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THE ALGEBRAIC-ANALYTIC STRUCTURE OF INTEGRABILITY BY QUADRATURES OF ABEL – RICCATI EQUATIONS

АЛГЕБРАЇЧНО-АНАЛІТИЧНА СТРУКТУРА ІНТЕГРОВНОСТІ В КВАДРАТУРАХ РІВНЯНЬ АБЕЛЯ – РІККАТІ

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More than one hundred and fifty years ago J. Liouville posed a problem of describing Riccati equations $dy/dx = y^2 + a(x)y + b(x)$, which are integrable by quadratures. Since this time, the Liouville problem was considered many times, and there are some partial solutions to it, but up to now there exists no effective theory answering the question whether a given Riccati equation is integrable or not. Only eighteen years ago there was made a new attempt to study the Liouville problem based on the theory of Lax type integrable dynamical systems. In this paper there is developed further an approach of investigating the integrability by quadratures of generalized Riccati – Abel equations that before was applied to a usual Riccati equation. We reduce a given Riccati – Abel equation to some equivalent nonlinear evolution partial differential equations with natural Cauchy – Goursat initial data, and prove further their Lax type integrability, connected via Liouville with the integrability by quadratures. This approach having backgrounds both in modern differential-geometric and Lie-algebraic techniques, gives rise to a partial solution to the Liouville problem, effective enough in the case of Riccati equation.

Більше ніж сто п'ятдесят років тому Ж. Ліувілль поставив проблему опису рівнянь Ріккаті $dy/dx = y^2 + a(x)y + b(x)$, інтегровних у квадратурах. З тих пір ця проблема досліджувалась багато разів, було отримано різні часткові її розв'язки, але до сьогодні немає ефективної теорії, яка б давала відповідь на питання: чи є дане рівняння Ріккаті інтегровним у квадратурах чи ні? П'ятнадцять років тому був запропонований новий підхід до проблеми Ліувілля, оснований на теорії інтегровних за Лаксом динамічних систем. У даній статті цей підхід розвивається далі для дослідження інтегровності узагальнених рівнянь Ріккаті зведенням їх до рівнянь з частинними похідними, інтегровних за Лаксом. Цей підхід, що грунтується як на диференціально-геометричних, так і на алгебраїчно-геометричних методах, приводить до часткового розвитку проблеми Ліувілля, досить ефективного для рівнянь Ріккаті.

1. Introduction. Our purpose is to describe the class of functions a and $b \in C^1(\mathbf{R};\mathbf{R})$ in the generalized Riccati – Abel equation

$$\frac{dy}{dx} = y^n + a\,(x)\,y + b\,(x)\,, \tag{1}$$

where $\mathbf{Z}_+ \ni n \geq 2$, for which this equation is integrable by quadratures (i. e. a solution of this equation can be expressed by means of elementary and algebraic functions as well as of integrals of them [1]). The backgrounds of the integrability theory via the differential-algebraic

Picard – Vessiot approach were founded in classical articles [2, 3] and further developed in [4 – 6].

First we consider the Cauchy problem for equation (1) with some fixed functions a and $b \in C^1(\mathbf{R};\mathbf{R})$:

$$\frac{dy}{dx} = y^n + a\,(x)\,y + b\,(x)\,, \qquad y\,(x_0) = y_0, \tag{2}$$

with $y_0 \in \mathbf{R}$ being an arbitrary Cauchy data at a point $x_0 \in \mathbf{R}$.

From the Regularity and Uniqueness Theorem [7] it follows that there exists a unique solution of (2), differentiable with respect to x_0 and $y_0 \in \mathbf{R}$ satisfying the conditions

$$\frac{\partial y}{\partial x_0}\Big|_{x=x_0} = -\left(y_0^n + a\,(x_0)\,y_0 + b\,(x_0)\right), \qquad \frac{\partial y}{\partial y_0}\Big|_{x=x_0} = 1. \tag{3}$$

Thus, differentiating (2) with respect to x_0 and $y_0 \in \mathbf{R}$, we obtain a system of nonlinear evolution equations in the form:

$$\frac{dy}{dt} = v, \qquad \frac{du}{dt} = n\,(n-1)\,y^{n-2}v \tag{4}$$

on the jet-submanifold $M_0^\infty = \left\{(u,v)^\tau \in J\left(\mathbf{R}\times\mathbf{R}^2;\mathbf{R}^3\right) : v_x = uv\right\}$, where $t := (x_0, y_0) \in \mathbf{R}^2$ -an evolution vector parameter, with the following Cauchy – Goursat data:

$$\begin{gathered}\frac{\partial y}{\partial x_0}\Big|_{x=x_0} = -\left(y_0^n + a\,(x_0)\,y_0 + b\,(x_0)\right), \qquad \frac{\partial y}{\partial y_0}\Big|_{x=x_0} = 1, \\ u|_{x=x_0} = n y_0^{n-1} + a\,(x_0)\,, \qquad v|_{x=x_0} = \frac{\partial y}{\partial x_0}\Big|_{x=x_0}.\end{gathered} \tag{5}$$

The solutions of (2) and (4) are characterized by the following simple but important lemma (see also [1]).

Lemma 1. *All solutions of equations (4) with conditions (5) that reduce to quadratures are also solutions of equation (2), reducible to quadratures.*

Our further main point of the analysis will be concerned with the problem of proving integrability by quadratures of the Cauchy – Goursat problem (5) for the system of partial differential equations (4) on the jet-submanifold M_0^∞. First, the detailed analysis will be carried out for the system (4).

2. General differential-geometric analysis. At the beginning we prove that evolution equations (4) are integrable by quadratures, being linearized via a Lax type representation. Via the gradient-holonomic algorithm [8, 9] the system (4) on the jet-submanifold $M_0^\infty \subset J\left(\mathbf{R}\times\mathbf{R}^2;\mathbf{R}^3\right)$ can be recast into a set of 2-forms $\{\alpha\} \subset \subset \Lambda^2\left(J^0\left(\mathbf{R}\times\mathbf{R}^2;\mathbf{R}^3\right)\right)$ upon the adjoint jet-manifold $J^0\left(\mathbf{R}\times\mathbf{R}^2;\mathbf{R}^3\right)$ as follows:

$$\begin{gathered}\{\alpha\} = \Big\{\alpha_1 := dy^{(0)}\wedge dx + v^{(0)}dx\wedge dt;\ \alpha_2 := dv^{(0)}\wedge dt - u^{(0)}v^{(0)}dx\wedge dt; \\ \alpha_3 := du^{(0)}\wedge dx + n\,(n-1)\,(y^{(0)})^{n-2}v^{(0)}dx\wedge dt : \left(t, x, u^{(0)}, v^{(0)}, y^{(0)}\right)^\tau \in M\Big\},\end{gathered} \tag{6}$$

where M is a finite-dimensional submanifold in $J^0\left(\mathbf{R}\times\mathbf{R}^2;\mathbf{R}^3\right)$ with local coordinates $(x,t,u^{(0)}=u,v^{(0)}=v,y^{(0)}=y)$. The set of 2-forms (6) generates a closed ideal $I(\alpha)$, that is $dI(\alpha)\subset I(\alpha)$, since

$$d\alpha_1=-dx\wedge\alpha_2,\qquad d\alpha_2=-v^{(0)}dt\wedge\alpha_3+u^{(0)}dx\wedge\alpha_2,$$
$$d\alpha_3=n(n-1)(n-2)v^{(0)}(y^{(0)})^{n-3}dt\wedge\alpha_1-n(n-1)(y^{(0)})^{n-2}dx\wedge\alpha_2. \tag{7}$$

Therefore, the ideal $I(\alpha)$ is Cartan – Frobenius integrable (due to the Cartan theorem) with the integral three-dimensional integral submanifold $N^3=\left\{(x,t)\in\mathbf{R}^3\right\}\subset M$, being defined locally by the condition $I(\alpha)=0$.

Integrability by quadratures of system (4) is equivalent [8, 10] to the vanishing on the integral submanifold $N^3\subset M$ the curvature Ω of the corresponding connection form Γ upon the principal fiber space $P(M,G)$:

$$\Omega=d\Gamma+\Gamma\wedge\Gamma\in I(\alpha)\otimes\mathcal{G}, \tag{8}$$

where $\mathcal{G}$ is the Lie algebra of a structure group G.

Now we shall look for this connection form $\Gamma\in\Lambda^1(M)\otimes\mathcal{G}$ belonging to some not still determined Lie algebra $\mathcal{G}$ of a structure group G. This 1-form can be represented using (6), as follows:

$$\Gamma:=\Gamma^{(x)}\left(u^{(0)},v^{(0)},y^{(0)}\right)dx+\Gamma^{(t)}\left(u^{(0)},v^{(0)},y^{(0)}\right)dt, \tag{9}$$

where elements $\Gamma^{(x)},\Gamma^{(t)}\in\mathcal{G}$ satisfy the following determining equations:

$$\Omega\equiv\frac{\partial\Gamma^{(x)}}{\partial u^{(0)}}du^{(0)}\wedge dx+\frac{\partial\Gamma^{(x)}}{\partial v^{(0)}}dv^{(0)}\wedge dx+\frac{\partial\Gamma^{(x)}}{\partial y^{(0)}}dy^{(0)}\wedge dx+\frac{\partial\Gamma^{(t)}}{\partial u^{(0)}}du^{(0)}\wedge dt+$$
$$+\frac{\partial\Gamma^{(t)}}{\partial v^{(0)}}dv^{(0)}\wedge dt+\frac{\partial\Gamma^{(t)}}{\partial y^{(0)}}dy^{(0)}\wedge dt+[\Gamma^{(x)},\Gamma^{(t)}]dx\wedge dt=$$
$$=g_1(dy^{(0)}\wedge dx+v^{(0)}dx\wedge dt)+g_2(dv^{(0)}\wedge dt-u^{(0)}v^{(0)}dx\wedge dt)+$$
$$+g_3(du^{(0)}\wedge dx+n(n-1)(y^{(0)})^{n-2}v^{(0)}dx\wedge dt)\in I(\alpha)\otimes\mathcal{G} \tag{10}$$

for some $\mathcal{G}$-valued functions g_1,g_2,g_3 on M. From (10) one easily finds that

$$\frac{\partial\Gamma^{(x)}}{\partial u^{(0)}}=g_3,\quad\frac{\partial\Gamma^{(x)}}{\partial v^{(0)}}=0,\quad\frac{\partial\Gamma^{(x)}}{\partial y^{(0)}}=g_1,\quad\frac{\partial\Gamma^{(t)}}{\partial u^{(0)}}=0,\quad\frac{\partial\Gamma^{(t)}}{\partial v^{(0)}}=g_2,$$
$$\frac{\partial\Gamma^{(t)}}{\partial y^{(0)}}=0,\quad\left[\Gamma^{(x)},\Gamma^{(t)}\right]=g_1v^{(0)}-g_2u^{(0)}v^{(0)}+g_3n(n-1)(y^{(0)})^{n-2}v^{(0)}. \tag{11}$$

The set (11) has the following unique global solution:

$$\Gamma^{(x)}=X_1u^{(0)}+\sum_{m=2}^{n}X_m(y^{(0)})^{n-m},\quad\Gamma^{(t)}=X_0v^{(0)}, \tag{12}$$

where $X_j \in \mathcal{G}, j = \overline{0, n}$, are some constant on M elements of the sought Lie algebra $\mathcal{G}$, satisfying the following structure equations:

$$[X_1, X_0] = -X_0,\ [X_2, X_0] = n\,(n-1)\,X_1,\ [X_{m+1}, X_0] = (n-m)\,X_m,\ n \geq m \geq 2. \quad (13)$$

3. Lie-algebraic analysis of the case $n = 2$***.3.1.*** We can use (8) for determining the Lie algebra structure of $\mathcal{G}$, taking into account the holonomy Lie group reduction theorem of Ambrose, Singer and Loos [11]. Namely, the holonomy Lie algebra $\mathcal{G}(h) \subset \mathcal{G}$ is generated by the covariant derivatives composition of the $\mathcal{G}$-valued curvature form $\Omega \in \Lambda^2(M) \otimes \mathcal{G}$:

$$\mathcal{G} := \mathrm{span}_{\mathbf{C}}\{\nabla_1^{j_1}\nabla_2^{j_2}\ldots\nabla_n^{j_n}\Omega_{si} \in \mathcal{G} : j_k \in \mathbf{Z}_+, s, i, k = \overline{1, n}\}, \quad (14)$$

where by definition, the covariant derivative $\nabla_j : \Lambda(M) \longrightarrow \Lambda(M), j = \overline{1, n}$, is given as follows $\nabla_j := \dfrac{\partial}{\partial z_j} + \Gamma^{(j)}(z)$, $z_j \in M$, $j = \overline{1, n}$. Therefore, reducing via the Ambrose – Singer theorem the associated principal fibered frame space $P(M, G)$ to the principal fiber bundle $P(M, G(h))$, where $G(h) \subset G$ is the corresponding holonomy Lie group of the connection Γ on P, we must verify the following conditions for the set $\mathcal{G}(h) \subset \mathcal{G}$ to be a subalgebra in $\mathcal{G}$: $\nabla_x^m\nabla_t^n\Omega \in \mathcal{G}(h)$ for all $m, n \in \mathbf{Z}_+$. To do this we shall try to close the above transfinite procedure. One can easily verify, that the simplest equality

$$\mathcal{G}(h) = \mathcal{G}(h)_1 := \mathrm{span}_{\mathbf{C}}\left\{\nabla_x^m\nabla_t^n\Omega \in \mathcal{G} : m + n = \overline{0, 1}\right\}$$

meets all of the conditions mentioned above. This means that one can put

$$\mathcal{G}(h) = \mathcal{G}(h)_1 := \mathrm{span}_{\mathbf{C}}\left\{\nabla_x^m\nabla_t^n g_j \in \mathcal{G} : j = \overline{1, 3}, m + n = \overline{0, 1}\right\} =$$

$$= \mathrm{span}_{\mathbf{C}}\left\{g_j \in \mathcal{G}; \frac{\partial g_j}{\partial x} + [g_j, \Gamma^{(x)}], \frac{\partial g_j}{\partial t} + [g_j, \Gamma^{(t)}] \in \mathcal{G} : j = \overline{1, 3}\right\} =$$

$$= \mathrm{span}_{\mathbf{C}}\{X_0, X_1, [X_1, X_2]\} = \mathrm{span}_{\mathbf{C}}\left\{X_j \in \mathcal{G} : j = \overline{0, 3}, j \neq 2\right\}, \quad (15)$$

where, by definition, $[X_1, X_2] = X_3 \in \mathcal{G}$. To satisfy the set of relations (13) we need to use expansions over the basis (15) of the external element $X_2 \in \mathcal{G}(h)$:

$$X_2 := \sum_{j=0, j\neq 2}^{3} q_j X_j. \quad (16)$$

Substituting the expansion (16) into (13) we obtain that $q_0 = -\lambda, q_1 = 0, q_3 = 1$, for an arbitrary complex parameter $\lambda \in \mathbf{C}$, that is $\mathcal{G}(h) = \mathrm{span}_{\mathbf{C}}\{X_0, X_1, X_3\}$, where

$$[X_0, X_3] = -2X_1, \quad [X_1, X_3] = -\lambda X_0 + X_3, \quad X_2 = -\lambda X_0 + X_3. \quad (17)$$

We can now state that this finite-dimensional holonomy Lie algebra $\mathcal{G}(h)$, being generated by comutator relationships (13) and (17), possesses the following a general solution:

$$X_0 = L_{-1} - 2L_0 + L_1, \quad X_1 = L_{-1} - L_0, \quad X_2 = \left(-\frac{\lambda}{2} - 1\right)L_{-1} + \lambda L_0 - \frac{\lambda}{2}L_1, \quad (18)$$

with L_{-1}, L_0, L_1 satisfying canonical $sl(2)$-commutation relations and $\lambda \in \mathbf{C}$ being a parameter.

It is easy to find such a (2×2)-matrix representation of L-elements of $\mathcal{G}$:

$$L_{-1}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\quad L_0=\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad L_1=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}. \tag{19}$$

Therefore, from (12), (18) and (19) we obtain the following $\Gamma^{(x)}, \Gamma^{(t)}$ expressions:

$$\Gamma^{(x)}=\begin{bmatrix}-\dfrac{u}{2}+\dfrac{\lambda}{2} & \dfrac{\lambda}{2}\\ u-\dfrac{\lambda}{2}-1 & \dfrac{u}{2}-\dfrac{\lambda}{2}\end{bmatrix},\quad \Gamma^{(t)}=\begin{bmatrix}-v & -v\\ v & v\end{bmatrix}, \tag{20}$$

and the following 1-form $\Gamma\in\Lambda^1(M)\otimes\mathcal{G}$

$$\Gamma=(X_1u+X_2)dx+X_0vdt, \tag{21}$$

generating parallel transporting of vectors $f\in\mathbf{C}^2$ from a linear representation space $F\simeq\mathbf{C}^2$ of the holonomy Lie algebra $\mathcal{G}(h)$:

$$df+\Gamma f=0 \tag{22}$$

along the integral submanifold $N^3\subset M$ of the ideal $I(\alpha)$, generated by the set of 2-forms (6). The result (22) means also that the dynamical system (4) is endowed with the standard Lax type representation, containing the spectral parameter $\lambda\in\mathbf{C}$, that is necessary for its integrability by quadratures [1, 8].

3.2. Now we shall proceed to consider the following generalized spectral problem for system (4):

$$Lf:=\left(\frac{\partial}{\partial x}+\Gamma^{(x)}[u,v,\lambda]\right)f=0, f\in L_\infty\left(\mathbf{R};\mathbf{C}^2\right), \tag{23}$$

posed in the Banach space $L_\infty\left(\mathbf{R};\mathbf{C}^2\right)$ with the manifold M_0^∞ being taken 2π-periodic in $x\in$ $\in\mathbf{R}$. We need to analyze [8, 12], in more detail, spectral properties of the problem (23).

Let $Y(x,x_0;\lambda)\in L(\mathbf{C}^2,\mathbf{C}^2)$ be the fundamental solution of equation (23), being normalized to the identity matrix at $x=x_0\in\mathbf{R}$, i. e. $Y(x_0,x_0;\lambda)=\mathbf{1}$ for all $x_0\in\mathbf{R},\ \lambda\in\mathbf{C}$. Any solution of (23) can be evidently represented as

$$f(x,x_0;\lambda)=Y(x,x_0;\lambda)f_0(\lambda), \tag{24}$$

where $f_0(\lambda)\in\mathbf{C}^2$ is some initial Cauchy data at $x=x_0\in\mathbf{R}$. Consider the value of $f(x,x_0;\lambda)\in$ $\in\mathbf{C}^2$ at $x=x_0+2\pi N$, where $N\in\mathbf{Z}$. Owing to the periodicity of the manifold M_0^∞ in the independent variable $x\in\mathbf{R}$, one obtains from (24) that

$$f(x_0+2\pi N,x_0;\lambda)=S^N(x_0;\lambda)f_0(\lambda),$$

where $S(x_0;\lambda):=Y(x_0+2\pi,x_0;\lambda):\mathbf{C}^2\to\mathbf{C}^2$ is the so called monodromy (transfer) matrix of the periodic differential equation (23).

The monodromy matrix $S(x_0;\lambda)$ possesses the following useful properties [8, 12]:

1°) the matrix $S(x_0;\lambda)$, $\lambda\in\mathbf{C}$, as a function of the parameter $x_0\in\mathbf{R}$ satisfies the following Novikov – Marchenko equation:

$$\frac{dS}{dx_0}=[-\Gamma^{(x)},S] \tag{25}$$

for all $x_0\in\mathbf{R}$, where $[.,.]$ is the usual matrix commutator;

2°) the eigenvalue $\varrho(\lambda)$ of the matrix $S(x_0;\lambda)$, $\lambda \in \mathbf{C}$, does not depend on the variable $x_0 \in \mathbf{R}$;

3°) the eigenvalue $\varrho(\lambda)$ of the matrix $S(x_0;\lambda)$, $\lambda \in \mathbf{C}$, as a functional on manifold M_0^∞, is independent of the evolution parameter $t \in \mathbf{R}^2$.

Since $\dfrac{d\varrho(\lambda)}{dt} = 0$, for all $t \in \mathbf{R}^2$, we claim that as $|\lambda| \to \infty$, $\varrho(\lambda) \in \mathcal{D}(M_0^\infty)$ is a generating functional of conservation laws of system (4). Hence, one can find an infinite hierarchy of conservation laws of system (4), making use of (23). We introduce the function $\sigma(x;\lambda) :=$ $:= \dfrac{\partial}{\partial x} \ln \overline{f_1}(x, x_0;\lambda)$, where $\overline{f_1}(x, x_0;\lambda)$ is the first coordinate of the vector-eigenfunction $\overline{f} \in$ $\in L_\infty\left(\mathbf{R};\mathbf{C}^2\right)$ of the monodromy matrix $S(x;\lambda)$:

$$S(x;\lambda)\overline{f}(x, x_0;\lambda) = \varrho(\lambda)\overline{f}(x, x_0;\lambda),$$

that is normalized at $x = x_0$ by identity, i. e. $\overline{f_1}(x = x_0, x_0;\lambda) = 1$. Substituting the function $\overline{f}$ into (23) one can find the following differential Riccati equation for the function $\sigma(x;\lambda)$:

$$\sigma_x = -\sigma^2 + \frac{1}{2}u_x - \left(-u + \frac{\lambda}{2} + 1\right)\frac{\lambda}{2} + \left(\frac{1}{2}u - \frac{\lambda}{2}\right)^2, \quad u = v_x/v, \tag{26}$$

where $(.)_{nx} := \dfrac{d^n(.)}{dx^n}$, $x \in \mathbf{R}$, $n \in \mathbf{Z}_+$. Assuming that $\sigma(x;\lambda)$ allows an asymptotic solution :

$$\sigma(x;\lambda) \cong \delta^m(\lambda) \sum_{j \in \mathbf{Z}_+} \sigma_j[u,v]\delta^{-j}(\lambda), \tag{27}$$

as $|\lambda| \to +\infty$, with respect to the parameter $\delta(\lambda)$ being analytic on $\mathbf{C}$, and $m \in \mathbf{Z}_+$ is some fixed number, one can find from (26) recurrent equations for $\sigma_j[u,v]$, $j \in \mathbf{Z}_+$:

$$-\sigma_0^2 = \frac{1}{2}, \;\; \sigma_{0,x} = -(\sigma_0\sigma_1)^2, \quad \sigma_{1,x} = -(2\sigma_0\sigma_2 + \sigma_1^2) + \frac{u_x}{2} + \frac{u^2}{4},$$

$$\sigma_{2,x} = -(2\sigma_0\sigma_3 + 2\sigma_1\sigma_2), \quad \sigma_{3,x} = -(2\sigma_0\sigma_4 + 2\sigma_1\sigma_3 + \sigma_2^2),$$

$$\sigma_{4,x} = -(2\sigma_0\sigma_5 + 2\sigma_1\sigma_4 + 2\sigma_2\sigma_3), \ldots$$

and from them the following polynomial functionals $\sigma_j[u,v]$, $j \in \mathbf{Z}_+$:

$$\sigma_0 = \frac{\sqrt{2}}{2}i, \quad \sigma_1 = 0, \quad \sigma_2 = -\frac{\sqrt{2}i}{4}\left(\frac{1}{2}u^2 + u_x\right), \quad \sigma_3 = \frac{1}{4}(u_{xx} + uu_x),$$

$$\sigma_4 = \frac{i\sqrt{2}}{16}\left(2uu_{xx} + 2u_{xxx} + u_x^2 - \frac{u^4}{4} - u_x u^2\right),$$

$$\sigma_5 = -\frac{1}{8}(u_{xxxx} + uu_{xxx} + u_x u_{xx} - u^3 u_x - u^2 u_{xx} - 2u_x^2 u),$$

$$\sigma_6 = \frac{\sqrt{2}}{32}i\left(-2u_{xxxxx} - 2u_x u_{xxx} - u_{xx}^2 - 2uu_{xxxx} + \frac{13}{2}u^2u_x^2 + 3u_{xx}u^3 + 3u^2u_{xxx} + \right.$$
$$\left. + 16uu_xu_{xx} + 5u_x^3 - \frac{3}{4}u^4u_x - \frac{u^6}{8}\right), \ldots .$$

Since $\varrho(\lambda) = \exp\left[\int_0^{2\pi} \sigma(x;\lambda)dx\right]$, from (27) and 3^o) one gets right away that all functionals $\gamma_j = \int_0^{2\pi} dx\sigma_j\,[u,v]$, $j \in \mathbf{Z}_+$, are conservation laws for (4). Thus we have explicit expression for γ_j, $j \in \mathbf{Z}_+$:

$$\gamma_0 = \pi i\sqrt{2}, \quad \gamma_1 = 0, \quad \gamma_2 = \frac{i\sqrt{2}}{8}\int_0^{2\pi} u^2 dx, \quad \gamma_3 = 0, \quad \gamma_4 = \frac{i\sqrt{2}}{16}\int_0^{2\pi}\left(uu_{xx} - \frac{1}{4}u^4\right)dx,$$

$$\gamma_5 = 0, \quad \gamma_6 = -\frac{i\sqrt{2}}{32}\int_0^{2\pi}\left(-u_x u_{xxx} + \frac{5}{2}u^2 u_x^2 + \frac{u^6}{8}\right)dx, \quad \gamma_7 = 0, \tag{28}$$

$$\gamma_8 = \frac{i\sqrt{2}}{64}\int_0^{2\pi}\left(-u_{xxxxx}u_x + \frac{7}{2}u^2 u_x u_{xxx} + \frac{7}{4}uu_x^2 u_{xx} + \frac{7}{8}u^5 u_{xx} - \frac{5}{64}u^8\right)dx, \ldots .$$

3.3. For a further analysis of system (4) we make use of the Novikov – Marchenko equation (25). Let us denote $\Delta(\lambda) :=\mathrm{Tr}S(x;\lambda)$, $\lambda \in \mathbf{C}$, as a normalized trace of the monodromy matrix $S(x;\lambda) : \mathbf{C}^2 \to \mathbf{C}^2$ at $x \in \mathbf{R}$. By virtue of the results stated above we can claim that the functional $\Delta(\lambda) \in \mathcal{D}(M_0^\infty)$ is a generating function of conservation laws for (4). The same is evidently true also for all functionals $\mathrm{Tr}S^k(x;\lambda)$, $k \in \mathbf{Z}$, $\lambda \in \mathbf{C}$, at $x \in \mathbf{R}$, but not all of them are obviously functionally independent.

Making now use of the Novikov – Marchenko equation (25) one can obtain [1, 12] that the following relation holds

$$\eta \mathrm{grad}\Delta(\lambda) = \lambda\vartheta \mathrm{grad}\Delta(\lambda), \tag{29}$$

for all $\lambda \in \mathbf{C}$, where (η, ϑ) is a pair of implectic operators [12] acting from $T^*(M_0^\infty)$ into $T(M_0^\infty)$. Since the system (4) is considered on the singular submanifold M_0^∞, we are forced to introduce new frame-regularizing coordinates $(\widetilde{x}, \widetilde{t}) \in \mathbf{R} \times \mathbf{R}^2$, defined as follows:

$$\widetilde{x} = x, \quad \widetilde{t} = x\mathbf{1} + t, \tag{30}$$

where $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In these coordinates, the system (4) takes the form of a nondegenerate evolution system on $J(\mathbf{R}^1;\mathbf{R}^2)$:

$$\begin{aligned} \frac{d\widetilde{u}}{d\widetilde{t}} &= 2\widetilde{v}, \\ \frac{d\widetilde{v}}{d\widetilde{t}} &= \widetilde{v}_{\widetilde{x}} - \widetilde{u}\widetilde{v}. \end{aligned} \tag{31}$$

The corresponding to (20) connection matrices $\Gamma^{(\widetilde{x})}, \Gamma^{(\widetilde{t})} \in \mathcal{G}$, are given as follows:

$$\Gamma^{(\widetilde{x})} = \begin{bmatrix} -\dfrac{\widetilde{u}}{2} - \widetilde{v} + \dfrac{\lambda}{2} & \dfrac{\lambda}{2} - \widetilde{v} \\ \widetilde{u} + \widetilde{v} - \dfrac{\lambda}{2} - 1 & \dfrac{u}{2} + \widetilde{v} - \dfrac{\lambda}{2} \end{bmatrix}, \quad \Gamma^{(\widetilde{t})} = \begin{bmatrix} -\widetilde{v} & -\widetilde{v} \\ \widetilde{v} & \widetilde{v} \end{bmatrix}. \tag{32}$$

Whence the Novikov – Marchenko equation (25) takes the form:

$$
\begin{aligned}
s_{\tilde{x}} &= \left(\tilde{v} - \frac{\lambda}{2}\right) s_{21} + \left(\tilde{v} + \tilde{u} - \frac{\lambda}{2} - 1\right) s_{12}, \\
s_{12,\tilde{x}} &= -2\left(\tilde{v} - \frac{\lambda}{2}\right) s + (2\tilde{v} + \tilde{u} - \lambda) s_{12}, \\
s_{21,\tilde{x}} &= -2\left(\tilde{v} + \tilde{u} - \frac{\lambda}{2} - 1\right) s + (-\tilde{u} - 2\tilde{v} + \lambda) s_{21},
\end{aligned}
\tag{33}
$$

for the components $s_{ij}(\tilde{x};\lambda)$, $i,j=\overline{1,2}$, of the monodromy matrix $S(\tilde{x};\lambda):\mathbf{C}^2\to\mathbf{C}^2$ at $\tilde{x}\in\mathbf{R}$ for all $\lambda\in\mathbf{C}$, where $s:=\frac{1}{2}(s_{11}-s_{22})$.

Having taken the operator $L:=\frac{\partial}{\partial\tilde{x}}+\Gamma^{(\tilde{x})}$, with $\Gamma^{(\tilde{x})}$ being given by (32), one finds easily that

$$
\operatorname{grad}\Delta(\lambda) = \frac{1}{2}\begin{pmatrix} s - s_{12} \\ 2s - s_{12} + s_{21} \end{pmatrix}, \tag{34}
$$

for all $\lambda\in\mathbf{C}$.

As a result of expression (34) and equations (33) one arrives at the following spectral gradient identity:

$$
\tilde{\eta}\operatorname{grad}\Delta(\lambda) = \lambda\tilde{\vartheta}\operatorname{grad}\Delta(\lambda), \tag{35}
$$

valid for all $\lambda\in\mathbf{C}$, where

$$
\tilde{\eta} = \begin{bmatrix} -2\partial & \partial^2 + \partial\tilde{u} \\ -\partial^2 + \tilde{u}\partial & \tilde{v}\partial + \partial\tilde{v} \end{bmatrix}, \quad \tilde{\vartheta} = \begin{bmatrix} 0 & 0 \\ 0 & \partial \end{bmatrix}, \tag{36}
$$

with $\partial := \frac{\partial}{\partial\tilde{x}}$.

Based on (34) and (35) one can further make the Moser reduction [8] of our problem (31) on a finite-dimensional nonlocal invariant submanifold carrying a natural symplectic structure. The latter gives rise to a new reduction of the problem to a one of Liouville – Arnold type.

3.4. Below we shall analyze, using some results of [12], the next useful for further representation of the holonomy Lie algebra $sl(2)$ by means of such vector fields on the circle $\mathbf{S}^1$:

$$
L_{-1} = \frac{\partial}{\partial\xi}, \quad L_0 = \xi\frac{\partial}{\partial\xi}, \quad L_1 = \xi^2\frac{\partial}{\partial\xi}, \qquad \xi\in\mathbf{S}^1\simeq\mathbf{R}/2\pi\mathbf{Z}.
$$

The corresponding connection operators $\Gamma^{(x)}$ and $\Gamma^{(t)}\in\mathcal{G}$ then take the form:

$$
\Gamma^{(x)} = \left(-\frac{\lambda}{2}\xi^2 - (u-\lambda)\xi + \left(u - \frac{\lambda}{2} - 1\right)\right)\frac{\partial}{\partial\xi}, \quad \Gamma^{(t)} = (v - 2v\xi - v\xi^2)\frac{\partial}{\partial\xi}. \tag{37}
$$

Consider now the following system:

$$
\frac{\partial f}{\partial x} = A\frac{\partial f}{\partial\xi}, \quad \frac{\partial f}{\partial t} = B\frac{\partial f}{\partial\xi} \tag{38}
$$

where $f \in H(\mathbf{C})$ is some complex analytical continuation of f from the circle $\mathbf{S}^1$ to the whole $\mathbf{C}$, and $A\dfrac{\partial}{\partial \xi} = -\Gamma^{(x)}$, $B\dfrac{\partial}{\partial \xi} = -\Gamma^{(t)}$, as defined by (37).

The following lemmas [13] are true:

Lemma 2. *The dynamical system (4) is equivalent to the following vector field Lax type representation:*

$$\frac{d\widetilde{L}}{dt} = \left[\widetilde{L}, \widetilde{P}(l)\right], \tag{39}$$

where $\widetilde{L} = \dfrac{\partial l}{\partial x}\dfrac{\partial}{\partial \xi} - \dfrac{\partial l}{\partial \xi}\dfrac{\partial}{\partial x}$, $\widetilde{P}(l) = \dfrac{\partial p}{\partial x}\dfrac{\partial}{\partial \xi} - \dfrac{\partial p}{\partial \xi}\dfrac{\partial}{\partial x}$ *and* $l := l(x;\xi) \cong \sum\limits_{j\geq 0} l_j[u,v]\,\xi^{-(j-s)} \in H(\mathbf{C})$ *is an element of the Lie algebra,* ξ *is symbol expressions,* $s \in \mathbf{Z}_+$ *is some nonnegative integer.*

A proof of (39) follows simply from the definition of the associative product in the space of reduced pseudo-differential expressions, which one verifies via a straightforward calculation.

Lemma 3. *The following equality holds:* $l \equiv f$*, where* f *is an analytical solution of (38).*

Proof. Since $\widetilde{L}f = 0$ with the operator $\widetilde{L}$ being defined in Lemma 2, using the usual characteristics method one arrives at the equality

$$\frac{d\xi}{dx} = -\frac{\partial l}{\partial x}\Big/\frac{\partial l}{\partial \xi}.$$

On the other hand, from the characteristics equations of the first equation in (38), one finds that $dx/1 = -d\xi/A$. Whence $A\dfrac{\partial l}{\partial \xi} = \dfrac{\partial l}{\partial x}$, so $l \in H(\mathbf{C})$ is a solution of (38), which proves the lemma.

Taking into account Lemmas 1 and 2, one can find the following asymptotic solution of (38)

$$l(x;\xi) \cong \sum_{j\geq 1} l_j[u,v]\,\xi^{-(j-1)}.$$

After simple calculations we find, for functionals $l_j \in H(\mathbf{C})$, $j \in \mathbf{Z}_+$, the following a recurrent chain of differential equations:

$$l_{1,x} = -\frac{\lambda}{2}l_2, \;\; l_{2,x} = (\lambda - u)l_2 - \lambda l_3, \;\; l_{3,x} = \left(u - \frac{\lambda}{2} - 1\right)l_2 + 2(\lambda - u)l_3 - \frac{3}{2}\lambda l_4, \ldots,$$
$$l_{k,x} = \left(u - \frac{\lambda}{2} - 1\right)(k-2)l_{k-1} + (k-1)(\lambda - u)l_k - \frac{k}{2}\lambda l_{k+1}, \ldots$$

giving rise to the following expressions:

$$l_1 = \varphi, \;\; l_2 = -\frac{2}{\lambda}\varphi_x, \;\; l_3 = \frac{2}{\lambda^2}((\lambda - u)\varphi_x + \varphi_{xx}),$$
$$l_4 = \frac{4}{3\lambda^3}\left[\left(-2u^2 - u_x + 3\lambda u - \frac{3}{2}\lambda^2 + \lambda\right)\varphi_x + 3(\lambda - u)\varphi_{xx} - \varphi_{xxx}\right],$$
$$l_5 = \frac{2}{3\lambda^4}\left[\left(-4\lambda^3 + (12u+4)\lambda^2 - (14u^2 + 4u + 6u_x)\lambda + 6u^3 + 7uu_x + u_{xx}\right)\varphi_x + \right.$$
$$\left. + (10\lambda^2 - 2\lambda(10u+1) + 11u^2 + 4u_x)\varphi_{xx} + 6(u - \lambda)\varphi_{xxx} + \varphi_{xxxx}\right], \ldots,$$

with $\varphi = \varphi(x;\lambda)$ being some $\mathbf{R}^2 \ni t$-independent function, accumulating the sought information about the integrable by quadratures Cauchy – Goursat conditions (5).

A relative analysis of this structure obtained above will be done as a separate subject of studying this problem elsewhere.

4. The algebraic-geometric properties of the integrable Riccati equations ($n = 2$). ***4.1.*** From Section 2 it easily follows that the following lemma holds.

Lemma 4. *The system (4) on M_0^∞ possesses the commutator Lax type matrix representation*

$$[X_\lambda, T_\lambda] = 0, \tag{40}$$

where

$$X_\lambda = \frac{\partial}{\partial x} - \begin{bmatrix} \frac{u}{2} - \frac{\lambda}{2} & -\frac{\lambda}{2} \\ -u + \frac{\lambda}{2} + 1 & -\frac{1}{2}u + \frac{\lambda}{2} \end{bmatrix}, \quad T_\lambda = \frac{\partial}{\partial t} - \begin{bmatrix} v & v \\ -v & -v \end{bmatrix}, \tag{41}$$

are linear matrix differential operators in the space of complex vector-valued functions with $\lambda \in \mathbf{C}$ being an arbitrary complex parameter.

The notation (40) means that the comutator of the operators (41) on the solutions of (4) equals identifically zero for all $\lambda \in \mathbf{C}$.

The Lax representation (40) enables us to find the solutions to equations (4) by means of the classical algebraic-geometric methods [1, 10, 12] reducing the problem to the Jacobi problem of inversion of Abelian integrals on hyperelliptic Riemann surfaces, effectively using for its solution the multidimensional Riemann ϑ-functions [14]. Following [12, 15], we consider the following compatible linear equations for the vector-valued complex function $g_\lambda = (g_1, g_2)^\tau \in \in L_\infty(\mathbf{R}^3; \mathbf{C}^2)$:

$$X_\lambda g_\lambda(x,t) = 0, \quad T_\lambda g_\lambda(x,t) = 0, \tag{42}$$

where $\lambda \in \mathbf{C}$ is an arbitrary parameter.

By a direct computation, from (42) one arrives to equations for $\zeta = g_1 g_2,\ \psi = -g_1^2,\ \chi = g_2^2$:

$$\begin{gathered} \frac{\partial \zeta}{\partial x} = -\frac{\lambda}{2}\chi + \psi\left(u - \frac{\lambda}{2} - 1\right), \quad \frac{\partial \zeta}{\partial t} = (\chi + \psi)v, \quad \frac{\partial \psi}{\partial x} = \psi(u - \lambda) + \zeta\lambda, \\ \frac{\partial \psi}{\partial t} = 2v(\psi - \zeta), \quad \frac{\partial \chi}{\partial x} = -\chi(u - \lambda) + 2\zeta\left(-u + \frac{\lambda}{2} + 1\right), \quad \frac{\partial \chi}{\partial t} = -2v(\chi + \zeta), \end{gathered} \tag{43}$$

compatible on the solution submanifold of the system (4).

The solutions of (43) are characterized by the following lemma [4, 9, 15].

Lemma 5. *The system of differential equations (43) possesses a polynomial in $\lambda \in \mathbf{C}$ solution*

$$\zeta = \sum_{k=0}^{N} \zeta_k(x,t)\lambda^k, \quad \psi = \sum_{k=0}^{N} \psi_k(x,t)\lambda^k, \quad \chi = \sum_{k=0}^{N} \chi_k(x,t)\lambda^k \tag{44}$$

with $N \in \mathbf{Z}_+$ fixed, iff the coefficients ζ_k, ψ_k, χ_k, $k = \overline{0, N}$, satisfy certain compatible, autonomous systems of nonlinear nonlocal ordinary differential equations, and

$$u(x,t) = \frac{\psi_{N-1} - \zeta_{N-1}}{\zeta_N}, \quad v_x = \frac{\psi_{N-1} - \zeta_{N-1}}{\zeta_N} v. \tag{45}$$

If, furthermore, the relations

$$\begin{aligned}
&\zeta(x',t',\lambda^\star) = \zeta^\star(x',t',\lambda), \quad \psi(x',t',\lambda^\star) = \psi^\star(x',t',\lambda), \\
&\psi_N(x',t') = \zeta_N(x',t') = -\chi_N(x',t'), \quad \chi(x',t',\lambda^\star) = \chi^\star(x',t',\lambda), \\
&\zeta_{N-1}(x',t') - \psi_{N-1}(x',t') = \zeta^\star_{N-1}(x',t') - \psi^\star_{N-1}(x',t')
\end{aligned} \tag{46}$$

are fulfilled at some point $(x',t') \in \mathbf{R} \times \mathbf{R}^2 (t' := (x'_0, y'_0))$, then these autonomous systems of nonlinear ordinary differential equations have a solution for all $x \in \mathbf{R}$, $t \in \mathbf{R}^2$ and the functions $u(x,t)$ and $v(x,t)$ found from (45) are real infinitely differentiable solutions of (4).

Proof. For the proof, we substitute solution (44) into system (43) and equate the coefficients at the same powers of $\lambda \in \mathbf{C}$, since it is arbitrary. As a result, we obtain systems both of differential and algebraic equations. Using these algebraic equations we obtain two autonomous nonlocal systems of nonlinear ordinary differential equations of the form:

$$\frac{\partial z_i}{\partial x} = F_{1i}(z_0, \ldots, z_{3N+2}), \quad \frac{\partial z_i}{\partial t} = F_{2i}(z_0, \ldots, z_{3N+2}), \tag{47}$$

where $z_i = \zeta_i$, $z_{N+1+i} = \psi_i$, $z_{2N+2+i} = \chi_i$, $i = \overline{0,N}$, and F_{ki}, $k = \overline{1,2}$, $i = \overline{0, 3N+2}$, are polynomials in $z \in \mathbf{R}^{3N+3}$. The compatibility condition of system (47) can be written as

$$\sum_{j=0}^{3N+2} \left(\frac{\partial F_{1i}}{\partial z_j} F_{2j} - \frac{\partial F_{2i}}{\partial z_j} F_{1j} \right) = 0, \quad i = \overline{0, 3N+2},$$

which is verified by a direct calculation.

On the other hand, from these autonomous systems one easily obtains the relationship on the function u:

$$u = \frac{\psi_{N-1} - \zeta_{N-1}}{\zeta_N},$$

which together with the condition $v_x = \dfrac{\psi_{N-1} - \zeta_{N-1}}{\zeta_N} v$ coincides with system (4). This proves the lemma.

Remark. As a consequence of system (43) one easily obtains the equalities $\dfrac{\partial}{\partial t}(\zeta^2 + \chi\psi) = 0$ and $\dfrac{\partial}{\partial x}(\zeta^2 + \chi\psi) = 0$, determining in the natural way from initial data for ζ, ψ, χ a certain polynomial $P_{2N-1}(\lambda)$, $\lambda \in \mathbf{C}$, with constant real coefficients (a consequence of Lemma 5) of the form

$$P_{2N-1}(\lambda) = \zeta^2 + \chi\psi = \sum_{k=0}^{2N-1} p_k \lambda^k = \prod_{j=0}^{2N-2} (\lambda - E_j), \tag{48}$$

with the conditions $P(0) \neq 0$ and $E_i \neq E_j \neq 0$ for $i \neq j = \overline{0, 2N-2}$.

Having used now the expansion of the polynomial solution $\psi(x,t,\lambda)$ with respect to zeros $\mu_j(x,t)$, $j = \overline{1,N}$,

$$\psi = \psi_N \prod_{j=1}^{N} (\lambda - \mu_j), \tag{49}$$

from (43), passing to the limit as $\lambda \longrightarrow \mu_j(x,t)$, for each $j = \overline{1,N}$, we get the following system of nonlinear equations for the zeros μ_j, $j = \overline{1,N}$:

$$\frac{\partial \mu_j}{\partial x} = \frac{-\mu_j\sqrt{P_{2N-1}(\mu_j)}}{\prod_{n\neq j}(\mu_j - \mu_n)}, \quad \frac{\partial \mu_j}{\partial t} = u_t \frac{\sqrt{P_{2N-1}(\mu_j)}}{\prod_{n\neq j}(\mu_j - \mu_n)}, \tag{50}$$

where $u_t := \dfrac{du}{dt}$. Since $\psi_N = \mathrm{const}$, we have assumed for convenience that $\psi_N = 1$.

4.2. Equations (50) belong to the class of equations integrable by means of the Abel transformation on the hyperelliptic Riemann surface $\mathcal{R}$ of the function $w = \sqrt{P_{2N-1}(\lambda)}$, $\lambda, w \in \mathbf{C}$. So we shall consider systems (50) as ones defined on the hyperelliptic Riemann surface $\mathcal{R}$ of genus $N-1$ of the function $\sqrt{P_{2N-1}(\lambda)}$. This surface can be realized as a two-sheeted covering surface of the extended complex plane $\mathbf{C}$ with cuts along the intervals $[E_{2j}, E_{2j+1}]$, $j = \overline{0, N-2}$, $[E_{2N-2}, \infty]$.

Let $\omega_j(\lambda),\ j = \overline{1,N}$, be the following Abelian integrals:

$$\omega_j(\lambda) = \int\limits_{\lambda_0}^{\lambda} \frac{q_j(\xi)d\xi}{\xi\sqrt{P_{2N-1}(\xi)}}, \quad q_j(\xi) = \sum_{k=1}^{N} C_{jk}\xi^{N-k}, \tag{51}$$

normalized by the conditions:

$$\oint\limits_{a_k} d\omega_j(\lambda) = \delta_{kj}, \tag{52}$$

where $a_k, k = \overline{1, N-1}$, are a-cycles of the Riemann surface $\mathcal{R}$, and a_N is the outline on the upper sheet of $\mathcal{R}$, surrounding point $0 \in \mathcal{R}$.

It easily follows from (52) [14] that the coefficients C_{jk}, $j,k = \overline{1,N}$, of (51) are unique.

The zeros E_j, $j = \overline{0, 2N-2}$, of the polynomial $P_{2N-1}(\lambda)$ are not singular, so the integrals $\omega_j(\lambda)$, $j = \overline{1,N}$, have singularities only in the points 0^+ and $0^- \in \mathcal{R}$ (zeros on the upper and lower sheet of $\mathcal{R}$).

The differentials $d\omega_j$, $j = \overline{1,N}$, in the neighbourhoods of the points $0^\pm \in \mathcal{R}$ have the form:

$$d\omega_j(\lambda) = \left(\pm\frac{C_{jN}}{\lambda\sqrt{P_{2N-1}(0)}} + \mathrm{reg}\,(\lambda)\right)d\lambda,$$

with logarythmic residua at 0^+ and $0^- \in \mathcal{R}$ equal to

$$\pm\frac{C_{jN}}{\sqrt{P_{2N-1}(0)}} = \pm\frac{\delta_{jN}}{2\pi i},$$

respectively. Thus $d\omega_j$, $j = \overline{1, N-1}$, are Abelian differentials of the first kind on $\mathcal{R}$, and $d\omega_N$ is an Abelian differential of the third kind on $\mathcal{R}$ with logarythmic singularities at 0^+ and $0^- \in \mathcal{R}$ and residua equal to 1 and -1, respectively. The differential $d\omega_N$ is normalized, since its a-periods are equal to zero.

The standard [14, 16] Abel substitution

$$\nu_j(x,t) = \sum_{k=1}^{N} \omega_j(\mu_k(x,t)), \tag{53}$$

turns (50) into

$$\nu_j(x,t) = C_{j1}x + \frac{C_{jN}}{(-1)^N}\gamma(t) + \nu_j(0,0), \tag{54}$$

where $j = \overline{1,N}$ and the introduced function $\gamma(t)$, $t \in \mathbf{R}^2$, satisfies the equations

$$\frac{\partial\gamma}{\partial t} = -u_t \prod_{j=1}^{N} \mu_j^{-1}, \quad \frac{\partial\gamma}{\partial x} = 0, \quad \gamma(0) = 0. \tag{55}$$

Considering (53) as expressions on $\mathcal{R}$, we get the following inversion problem: given $\nu_j(x,t), j = \overline{1,N}$, find $\mu_j(x,t), j = \overline{1,N}$, making use of equations (55).

This Jacobi inversion problem is not evidently standard, being considered on the Riemann surface $\mathcal{R}$ of genus $N-1$ with the normalized base of Abelian integrals of the first kind $\omega_j,\ j = \overline{1,N-1}$, and the one normalized Abelian integral of the third kind ω_N.

To solve this problem we consider the ε-deformed hyperelliptic Riemann surface $\mathcal{R}_\varepsilon, \varepsilon \in \mathbf{C}$, of the function $\sqrt{(\lambda^2+\varepsilon^2)P_{2N-1}(\lambda)}$, having just the genus N, and the following augmented equations on this surface:

$$\frac{\partial\mu_{j,\varepsilon}}{\partial x} = -\frac{\sqrt{(\mu_{j,\varepsilon}^2+\varepsilon^2)P_{2N-1}(\mu_{j,\varepsilon})}}{\prod_{n\neq j}(\mu_{j,\varepsilon}-\mu_{n,\varepsilon})}, \quad \frac{\partial\mu_{j,\varepsilon}}{\partial t} = \frac{u_t\sqrt{(\mu_{j,\varepsilon}^2+\varepsilon^2)P_{2N-1}(\mu_{j,\varepsilon})}}{\mu_{j,\varepsilon}\prod_{n\neq j}(\mu_{j,\varepsilon}-\mu_{n,\varepsilon})}, \tag{56}$$

where $\mu_{j,\varepsilon} \in \mathcal{R}_\varepsilon, j = \overline{1,N}$.

The following lemma is true.

Lemma 6. *Let the initial data for equations (50) and (56) satisfy the inequalities:*

$$\begin{aligned} &\max\nolimits_{1\le j\le N}|\mu_j(0,0)| < M, \quad \min\nolimits_{i\neq j}|\mu_i(0,0)-\mu_j(0,0)| > m_2, \\ &\min\nolimits_{1\le j\le N}|\mu_j(0,0)| > m_1, \quad \mu_j(0,0) = \mu_{j,\varepsilon}(0,0), \end{aligned} \tag{57}$$

for all $\varepsilon \in \mathbf{C}$, where $M, m_1, m_2 \in \mathbf{R}_+$ are some positive values. Then, there exists a convergent to zero sequence $\{\varepsilon_k : k \in \mathbf{Z}_+\}, \lim_{k\to\infty}\varepsilon_k = 0$, such that $\mu_{j,\varepsilon_k}(x,t)$, for each $j = \overline{1,N}$, uniformly tends to $\mu_j(x,t)$, $j = \overline{1,N}$, as $k\to\infty$ for small enough $x \in \mathbf{R}$ and $t \in \mathbf{R}^2$.

Proof. From (56) and (57), for small enough $x \in \mathbf{R}$ and $t \in \mathbf{R}^2$, one finds that there exist real positive constants C_1 and C_2 being independent on $\varepsilon \in \mathbf{C}$, $(x,t) \in \mathbf{R}\times\mathbf{R}^2$, such that

$$\left|\frac{\partial}{\partial x}\mu_{j,\varepsilon}(x,t)\right| \le C_1, \quad \left|\frac{\partial}{\partial t}\mu_{j,\varepsilon}(x,t)\right| \le C_2,$$

for all $j = \overline{1,N}$. Therefore, a set of functions $\mu_{j,\varepsilon_k}(x,t) : j = \overline{1,N}$, is compact in the uniform metric. Whence there exists a sequence of numbers $\{\varepsilon_k : k \in \mathbf{Z}_+\}$, $\lim_{k\to\infty}\varepsilon_k = 0$, such that for any $j = \overline{1,N}$ the sequence of functions $\{\mu_{j,\varepsilon_k}(x,t) : k \in \mathbf{Z}_+\}$ tends uniformly with regard to $x \in \mathbf{R}$ and $t \in \mathbf{R}^2$ to $\mu_j(x,t)$, $j = \overline{1,N}$, satisfying equations (50) and the chosen initial data. Based on (57), the solution of (50) is unique and thus for all subsequences $\{\varepsilon_k : k \in \mathbf{Z}_+\}$, $\lim_{k\to\infty}\varepsilon_k = 0$, there exists the same $\lim_{k\to\infty}\mu_{j,\varepsilon_k}(x,t)$, which finishes the proof.

Let $\omega_{j,\varepsilon}(z),\ j = \overline{1,N}$, be a normalized base of Abelian integrals of the first kind in $\mathcal{R}_\varepsilon$

$$\omega_{j,\varepsilon}(\lambda) = \int_{\lambda_0}^{\lambda} \frac{q_{j,\varepsilon}(\xi)d\xi}{\sqrt{(\xi^2+\varepsilon^2)P_{2N-1}(\xi)}}, \quad q_{j,\varepsilon}(\xi) = \sum_{k=1}^{N} C_{jk,\varepsilon}\xi^{N-k}. \tag{58}$$

Then the Jacobi inversion problem

$$\nu_{j,\varepsilon}(x,t) = \sum_{k=1}^{N} \omega_{j,\varepsilon}(\mu_{k,\varepsilon}(x,t)) \tag{59}$$

recasts (56) into

$$\nu_{j,\varepsilon}(x,t) = C_{j1\varepsilon}x + \frac{C_{jN,\varepsilon}}{(-1)^N}\gamma_\varepsilon(t) + \nu_{j,\varepsilon}(0,0), \tag{60}$$

where $j=\overline{1,N}$, and $\dfrac{\partial\gamma_\varepsilon}{\partial t} = -u_t\prod_{j=1}^{N}\mu_{j,\varepsilon}^{-1},\ \dfrac{\partial\gamma_\varepsilon}{\partial x}=0,\ \gamma_\varepsilon(0)=0$ for all $x\in\mathbf{R}$, $t\in\mathbf{R}^2$, due to the linearity of (60) in $x\in\mathbf{R}$ and $\gamma_\varepsilon(t)\in\mathbf{C}$.

4.3. For studying the convergence of Abelian integrals (58) we make canonical cuts of the Riemann surface $\mathcal{R}_\varepsilon$ by a- and b-cycles. Arbitrary branch of integral $\omega_{j,\varepsilon}(\lambda)$, $j=\overline{1,N}$, is regular on the cut region $\mathcal{R}_\varepsilon^*$, and as $\varepsilon\to 0$, it tends continuously (behind the points 0^+ and 0^-), to the respective branch of the integrals $\omega_j(\lambda)$, $j=\overline{1,N}$, on $\mathcal{R}^*$, the region obtained after canonical cuts of the surface $\mathcal{R}$.

Since the set of the regular Abelian integrals $\{\omega_{j,\varepsilon}(\lambda): j=\overline{1,N}\}$ is uniformly continuous and bounded with respect to $\varepsilon\in\mathbf{C}$, small enough on an arbitrary compact region $K\subset\mathcal{R}^*$ located at a positive distance from the points 0^+ and $0^-\in\mathcal{R}^*$, the Abelian integrals $\omega_{j,\varepsilon}(\lambda)$, $j = \overline{1,N}$, tend uniformly to $\omega_j(\lambda)$ on $K\subset\mathcal{R}^*$, as $\varepsilon\to 0$. From this, it follows that $\lim\limits_{\varepsilon\to 0}\nu_{j,\varepsilon}(x,t) = \nu_j(x,t)$, $j=\overline{1,N}$, iff the initial data $\mu_j(0,0)$, $j=\overline{1,N}$, don't belong to the neighbourhoods of 0^+ and 0^-. So we have shown, that the problem (59) approximates the problem (53) as $\varepsilon\to 0$. The problem (53) evidently is the standard Jacobi inversion problem on the Riemann surface $\mathcal{R}_\varepsilon$. Now we can formulate the following important lemma.

Lemma 7. *If $\varepsilon\to 0$, then the Jacobi inversion problem (59) tends to the Jacobi problem (53), where $\gamma(t):=\lim_{\varepsilon\to 0}\gamma_\varepsilon(t),\ t\in\mathbf{R}^2$, and $\omega_j(\lambda)=\lim\limits_{\varepsilon\to 0}\omega_{j,\varepsilon}(\lambda)$, $j=\overline{1,N}$.*

Assuming that $\mu_k(0,0)\in\mathcal{R}$, $k=\overline{1,N}$, are pairwise distinet, based on the well known Riemann theorem one gets that there exists a nontrivial Riemann θ-function $\theta_\varepsilon(\lambda)=\vartheta_\varepsilon(\omega_\varepsilon(\lambda) - \mathbf{e}_\varepsilon)$, where by definition

$$\vartheta_\varepsilon(\mathbf{u}) = \sum_{\mathbf{m}\in\mathbf{Z}^N}\exp[\pi i < B_\varepsilon\mathbf{m},\mathbf{m}> + 2\pi i<\mathbf{u},\mathbf{m}>],$$

with B_ε being the matrix of B-periods of the base $\{\omega_{j,\varepsilon}(\lambda): j=\overline{1,N}\}$ on $\mathcal{R}_\varepsilon$, $\mathbf{u}\in\mathbf{C}^N$, and $<.,.>$ is the usual inner product in $\mathbf{C}^N$, $\mathbf{e}_\varepsilon=(\nu_{j,\varepsilon}(x,t)+k_{j,\varepsilon}: j=\overline{1,N})$, $k_{j,\varepsilon}=\dfrac{1}{2}\sum\limits_{r=1}^{N}B_{rj,\varepsilon}-\dfrac{j}{2}$, $j=\overline{1,N}$. It is regular on the (a,b)-cycles cut surface $\mathcal{R}_\varepsilon^\star$ and has just N zeros $\mu_{j,\varepsilon}(x,t)$, $j=\overline{1,N}$, namely the solution of the Jacobi inversion problem.

After a direct computation we get that there exists the limit

$$T(\lambda):=\lim_{\varepsilon\to 0}\theta_\varepsilon(\lambda)=\vartheta_{N-1}(\omega(\lambda)-\mathbf{e}^+)+\exp[2\pi i(\omega_N(\lambda)-e_N)]\vartheta_{N-1}(\omega(\lambda)-\mathbf{e}^-) \tag{61}$$

where $\mathbf{e}^\pm=(\nu_j(x,t)+\widehat{k}_j\pm\dfrac{1}{2}B_{Nj}: j=\overline{1,N-1})$, $e_N=\nu_N(x,t)+\widehat{k}_N$, $\widehat{k}_j=\dfrac{1}{2}\sum_{k=1}^{N-1}B_{kj} - \dfrac{j}{2}$, $\omega(\lambda)=(\omega_j(\lambda): j=\overline{1,N-1})$, and $\vartheta_{N-1}(\mathbf{u})$ is the usual Rimann ϑ-function of $\mathbf{u}\in C^{N-1}$, B_{Nj} are B-periods of the Abelian integral $\omega_N(\lambda)$ on the surface $\mathcal{R}$.

Let us consider also the following associated with (61) τ-function on $\mathbf{C}^N$:

$$\tau(\mathbf{u}) = \tau(u_1, \dots, u_N) = \vartheta_{N-1}\left(u_1 - \frac{B_{N1}}{2}, \dots, u_{N-1} - \frac{B_{N,N-1}}{2}\right) +$$

$$+ \exp(2\pi i u_N)\vartheta_{N-1}\left(u_1 + \frac{B_{N1}}{2}, \dots, u_{N-1} + \frac{B_{N,N-1}}{2}\right), \tag{62}$$

where $\mathbf{u} \in \mathbf{C}^N$, and $B_{Nj},\ j = \overline{1, N-1}$, are defined as above.

It is easy to see that

$$T(\lambda) = \tau(\omega(\lambda) - \nu(x,t) - \mathbf{e}),$$

with $\nu(x,t) = (\nu_j(x,t) : j = \overline{1, N-1})$, and ω, $\mathbf{e} \in \mathbf{C}^{N-1}$ defined as above. Some more properties of the function $T(\lambda)$ one can also find in [12].

Lemma 8. *Zeros of the T-function (61) are the solution to the Jacobi inversion problem (59).*

Proof. For the proof, it is sufficient to show that $\mu_j(x,t) = \lim\limits_{\varepsilon\to 0} \mu_{j,\varepsilon}(x,t)$, $j = \overline{1, N}$, that immediately follows from the fact that the functions $\theta_\varepsilon(\lambda)$ tend continuously to the function $T(\lambda)$, $\lambda \in K$, bounded on an arbitrary compact K in $\mathcal{R}^*$ that lies at a positive distance from 0^-.

4.4. Following [1, 12, 14], one can find explicit expressions for the wanted above symmetric functions $\sum \ln \mu_j$ and $\sum \mu_j$.

To find the first sum, we consider the region $\mathcal{R}^*$ with one more intersection along the curve L starting at the point $0^+ \in \mathcal{R}^*$ and ending at $\infty \in \mathcal{R}^*$, and the function $\beta(\lambda) = \ln\lambda + 2\pi i \omega_N(\lambda)$, being the Abelian integral of the third kind with logarythmic residua at 0^+ and $\infty \in \mathcal{R}$ equal to 2 and -2, respectively.

Consider now the integral

$$\mathcal{I} = \frac{1}{2\pi i} \int\limits_{\partial\mathcal{R}^* \cup L^+ \cup L^-} \beta(\lambda) d\ln T(\lambda).$$

By the residua classical theorem,

$$\mathcal{I} = \sum_{k=1}^{N} \beta(\mu_k) - \beta(0^-) = \sum_{k=1}^{N} \ln\mu_k + 2\pi i \sum_{k=1}^{N} \omega_N(\mu_k) - \beta(0^-) =$$

$$= \sum_{k=1}^{N} \ln\mu_k + 2\pi i \nu_N(x,t) - \beta(0^-).$$

On the other hand, since $\beta(\lambda)$ and $T(\lambda)$ are continuous on b-cycles, $\beta^+(\lambda) - \beta^-(\lambda) = -B_j$, $B_j = \oint\limits_{b_j} d\ln\lambda + 2\pi i \oint\limits_{b_j} d\omega_N(\lambda)$, $\lambda \in a_j$, $j = \overline{1, N-1}$, and $\beta^+(\lambda) - \beta^-(\lambda) = -2\pi i$, $\lambda \in L$,

$$\mathcal{I} = \frac{1}{2\pi i}\sum_{k=1}^{N-1}\left(\oint_{a_k^+}\beta^+(\lambda)d\ln T^+(\lambda) - \oint_{a_k^-}\beta^-(\lambda)d\ln T^-(\lambda)\right)+$$

$$+\frac{1}{2\pi i}\left(\int_{L^+}\beta^+(\lambda)d\ln T^+(\lambda) - \int_{L^-}\beta^-(\lambda)d\ln T^-(\lambda)\right) =$$

$$= \sum_{k=1}^{N-1}\left(\oint_{a_k}\beta^-(\lambda)d\ln\frac{T^+(\lambda)}{T^-(\lambda)}\right) - \frac{1}{2\pi i}\sum_{k=1}^{N-1}B_k\oint_{a_k}d\ln T^-(\lambda) - \int_{L^+}d\ln T(\lambda).$$

Therefore,

$$\sum_{k=1}^{N}\ln\mu_k(x,t) = -2\pi i\nu_N(x,t) + \beta(0^-) +$$

$$+\sum_{j=1}^{N-1}\oint_{a_j}\beta^-(\lambda)d\omega_j(\lambda) - \sum_{j=1}^{N-1}B_jn_j + 2\ln\frac{T(\infty)}{T(0^+)}, \qquad (63)$$

where $n_j = \oint_{a_j} d\ln T^-(\lambda),\ j=\overline{1,N}$.

Following [12, 14] we find similarly that

$$\sum_{k=1}^{N}\mu_k(x,t) = \sum_{j=1}^{N-1}\oint_{a_j}\lambda d\omega_j(\lambda) - \mathrm{res}_{\lambda=\infty}(\lambda d\ln T(\lambda)). \qquad (64)$$

First we find the residuum at the right side of (64), making use of expansion of the function $\frac{d}{d\tau}\ln T(\lambda)$ in the series with respect to $\tau = \lambda^{-\frac{1}{2}}$:

$$\frac{d}{d\tau}\ln T(\lambda) = -2\lambda^{\frac{3}{2}}\frac{d}{d\lambda}\ln T(\lambda) = -2\lambda^{\frac{3}{2}}\sum_{j=1}^{N-1}\frac{\partial\ln T(\lambda)}{\partial\overline{\omega}_j}\frac{d\omega_j(\lambda)}{d\lambda} =$$

$$-2\lambda^{\frac{3}{2}}\sum_{j=1}^{N-1}\frac{q_j(\lambda)}{\lambda\sqrt{P_{2N-1}(\lambda)}}\frac{\partial\ln T(\lambda)}{\partial\overline{\omega}_j}.$$

Since $q_j(\lambda) = \sum_{j=1}^{N} C_{jk}\lambda^{N-k},\ j=\overline{1,N-1}$ (see (51))

$$\frac{q_j(\lambda)}{\lambda\sqrt{P_{2N-1}}(\lambda)} = C_{j1}\lambda^{-\frac{3}{2}}\left(1-\frac{d_0}{2}\lambda^{-1}-\frac{d_1}{2}\lambda^{-2}+\cdots\right)+$$

$$+C_{j2}\lambda^{-\frac{5}{2}}\left(1-\frac{d_0}{2}\lambda^{-1}-\frac{d_1}{2}\lambda^{-2}+\cdots\right)+\cdots,$$

with d_0, d_1 being some constants.

Whence

$$\lambda\frac{d}{d\tau}\ln T(\lambda) = -2\sum_{j=1}^{N-1}\left[\frac{C_{j1}}{\tau^2}+\dots\right]\frac{\partial\ln T(\lambda)}{\partial\overline{\omega}_j}$$

and

$$\operatorname{res}_{\lambda=\infty}\{\lambda d\ln T(\lambda)\} = -2\sum_{j=1}^{N-1}C_{j1}\frac{\partial}{\partial\tau}\left(\frac{\partial\ln T(\lambda)}{\partial\overline{\omega}_j}\right)\Big|_{\tau=0} = 4\sum_{i,j=1}^{N-1}C_{j1}C_{i1}\frac{\partial^2\ln T(\lambda)}{\partial\overline{\omega}_j\partial\overline{\omega}_i}.$$

On the other hand, based on (54) one can find that

$$\frac{\partial}{\partial x^2}\ln T(\lambda) = \sum_{i,j=1}^{N-1}C_{j1}C_{i1}\frac{\partial^2\ln T(\lambda)}{\partial\overline{\omega}_j\partial\overline{\omega}_i}.$$

Thus from the results obtained above one gets the following equality:

$$\sum_{k=1}^{N}\mu_k(x,t) = \sum_{j=1}^{N-1}\oint_{a_j}\lambda d\omega_j(\lambda) - 4\frac{\partial^2}{\partial x^2}\ln T(\infty). \tag{65}$$

From (45) and (4) we easily find an exact expression for the solution $y(x,t)$ of (2):

$$y(x,t) = -\frac{a(x)}{2} + \frac{1}{2}\frac{\partial}{\partial x}\ln\prod_{k=1}^{N}\mu_k. \tag{66}$$

Furthermore, in accordance with (5) the following relations are valid for $\gamma(x_0,y_0)$:

$$\frac{\partial\gamma}{\partial x_0} = 2\frac{y_0^2+a(x_0)y_0+b(x_0)}{N(x_0;\gamma)}, \qquad \frac{\partial\gamma}{\partial y_0} = -\frac{2}{N(x_0;\gamma)}, \tag{67}$$

where $N(x;\gamma(x_0,y_0)) = \prod_{j=1}^{N}\mu_j(x;x_0,y_0)$, $(x_0,y_0)\in\mathbf{R}^2$, is determined as

$$N(x;\gamma(x_0,y_0)) =$$

$$= \frac{T^2(\infty)}{T^2(0^+)}\exp\left\{-2\pi i\nu_N(x,t) + \beta(0^-) + \sum_{j=1}^{N-1}\oint_{a_j}\beta^-(\lambda)d\omega_j(\lambda) - \sum_{j=1}^{N-1}B_jn_j\right\}. \tag{68}$$

Lemma 9. *The following identity*

$$\gamma(x,y) = \gamma(x_0,y_0) \tag{69}$$

holds for any solution to the Cauchy – Goursat problem (4), (5).

Proof. One easily finds that the function $\gamma:\mathbf{R}^2\to\mathbf{R}$ satisfies the equation

$$\frac{d\gamma(x,y(x;t))}{dx} = \frac{\partial\gamma}{\partial x} + \frac{\partial\gamma}{\partial y}\frac{dy(x;t)}{dx} = 2\frac{y^2+a(x)y+b(x)}{N(x;\gamma)} - \frac{2}{N(x;\gamma)}\frac{dy(x)}{dx} \equiv 0,$$

on the solutions to (4), where $t=(x_0,y_0)\in\mathbf{R}^2$, that is $\gamma(x,y)=\gamma(x_0,y_0)$ for all $(x_0,y_0)\in\mathbf{R}^2$.

The equality of the mixed derivatives for $\gamma(x_0, y_0)$ yields the expression

$$y_0 = \frac{1}{2}\frac{\partial}{\partial x}\ln N(x = x_0; \gamma(x_0, y_0)) - \frac{a(x_0)}{2}, \tag{70}$$

determining apparently the function $\gamma(x_0, y_0)$ implicitly. Now from (4) and (70) one can find additionaly that

$$u = \frac{\partial}{\partial x}\ln N(x;\gamma), \quad \mathcal{L}_N(x;\gamma(x_0,y_0)) := \frac{u_x}{2} - \frac{1}{4}u^2 =$$

$$= -\frac{a^2(x)}{4} + \frac{a'(x)}{2} + b(x) =: \mathcal{L}(x), \tag{71}$$

that is

$$\mathcal{L}_N(x;\gamma(x_0,y_0)) = \mathcal{L}(x), \tag{72}$$

for all $t := (x_0, y_0) \in \mathbf{R}^2$.

Since from (4), (43), and (49) one can obtain that $v = \psi_{N-1,t}/2(\psi_{N-1} - \zeta_{N-1})$ and thus $u_t = -\sum \mu_{j,t}/u$, making further use of the condition (72) one can easily reduce it to the following characteristic expression

$$\frac{\partial}{\partial \gamma}\left(\sum_{j=1}^{N}(\mu_j + \frac{\partial}{\partial x}\ln \mu_j)\right) = 0. \tag{73}$$

The latter can be used effectively, based on exact formulas (63) and (65), for detecting the Riemann surface parameters generating integrable Riccati equations.

Thus we are in a position to formulate the following theorem.

Theorem. *The Riemann surface $\mathcal{R}$ of the function $w^2 = P_{2N-1}(\lambda)$ satisfying the condition (73), generates solutions to (2) representable by quadratures via (66) for given differentiable functions $a(x)$ and $b(x)$ only if the function $z := \ln N(x;\gamma)$, with $N(x;\gamma)$ being given by (68) and defining via (66) and (71) the sought functions $a, b \in C^1(\mathbf{R};\mathbf{R})$, satisfies the well known integrable by quadratures Liouville equation*

$$\partial^2 z/\partial x \partial \gamma + e^z = 0, \tag{74}$$

equivalent to (73).

At the same time equation (67) makes it possible to find a still unknown function $\gamma : \mathbf{R}^2 \to \mathbf{R}$ incorporated implicitly into the solution expression (66) to the Riccati equation (2).

5. The analysis of the case $n = 3$. For the case $n = 3$ one follows that the holonomy Lie algebra $\mathcal{G}(h)$ is strictly infinite dimensional. This means that there exists no finite-dimensional representation of the holonomy Lie algebra. The simplest infinite dimensional Lie algebra, containing the $sl(2)$-subalgebra, is the Lie algebra of the group of diffeomorphisms of the circle $\mathbf{S}^1$ or the Virasoro algebra:

$$L_j = \xi^{j+1}\frac{\partial}{\partial \xi}, \quad [L_i, L_j] = (j - i)\, L_{i+j}, \tag{75}$$

where $\xi \in \mathbf{S}^1 \simeq \mathbf{R}/2\pi\mathbf{Z}$ and $j \in \mathbf{Z}_+$.

The associated Lie algebra $\mathcal{G}(h)$ being generated by system (13), possesses the following general representation:

$$X_0 = -L_{-1}, \quad X_1 = L_{-1} + L_0, \quad X_2 = \left(3 + \frac{\lambda}{2}\right) L_{-1} + 6L_0 + 3L_1,$$
$$X_3 = \left(2 + \frac{\lambda}{6}\right) L_{-1} + \left(3 + \frac{\lambda}{2}\right) L_0 + 3L_1 + L_2, \tag{76}$$

where $\lambda \in \mathbf{C}$ is a spectral parameter. Having substituted representation (75) into (76), one arrives at the following (A, B) expressions of the induced Cartan – Heresmann connection Γ :

$$A = \left(-\frac{\lambda}{2} y - \frac{1}{3}\right) - \left(u + \frac{\lambda}{2}\right) \xi - 3y\xi^2 - \xi^3, \quad B = v, \tag{77}$$

where $A\frac{\partial}{\partial \xi} = -\Gamma^{(x)}, B\frac{\partial}{\partial \xi} = -\Gamma^{(t)}$. The corresponding symbol expression $l(x;\xi) \cong \cong \sum_{j\geq 1} l_j \xi^{-(j-1)}$ solving equation (37) is described by the following series of coefficients:

$$l_1 = \varphi, \quad l_2 = 0, \quad l_3 = \frac{\varphi_x}{2}, \quad l_4 = -y\varphi_x, \quad l_5 = \frac{1}{4}\left(\frac{1}{2}\varphi_{xx} - \left(u + \frac{\lambda}{2}\right)\varphi_x + 9u^2\varphi_x\right), \ldots ,$$

where $\varphi = \varphi(x;\lambda)$ is also some $\mathbf{R}^2 \ni t$-independent function as was stated before for $n = 2$ containing an information about the integrable by quadratures Cauchy – Goursat conditions (5).

One can also easily verify that for $n = 3$ assertions similar to Lemma 2 and Lemma 3 hold too.

6. Concluding remarks. The next problem under investigation of the integrability of the Riccati – Abel equation (1) consists in analyzing the Lie-algebraic properties of the corresponding solution manifolds for the $n = 3$ case, related to the Cauchy – Goursat conditions (5), and studying their corresponding finite-dimensional Moser type reductions via the momentum mapping approach and methods of the modern symplectic theory.

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