2k^2\right)}{8} + A_0 + B_0, \tag{10}$$

$$C_3^2 = \frac{k^2l^2\left(3k^2l^2 - 8B_0\right)^3}{2048\left(k^2l^2 - 2B_0\right)^4}\left(-\lambda_2^2 + \frac{k^4l^4\left(3k^2l^2 - 8B_0\right)^3}{256\left(k^2l^2 - 2B_0\right)^4}\right) \times$$

$$\times \left(\frac{k^2l^4\left(1 - k^2\right)\left(8B_0 - 3k^2l^2\right)^3}{4096\left(k^2l^2 - 2B_0\right)^4} - \lambda_2^2\right).$$

The freedom in choosing the potential gives five free parameters, λ_2, A_0, B_0, k, and l. The requirements that $r_3^2(x)$, B_3^2, and C_3^2 be nonnegative imposes conditions on the domain of these parameters. For the positivity of both $r_3^2(x)$ and B_3^2, we must have $\frac{3k^2l^2}{8} > B_0$ and $\lambda_2 \geq \frac{4\left(k^2l^2 - 2B_0\right)}{\sqrt{2\left(3k^2l^2 - 8B_0\right)}} > 0$. This means that the $\operatorname{cn}(lx, k)$ solutions do not exist for the BECs with attractive two-body-interaction. Other restrictions on the region of validity of solution (9), (10) are obtained by imposing the nonnegativity of C_3^2.

The null angular momentum solutions case. Null angular momentum corresponds to $C_j = 0$. Since for each of solutions $r_j(x)$, $j = 1, 2, 3$, C_j^2 has three factors, one of which is different from zero (because $B_j \neq 0$ for non plane wave solutions), and each factor that can be zero is a quadratic function of λ_2 (see the third equation of Eqs. (6), (8), and (10), there are at most four possible choices of λ_2 for which this occurs.

(I) For the solutions $r_1(x)$, we have

$$\lambda_2 \in \left\{\pm\frac{2\left(k^2l^2 - 2B_0\right)}{k\left(3k^2l^2 - 8B_0\right)}\sqrt{6\left(l^2k^2 - 4B_0\right)}, \pm\frac{2\left(k^2l^2 - 2B_0\right)}{k}\sqrt{\frac{2\left(1 - k^2\right)}{3k^2l^2 - 8B_0}}\right\} \text{ if } l^2k^2/4 > B_0,$$

and

$$\lambda_2 \in \left\{ \pm \frac{2\left(k^2l^2 - 2B_0\right)}{k} \sqrt{\frac{2\left(1-k^2\right)}{3k^2l^2 - 8B_0}} \right\} \quad \text{if} \quad l^2k^2/4 \le B_0 < 3k^2l^2/8.$$

(II) For the solutions $r_2(x)$ with condition $r_2^2(x) \ge 0$, we obtain

$$\lambda_2 = 2\left(2B_0 - k^2l^2\right)\sqrt{\frac{2\left(k^2l^2 - 2B_0\right)}{8B_0 - 3k^2l^2}}$$

if $\dfrac{k^2l^2 - 1}{2} \ge B_0 > \dfrac{3}{8}k^2l^2$ and $k^2l^2 - 4 > 0$, and

$$\lambda_2 = \frac{2\left(2B_0 - k^2l^2\right)}{k}\sqrt{\frac{2\left(k^2l^2 - 2B_0\right)}{8B_0 - 3k^2l^2}}$$

if $\dfrac{k^2\left(l^2-1\right)}{2} \ge B_0 > \dfrac{3}{8}k^2l^2$ and $l^2 - 4 > 0$.

We note that Eqs. (9) and (10) do not define any null angular momentum $\mathrm{cn}(lx, k)$ solutions with $r_3^2(x) \ge 0$ for all x.

The trigonometric limit. In the limit as $k \to +0$, the Jacobian elliptic functions reduce to trigonometric functions and $V(x) = A_0 + B_0\sin^2(lx)$. In the limit as $k \to +0$, Eq. (8) gives a negative C_2^2, and this means that Eqs. (7) and (8) do not give any trigonometric solution. Passing in the limit as $k \to +0$ in Eqs. (9) and (10), we find the trigonometric solutions

$$r_{3\pm}^2(x) = \frac{\lambda_2}{2} \pm \sqrt{-B_0}\cos(lx), \qquad \omega_3 = \frac{\lambda_2^2}{4} + \frac{l^2}{8} + A_0 + B_0, \quad B_0 < 0,$$

which is a null angular momentum solution. For the positivity of $r_{3\pm}^2(x)$, we must have $\lambda_2 \ge$ $\ge 2\sqrt{-B_0}$. Some trigonometric solutions are illustrated in Fig. 1.