UDC 517 . 9

QUANTUM MATHEMATICS: BACKGROUNDS AND SOME APPLICATIONS TO NONLINEAR DYNAMICAL SYSTEMS

КВАНТОВА МАТЕМАТИКА: ВИТОКИ І ДЕЯКІ ЗАСТОСУВАННЯ ДО НЕЛІНІЙНИХ ДИНАМІЧНИХ СИСТЕМ

N. N. Bogoliubov (jr.)

V. A. Steklov Math. Inst. RAN, Moscow, Russia
and Abdus Salam Int. Center Theor. Physics, Trieste, Italy
e-mail: nickolai_bogolubov@hotmail.com

J. Golenia

AGH Univ. Sci. and Technol.,
Kraków 30-059 Poland
e-mail: goljols@tlen.pl

A. K. Prykarpatsky

Inst. Appl. Problem Mech. and Math.
Nat. Acad. Sci., Lviv, Ukraine
and Abdus Salam Int. Center Theor. Phys., Trieste, Italy
e-mail: pryk.anat@ua.fm, prykanat@cybergal.com

U. Taneri

Eastern Mediterranean Univ. EMU, Famagusta, North Cyprus
and Kyrenia Amer. Univ. GAU,
Inst. Graduate Studies, Kyrenia, North Cyprus
e-mail: ufuk.taneri@gmail.com

Backgrounds of a new mathematical physics discipline 'Quantum Mathematics' are discussed and analyzed from both historical and analytical points of view. The magic properties of the second quantization method, invented by V. Fock in 1934, are demonstrated, and an impressive application to the nonlinear dynamical systems theory is considered.

Обговорюються витоки нової дисципліни в математичній фізиці — квантової математики — як з історичної, так і з аналітичної точок зору. Проілюстровано дивовижні властивості методу вторинного квантування, який був запропонований Фоком у 1934 р., та розглянуто вражаючі застосування до теорії нелінійних динамічних систем.

1. Introduction. Let us look at the beginning of the XXth century and trace the way mathematics has in ifluenced the modern and classical quantum physics, and next observe the way the modern quantum physics is nowadays impressively influencing the modern mathematics. The latter will in part be a main topic of our present work, devoted to the application of the modern quantum mathematics to studying nonlinear dynamical systems in functional spaces. We will begin with a brief history of quantum mathematics.

In the beginning of the XXth century:

P.A.M. Dirac was the first who realized and used the fact that the commutator operation $D_a\colon \mathcal{A} \ni b \longrightarrow [a,b] \in \mathcal{A}$, where $a \in \mathcal{A}$ is fixed and $b \in \mathcal{A}$, is a differentiation of any operator algebra $\mathcal{A}$; moreover, he first constructed a spinor matrix realization of the Poincaré symmetry group $\mathcal{P}(1,3)$ [1] (1920 – 1926);

J. von Neumann first applied the spectral theory of self-adjoint operators on Hilbert spaces to explain the radiation spectra of atoms and the related matter stability [2] (1926);

V. Fock was the first to introduce the notion of many-particle Hilbert space, named Fock space, and introduced the related creation and annihilation operators acting on it [3] (1932);

H. Weyl understood the fundamental role of the notion of symmetry in physics and developed a physics-oriented group theory; moreover he showed the importance of different representations of the classical matrix groups for physics and studied the unitary representations of the Heisenberg – Weyl group related to the creation and annihilation operators on a Fock space [4] (1931).

In the end of the XXth century, new developments are due to

L. Faddeev with co-workers (quantum inverse spectral theory transform [5], 1978);

V. Drinfeld, S. Donaldson, E. Witten (quantum groups and algebras, quantum topology, quantum super-analysis [6 – 8], 1982 – 1994);

Yu. Manin, R. Feynman (quantum information theory [9 – 11], 1980 – 1986);

P. Shor, E. Deutsch, L. Grover and others (quantum computer algorithms [12 – 14], 1985 – 1997).

As one can observe, many exciting and highly important mathematical achievements were strictly motivated by the impressive and deep influence of quantum physics ideas and ways of thinking, leading nowadays to an alltogether new scientific field, often called quantum mathematics.

Following this quantum mathematical way of thinking, we will demonstrate below that a wide class of strictly nonlinear dynamical systems in functional spaces can be treated as a natural object in specially constructed Fock spaces in which the corresponding evolution flows are completely linearized. Thereby, the powerful machinery of classical mathematical tools can be applied to studying the analytical properties of exact solutions to suitably well posed Cauchy problems.

2. Mathematical preliminaries: Fock space and its realizations. Let Φ be a separable Hilbert space, F a topological real linear space, and $\mathcal{A} := \{A(\varphi)\colon \varphi \in F\}$ a family of commuting self-adjoint operators on Φ (i.e., these operators commute in the sense of their resolutions of the identity). Consider the Gelfand rigging [15] of the Hilbert space Φ, i.e., a chain

$$\mathcal{D} \subset \Phi_+ \subset \Phi \subset \Phi_- \subset \mathcal{D}' \tag{2.1}$$

in which Φ_+ and Φ_- are Hilbert spaces, and the inclusions are dense and continuous, i.e., Φ_+ is topologically (densely and continuously) and quasi-nuclearly (the inclusion operator $i\colon \Phi_+ \longrightarrow \Phi$ is of the Hilbert – Schmidt type) embedded into Φ, Φ_- is the dual of Φ_+ with respect to the scalar product $< .,. >_\Phi$ in Φ, and $\mathcal{D}$ is a separable projective limit of Hilbert spaces, topologically embedded in Φ_+. Then, the following structural theorem [15, 16] holds.

Theorem 2.1. *Assume that the family of operators $\mathcal{A}$ satisfies the following conditions:*

a) $\mathcal{D} \subset \operatorname{Dom} A(\varphi)$, $\varphi \in F$, and the closure of the operator $A(\varphi) \upharpoonright \mathcal{D}$ coincides with $A(\varphi)$ for any $\varphi \in F$, that is $A(\varphi) \upharpoonright \mathcal{D} = A(\varphi)$ on Φ;

b) the Range $A(\varphi)\uparrow\mathcal{D}\subset\Phi_+$ *for any* $\varphi\in F$;

c) for every $f\in\mathcal{D}$ *the mapping* $F\ni\varphi\longrightarrow A(\varphi)f\in\Phi_+$ *is linear and continuous;*

d) there exists a strong cyclic (vacuum) vector $|\Omega\rangle\in\bigcap_{\varphi\in F}\operatorname{Dom}A(\varphi)$, *such that the set of all vectors* $|\Omega\rangle$, $\prod_{j=1}^{n}A(\varphi_j)|\Omega\rangle$, $n\in\mathbb{Z}_+$, *is total in* Φ_+ *(i.e., their linear hull is dense in* Φ_+*).*

Then there exists a probability measure μ *on* $(F', C_\sigma(F'))$, *where* F' *is the dual of* F *and* $C_\sigma(F')$ *is the* σ*-algebra generated by cylinder sets in* F' *such that, for* μ*-almost every* $\eta\in F'$ *there is a generalized joint eigenvector* $\omega(\eta)\in\Phi_-$ *of the family* $\mathcal{A}$, *corresponding to the joint eigenvalue* $\eta\in F'$, *that is,*

$$\langle\omega(\eta),A(\varphi)f\rangle_\Phi=\eta(\varphi)\langle\omega(\eta),f\rangle_\Phi \tag{2.2}$$

with $\eta(\varphi)\in\mathbb{R}$ *denoting the pairing between* F *and* F'.

The mapping

$$\Phi_+\ni f\longrightarrow\langle\omega(\eta),f\rangle_\Phi:=\hat{f}(\eta)\in\mathbb{C}, \tag{2.3}$$

for any $\eta\in F'$, *can be continuously extended to a unitary surjective operator* $\mathcal{F}\colon\Phi\to L_2^{(\mu)}(F';\mathbb{C})$, *where*

$$\mathcal{F}\,f(\eta):=\hat{f}(\eta) \tag{2.4}$$

for any $\eta\in F'$ *is a generalized Fourier transform corresponding to the family* $\mathcal{A}$. *Moreover, the image of the operator* $A(\varphi)$, $\varphi\in F'$, *under the* $\mathcal{F}$*-mapping is the operator of multiplication by the function* $F'\ni\eta\to\eta(\varphi)\in\mathbb{C}$.

We assume additionally that the main Hilbert space Φ possesses the standard Fock space (bose)-structure [17–19], that is,

$$\Phi=\oplus_{n\in\mathbb{Z}_+}\Phi_n, \tag{2.5}$$

where the subspaces $\Phi_n:=\Phi_{(s)}^{\otimes n}$, $n\in\mathbb{Z}_+$, are the symmetrized tensor products of a Hilbert space $\mathcal{H}:=L_2(\mathbb{R}^m;\mathbb{C})$. If a vector $g:=(g_0,g_1,\ldots,g_n,\ldots)\in\Phi$, its norm is

$$\|g\|_\Phi:=\left(\sum_{n\in\mathbb{Z}_+}\|g_n\|_n^2\right)^{1/2}, \tag{2.6}$$

where $g_n\in\Phi_{(s)}^{\otimes n}\simeq L_{2,(s)}((\mathbb{R}^m)^n;\mathbb{C})$ and $\|\ \ldots\ \|_n$ is the corresponding norm in $\Phi_{(s)}^{\otimes n}$ for all $n\in\mathbb{Z}_+$. Note that here the rigging structure (2.1) gives rise to the corresponding rigging for the Hilbert spaces $\Phi_{(s)}^{\otimes n}$, $n\in\mathbb{Z}_+$, that is

$$\mathcal{D}_{(s)}^n\subset\Phi_{(s),+}^{\otimes n}\subset\Phi_{(s)}^{\otimes n}\subset\Phi_{(s),-}^{\otimes n} \tag{2.7}$$

with some suitably chosen dense and separable topological spaces of symmetric functions $\mathcal{D}_{(s)}^n$, $n\in\mathbb{Z}_+$. Using expansion (2.5) we obtain, by means of projective and inductive limits [15, 16, 18], the quasi-nucleous rigging of the Fock space Φ in the form (2.1),

$$\mathcal{D}\subset\Phi_+\subset\Phi\subset\Phi_-\subset\mathcal{D}'.$$

Consider now any vector $|(\alpha)_n\rangle\in\Phi_{(s),}^{\otimes n}$, $n\in\mathbb{Z}_+$, which can be written [15, 17, 20] in the following canonical Dirac ket-form:

$$\big|(\alpha)_n\big\rangle:=|\alpha_1,\alpha_2,\ldots,\alpha_n\rangle, \tag{2.8}$$

where, by definition,

$$|\alpha_1, \alpha_2, \ldots, \alpha_n\rangle := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} |\alpha_{\sigma(1)}\rangle \otimes |\alpha_{\sigma(2)}\rangle \ldots |\alpha_{\sigma(n)}\rangle \tag{2.9}$$

and $|\alpha_j\rangle \in \Phi_{(s)}^{\otimes 1}(\mathbb{R}^m; \mathbb{C}) := \mathcal{H}$ for any fixed $j \in \mathbb{Z}_+$. The corresponding scalar product of the base vectors (2.9) is given as follows:

$$\langle(\beta)_n|(\alpha)_n\rangle := \langle\beta_n, \beta_{n-1}, \ldots, \beta_2, \beta_1|\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n\rangle =$$

$$= \sum_{\sigma \in S_n} \langle\beta_1|\alpha_{\sigma(1)}\rangle \ldots \langle\beta_n|\alpha_{\sigma(n)}\rangle := \text{per}\,\{\langle\beta_i|\alpha_j\rangle\colon i, j = \overline{1, n}\},$$

where "per" denotes the permanent of matrix and $\langle.|.\rangle$ is the corresponding product in the Hilbert space $\mathcal{H}$. Based now on representation (2.8) one can define an operator $a^+(\alpha)\colon \Phi_{(s)}^{\otimes n} \rightarrow \Phi_{(s)}^{\otimes(n+1)}$ for any $|\alpha\rangle \in \mathcal{H}$ as follows:

$$a^+(\alpha)|\alpha_1, \alpha_2, \ldots, \alpha_n\rangle := |\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n\rangle,$$

which is called the "creation" operator in the Fock space Φ. The adjoint operator $a(\beta) := (a^+(\beta))^*\colon \Phi_{(s)}^{\otimes(n+1)} \rightarrow \Phi_{(s)}^{\otimes n}$ with respect to the Fock space Φ (2.5) for any $|\beta\rangle \in \mathcal{H}$, called the "annihilation" operator, acts as follows:

$$a(\beta)|\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\rangle := \sum_{\sigma \in S_n} \langle\beta, \alpha_j\rangle|\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \hat{\alpha}_j, \alpha_{j+1}, \ldots, \alpha_{n+1}\rangle,$$

where the "hat" over a vector denotes that it should be omitted from the sequence.

It is easy to check that the commutator relationship

$$[a^+(\alpha), a(\beta)] = \langle\alpha, \beta\rangle \tag{2.10}$$

holds for any vectors $|\alpha\rangle \in \mathcal{H}$ and $|\beta\rangle \in \mathcal{H}$. Expression (2.10), owing to the rigged structure (2.1), can be naturally extended preserving the form to the general case where $|\alpha\rangle$ and $|\beta\rangle \in \mathcal{H}_-$. In particular, if we take $|\alpha\rangle := |\alpha(x)\rangle = \frac{1}{\sqrt{2\pi}} e^{i\langle\lambda, x\rangle} \in \mathcal{H}_- := L_{2,-}(\mathbb{R}^m; \mathbb{C})$ for any $x \in \mathbb{R}^m$, one easily gets from (2.10) that

$$[a^+(x), a(y)] = \delta(x - y),$$

where we put, by definition, $a^+(x) := a^+(\alpha(x))$ and $a(y) := a(\alpha(y))$ for all $x, y \in \mathbb{R}^m$ and denote by $\delta(\cdot)$ the classical Dirac delta-function.

The construction above makes it possible to easily observe that there exists a unique vacuum vector $|\Omega\rangle \in \mathcal{H}_+$ such that, for any $x \in \mathbb{R}^m$,

$$a(x)|\Omega\rangle = 0,$$

and the set of vectors

$$\left(\prod_{j=1}^{n} a^{+}(x_j)\right)|\Omega\rangle \in \Phi_{(s)}^{\otimes n}$$

is total in $\Phi_{(s)}^{\otimes n}$, that is, their linear integral hull over the dual functional spaces $\hat{\Phi}_{(s)}^{\otimes n}$ is dense in the Hilbert space $\Phi_{(s)}^{\otimes n}$ for every $n \in \mathbb{Z}_+$. This means that for any vector $g \in \Phi$ the representation

$$g = \oplus_{n\in\mathbb{Z}_+} \int\limits_{(\mathbb{R}^m)^n} \hat{g}_n(x_1,\ldots,x_n)a^{+}(x_1)a^{+}(x_2)\ldots a^{+}(x_n)|\Omega\rangle$$

takes place with the Fourier type coefficients $\hat{g}_n \in \hat{\Phi}_n := \hat{\Phi}_{(s)}^{\otimes n}$ for all $n \in \mathbb{Z}_+$, $\hat{\Phi}_{(s)}^{\otimes 1} := \hat{\mathcal{H}} \simeq \simeq L_2(\mathbb{R}^m;\mathbb{C})$. The latter is naturally endowed with the dual to (2.1) Gelfand type quasi-nuclear rigging,

$$\hat{\mathcal{H}}_+ \subset \hat{\mathcal{H}} \subset \hat{\mathcal{H}}_-, \qquad (2.11)$$

making it possible to construct a quasi-nuclear rigging of the dual Fock space $\hat{\Phi} := \oplus_{n\in\mathbb{Z}_+}\hat{\Phi}_n$. Thereby, chain (2.11) generates the dual Fock space quasi-nuclear rigging

$$\hat{\mathcal{D}} \subset \hat{\Phi}_+ \subset \hat{\Phi} \subset \hat{\Phi}_-\hat{\mathcal{D}}'$$

with respect to the central Fock type Hilbert space $\hat{\Phi}$, where $\hat{\mathcal{D}} \simeq \mathcal{D}$, easily following from (2.1) and (2.11).

Construct now the following self-adjoint operator

$$a^{+}(x)a(x) := \rho(x) : \Phi \to \Phi,$$

called the density operator at the point $x \in \mathbb{R}^m$, satisfying the commutation relations

$$\begin{aligned} [\rho(x),\rho(y)] &= 0,\\ [\rho(x),a(y)] &= -a(y)\delta(x-y),\\ [\rho(x),a^{+}(y)] &= a^{+}(y)\delta(x-y) \end{aligned} \qquad (2.12)$$

for all $y \in \mathbb{R}^m$.

Now, constructing the following self-adjoint family $\mathcal{A} := \left\{\int_{\mathbb{R}^m}\rho(x)\varphi(x)dx\colon \varphi \in F\right\}$ of linear operators on the Fock space Φ, where $F := \mathcal{S}(\mathbb{R}^m;\mathbb{R})$ is the Schwartz functional space, one can derive, making use of Theorem , that there exists the generalized Fourier transform (2.4) such that

$$\Phi(\mathcal{H}) = L_2^{(\mu)}(\mathcal{S}';\mathbb{C}) \simeq \int\limits_{\mathcal{S}'}^{\oplus} \Phi_\eta d\mu(\eta)$$

for some Hilbert space sets Φ_η, $\eta \in F'$, and a suitable measure μ on $\mathcal{S}'$, with respect to which the corresponding joint eigenvector $\omega(\eta) \in \Phi_+$ for any $\eta \in F'$ generates the Fourier transformed family $\hat{A} = \{\eta(\varphi) \in \mathbb{R}\colon \varphi \in \mathcal{S}\}$. Moreover, if $\dim \Phi_\eta = 1$ for all $\eta \in F$, the Fourier transformed eigenvector $\hat{\omega}(\eta) := \Omega(\eta) = 1$ for all $\eta \in F^{'}$.

Now we will consider the family of self-adjoint operators $\mathcal{A}$ as generating a unitary family $\mathcal{U} := \{U(\varphi)\colon \varphi \in F\} = \exp(i\mathcal{A})$, where for any $\rho(\varphi) \in \mathcal{A}$, $\varphi \in F$, the operator

$$U(\varphi) := \exp\left[i\rho(\varphi)\right]$$

is unitary, satisfying the abelian commutation relation

$$U(\varphi_1)U(\varphi_2) = U(\varphi_1 + \varphi_2)$$

for any $\varphi_1, \varphi_2 \in F$.

Since, in general, the unitary family $\mathcal{U} = \exp(i\mathcal{A})$ is defined on some Hilbert space Φ, not necessarily being of Fock type, an important problem of describing its Hilbert cyclic representation spaces arises, such that the factorization

$$\rho(\varphi) = \int\limits_{\mathbb{R}^m} a^+(x)a(x)\varphi(x)dx, \tag{2.13}$$

jointly with relationships (2.12), hold for any $\varphi \in F$. This problem can be treated using mathematical tools devised both within the representation theory of C^*-algebras [21] and Gelfand – Vilenkin [22] approach. Below we will describe main features of the Gelfand – Vilenkin formalism, being much more suitable for the task, providing a reasonably unified framework for constructing the corresponding representations.

Definition 2.1. *Let F be a locally convex topological vector space, $F_0 \subset F$ be a finite dimensional subspace of F. Let $F^0 \subseteq F'$ be defined by*

$$F^0 := \{\xi \in F'\colon \xi|_{F_0} = 0\},$$

and called the annihilator of F_0.

The quotient space $F'^0 := F'/F^0$ may be identified with $F_0' \subset F'$, the adjoint space of F_0.

Definition 2.2. *Let $A \subseteq F'$. Then the subset*

$$X^{(A)}_{F^0} := \{\xi \in F'\colon \xi + F^0 \subset A\}$$

is called the cylinder set with base A and generating subspace F^0.

Definition 2.3. *Let $n = \dim F_0 = \dim F_0' = \dim F'^0$. One says that a cylinder set $X^{(A)}$ has Borel base, if A is Borel, when regarded as a subset of $\mathbb{R}^n$.*

The family of cylinder sets with Borel base forms an algebra of sets.

Definition 2.4. *The measurable sets in F' are elements of the σ-algebra generated by the cylinder sets with Borel base.*

Definition 2.5. *A cylindrical measure on F' is a real-valued σ-pre-additive function μ defined on the algebra of cylinder sets with Borel base and satisfying the conditions $0 \leq \mu(X) \leq 1$ for* *any* X, $\mu(F')=1$, *and* $\mu\left(\coprod_{j\in\mathbb{Z}_+}X_j\right)=\sum_{j\in\mathbb{Z}_+}\mu(X_j)$, *if all the sets* $X_j\subset F'$, $j\in\mathbb{Z}_+$, *have a common generating subspace* $F_0\subset F$.

Definition 2.6. *A cylindrical measure* μ *satisfies the commutativity condition if and only if for any bounded continuous function* $\alpha\colon\mathbb{R}^n\to\mathbb{R}$ *of* $n\in\mathbb{Z}_+$ *real variables the function*

$$\alpha[\varphi_1,\varphi_2,\ldots,\varphi_n]:=\int\limits_{F'}\alpha(\eta(\varphi_1),\eta(\varphi_2),\ldots,\eta(\varphi_n))d\mu(\eta)$$

is sequentially continuous in $\varphi_j\in F$, $j=\overline{1,m}$. (*It is well known [22, 23] that in countably normed spaces the properties of sequential and ordinary continuity are equivalent*).

Definition 2.7. *A cylindrical measure* μ *is countably additive if and only if for any cylinder set* $X=\coprod_{j\in\mathbb{Z}_+}X_j$, *which is the union of countably many mutually disjoints cylinder sets* $X_j\subset F'$, $j\in\mathbb{Z}_+$, $\mu(X)=\sum_{j\in\mathbb{Z}_+}\mu(X_j)$.

The following propositions hold.

Proposition 2.1. *A countably additive cylindrical measure* μ *can be extended to a countably additive measure on the* σ*-algebra, generated by the cylinder sets with Borel base. Such a measure will also be called a cylindrical measure.*

Proposition 2.2. *Let* F *be a nuclear space. Then any cylindrical measure* μ *on* F', *satisfying the continuity condition, is countably additive.*

Definition 2.8. *Let* μ *be a cylindrical measure on* F'. *The Fourier transform of* μ *is the nonlinear functional*

$$\mathcal{L}(\varphi):=\int\limits_{F'}\exp[i\eta(\varphi)]d\mu(\eta). \tag{2.14}$$

Definition 2.9. *The nonlinear functional* $\mathcal{L}\colon F\to\mathbb{C}$ *on* F, *defined by (2.14), is called positive definite, if and only if for all* $f_j\in F$ *and* $\lambda_j\in\mathbb{C}$, $j=\overline{1,n}$, *the condition*

$$\sum_{j,k=1}^{n}\bar{\lambda}_j\mathcal{L}(f_k-f_j)\lambda_k\geq 0$$

holds for any $n\in\mathbb{Z}_+$.

Proposition 2.3. *The functional* $\mathcal{L}\colon F\to\mathbb{C}$ *on* F, *defined by (2.14), is the Fourier transform of a cylindrical measure on* F', *if and only if it is positive definite, sequentially continuous, and satisfies the condition* $\mathcal{L}(0)=1$.

Suppose now that we have a continuous unitary representation of the unitary family $\mathcal{U}$ on a Hilbert space Φ with a cyclic vector $|\Omega\rangle\in\Phi$. Then we can put

$$\mathcal{L}(\varphi):=\big\langle\Omega|U(\varphi)|\Omega\big\rangle \tag{2.15}$$

for any $\varphi \in F := \mathcal{S}$, which is the Schwartz space on $\mathbb{R}^m$, and observe that functional (2.15) is continuous on F owing to the continuity of the representation. Therefore, this functional is the generalized Fourier transform of a cylindrical measure μ on $\mathcal{S}'$,

$$\langle\Omega|U(\varphi)|\Omega\rangle = \int\limits_{\mathcal{S}'} \exp\left[i\eta(\varphi)\right] d\mu(\eta). \tag{2.16}$$

From the spectral point of view, Theorem shows that there is an isomorphism between the Hilbert spaces Φ and $L_2^{(\mu)}(\mathcal{S}';\mathbb{C})$, defined by $|\Omega\rangle \to \Omega(\eta) = 1$ and $U(\varphi)|\Omega\rangle \to \exp[i\eta(\varphi)]$ and then extended by linearity upon the whole Hilbert space Φ.

In the non-cyclic case there exists a finite or countably infinite family of measures $\{\mu_k : k \in \mathbb{Z}_+\}$ on $\mathcal{S}'$, with $\Phi \simeq \oplus_{k\in\mathbb{Z}_+} L_2^{(\mu_k)}(\mathcal{S}';\mathbb{C})$ and the unitary operator $U(\varphi)\colon \Phi \to \Phi$ for any $\varphi \in \mathcal{S}'$ corresponding in all $L_2^{(\mu_k)}(\mathcal{S}';\mathbb{C})$, $k \in \mathbb{Z}_+$, to $\exp[i\eta(\varphi)]$. This means that there exists a single cylindrical measure μ on $\mathcal{S}'$ and a μ-measurable field of Hilbert spaces Φ_η on $\mathcal{S}'$ such that

$$\Phi \simeq \int\limits_{\mathcal{S}'}^{\oplus} \Phi_\eta d\mu(\eta), \tag{2.17}$$

with $U(\varphi)\colon \Phi \to \Phi$, corresponding [22] to the operator of multiplication by $\exp[i\eta(\varphi)]$ for any $\varphi \in \mathcal{S}$ and $\eta \in \mathcal{S}'$. Thereby, having constructed the nonlinear functional (2.14) in an exact analytical form, one can retrieve the representation of the unitary family $\mathcal{U}$ on the corresponding Hilbert space Φ of the Fock type, making use of the suitable factorization (2.13) as follows: $\Phi = \oplus_{n\in\mathbb{Z}_+}\Phi_n$, where

$$\Phi_n = \underset{f_n \in L_{2,s}((R^m)^n;\mathbb{C})}{\operatorname{span}} \left\{ \prod_{j=\overline{1,n}} a^+(x_j)|\Omega\rangle \right\},$$

for all $n \in \mathbb{Z}_+$. The cyclic vector $|\Omega\rangle \in \Phi$ can be, in particular, obtained as the ground state vector of some unbounded self-adjoint positive definite Hamilton operator $\mathbb{H}\colon \Phi \to \Phi$, commuting with the self-adjoint particles number operator

$$\mathbb{N} := \int\limits_{\mathbb{R}^m} \rho(x) dx,$$

that is $[\mathbb{H}, \mathbb{N}] = 0$. Moreover, the conditions

$$\mathbb{H}|\Omega\rangle = 0$$

and

$$\inf_{g\in \operatorname{dom}\mathbb{H}} \langle g, \mathbb{H}g\rangle = \langle\Omega|\mathbb{H}|\Omega\rangle = 0$$

hold for the operator $\mathbb{H}\colon \Phi \to \Phi$, where $\operatorname{dom}\mathbb{H}$ denotes its domain of definition.

To find the functional (2.15), which is called the generating Bogolubov type functional for moment distribution functions

$$F_n(x_1, x_2, \ldots, x_n) := \langle\Omega| : \rho(x_1)\rho(x_2)\ldots\rho(x_n) : |\Omega\rangle,$$

where $x_j \in \mathbb{R}^m$, $j = \overline{1, n}$, and the normal ordering operation $: \cdot :$ is defined as

$$: \rho(x_1)\rho(x_2)\ldots\rho(x_n) : = \prod_{j=1}^{n}\left(\rho(x_j) - \sum_{k=1}^{j}\delta(x_j - x_k)\right),$$

it is convenient to choose the Hamilton operator $\mathbb{H}\colon \Phi \to \Phi$ in the following [23–25] algebraic form:

$$\mathbb{H} := \frac{1}{2}\int_{\mathbb{R}^m} K^{+}(x)\rho^{-1}(x)K(x)dx + V(\rho),$$

being equivalent in the Hilbert space Φ to the positive definite operator expression

$$\mathbb{H} := \frac{1}{2}\int_{\mathbb{R}^m} \big(K^{+}(x) - A(x;\rho)\big)\rho^{-1}(x)(K(x) - A(x;\rho))dx, \qquad (2.18)$$

where $A(x;\rho)\colon \Phi \to \Phi$, $x \in \mathbb{R}^m$, is some specially chosen linear self-adjoint operator. The "potential" operator $V(\rho)\colon \Phi \to \Phi$ is, in general, a polynomial (or analytical) functional of the density operator $\rho(x)\colon \Phi \to \Phi$ and the operator is given by

$$K(x) := \nabla_x\rho(x)/2 + iJ(x),$$

where the self-adjoint "current" operator $J(x)\colon \Phi \to \Phi$ can be defined (but non-uniquely) from the equality

$$\frac{\partial\rho}{\partial t} = \frac{1}{i}[\mathbb{H}, \rho(x)] = -\langle\nabla_x \cdot J(x)\rangle, \qquad (2.19)$$

which holds for all $x \in \mathbb{R}^m$. Such an operator $J(x)\colon \Phi \to \Phi$, $x \in \mathbb{R}^m$ can exist owing to the commutation condition $[\mathbb{H}, \mathbb{N}] = 0$, giving rise to the continuity relationship (2.19), if to take into account that supports $\operatorname{supp}\,\rho$ of the density operator $\rho(x)\colon \Phi \to \Phi$, $x \in \mathbb{R}^m$, can be chosen arbitrarily owing to the independence of (2.19) on the potential operator $V(\rho)\colon \Phi \to \to \Phi$, but its strict dependence on it of the corresponding representation (2.17). Note also that representation (2.18) holds only under the condition that there exists a self-adjoint operator $A(x;\rho)\colon \Phi \to \Phi$, $x \in \mathbb{R}^m$, such that

$$K(x)|\Omega\rangle = A(x;\rho)|\Omega\rangle$$

for all ground states $|\Omega\rangle \in \Phi$, corresponding to suitably chosen potential operators $V(\rho)\colon \Phi \to \Phi$.

The self-adjointness of the operator $A(x;\rho)\colon \Phi \to \Phi$, $x \in \mathbb{R}^m$, can be stated following schemes from the works [24, 25], under the additional condition of existence of such a linear anti-unitary mapping $T\colon \Phi \to \Phi$ that the following invariant conditions hold:

$$T\rho(x)T^{-1} = \rho(x), \qquad T\ J(x)\ T^{-1} = -J(x), \qquad T|\Omega\rangle = |\Omega\rangle \tag{2.20}$$

for any $x \in \mathbb{R}^m$. Thereby, owing to conditions (2.20), the expressions

$$K^*(x)|\Omega\rangle = A(x;\rho)|\Omega\rangle = K(x)|\Omega\rangle$$

hold for any $x \in \mathbb{R}^m$, giving rise to self-adjointness of the operator $A(x;\rho)\colon \Phi \to \Phi$, $x \in \mathbb{R}^m$.

Based now on the construction above one easily deduces from expression (2.19) that the generating Bogolubov type functional (2.15) obeys for all $x \in \mathbb{R}^m$ the following functional-differential equation:

$$[\nabla_x - i\nabla_x\varphi]\,\frac{1}{2i}\frac{\delta\mathcal{L}(\varphi)}{\delta\varphi(x)} = A\left(x;\frac{1}{i}\frac{\delta}{\delta\varphi}\right)\mathcal{L}(\varphi), \tag{2.21}$$

whose solutions should satisfy the Fourier transform representation (2.16). In particular, a wide class of special so-called Poissonian white noise type solutions to the functional-differential equation (2.21) was obtained in [24, 25] by means of functional-operator methods in the following generalized form:

$$\mathcal{L}(\varphi) = \exp\left\{A\left(\frac{1}{i}\frac{\delta}{\delta\varphi}\right)\right\}\exp\left(\bar{\rho}\int\limits_{\mathbb{R}^m}\{\exp[i\varphi(x)]-1\}dx\right),$$

where $\bar{\rho} := \langle\Omega|\rho|\Omega\rangle \in \mathbb{R}_+$ is a Poisson distribution density parameter.

Consider now the case when the basic Fock space $\Phi = \otimes_{j=1}^{s}\Phi^{(j)}$, where $\Phi^{(j)}$, $j = \overline{1,s}$, are Fock spaces corresponding to the different types of independent cyclic vectors $|\Omega_j\rangle \in \Phi^{(j)}$, $j = \overline{1,s}$. This, in particular, means that the suitably constructed creation and annihilation operators $a_j(x), a_k^+(y)\colon \Phi \to \Phi$, $j,k = \overline{1,s}$, satisfy the following commutation relations:

$$[a_j(x), a_k(y)] = 0,$$
$$[a_j(x), a_k^+(y)] = \delta_{jk}\delta(x-y)$$

for any $x, y \in \mathbb{R}^m$.

Definition 2.10. *A vector $|u\rangle \in \Phi$, $x \in \mathbb{R}^m$, is called coherent with respect to a mapping $u \in L_2(\mathbb{R}^m;\mathbb{R}^s) := M$, if it satisfies the eigenfunction condition*

$$a_j(x)|u\rangle = u_j(x)|u\rangle \tag{2.22}$$

for each $j = \overline{1,s}$ and all $x \in \mathbb{R}^m$.

It is easy to check that coherent vectors $|u\rangle \in \Phi$ exist. Indeed, the following vector expression

$$|u\rangle := \exp\{(u, a^+)\}|\Omega\rangle, \tag{2.23}$$

where $(.,.)$ is the standard scalar product in the Hilbert space M, satisfies the defining condition (2.22), and, moreover,

$$\|u\|_{\Phi} := \langle u|u\rangle^{1/2} = \exp\left(\frac{1}{2}\|u\|^2\right) < \infty, \tag{2.24}$$

since $u \in M$ and its norm $\|u\| := (u,u)^{1/2}$ is bounded.

3. The Fock space embedding method, nonlinear dynamical systems and their complete linearization. Consider any function $u \in M := L_2(\mathbb{R}^m;\mathbb{R}^s)$ and observe that the Fock space embedding mapping

$$\xi\colon M \ni u \longrightarrow |u\rangle \in \Phi, \tag{3.1}$$

defined by means of the coherent vector expression (2.23) realizes a smooth isomorphism between the Hilbert spaces M and Φ. The inverse mapping $\xi^{-1}\colon \Phi \to M$ is given by the following exact expression:

$$u(x) = \langle\Omega|a(x)|u\rangle, \tag{3.2}$$

holding for almost all $x \in \mathbb{R}^m$. Owing to condition (2.24) one finds from (3.2) that the corresponding function $u \in M$.

Let now on the Hilbert space M there be defined a nonlinear dynamical system (which, in general, can be non-autonomous) in partial derivatives

$$\frac{du}{dt} = K[u], \tag{3.3}$$

where $t \in \mathbb{R}_+$ is the corresponding evolution parameter, $[u] := (t, x; u, u_x, u_{xx}, \ldots, u_{rx}), r \in \mathbb{Z}_+$, and a mapping $K\colon M \to T(M)$ is Frechet smooth. Assume also that the Cauchy problem

$$u|_{t=+0} = u_0 \tag{3.4}$$

is solvable for any $u_0 \in M$ in an interval $[0,T) \subset \mathbb{R}^1_+$ for some $T > 0$. Thereby, there is a smooth evolution mapping,

$$T_t\colon M \ni u_0 \to u(t|u_0) \in M, \tag{3.5}$$

for all $t \in [0,T)$.

It is now natural to consider the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\xi} & \Phi \\ T_t \downarrow & & \downarrow \mathbb{T}_t \\ M & \xrightarrow{\xi} & \Phi, \end{array}$$

where the mapping $\mathbb{T}_t\colon \Phi \to \Phi$, $t \in [0,T)$, is defined from the conjugation relationship

$$\xi \circ T_t = \mathbb{T}_t. \circ \xi.$$

Take now the corresponding to $u_0 \in M$ coherent vector $|u_0\rangle \in \Phi$ and construct the vector

$$|u\rangle := \mathbb{T}_t \cdot |u_0\rangle \tag{3.6}$$

for all $t \in [0, T)$. Since, by the construction, vector (3.6) is coherent, that is,

$$a_j(x)|u\rangle := u_j(x, t|u_0)|u\rangle$$

for each $j = \overline{1, s}$, $t \in [0, T)$ and almost all $x \in \mathbb{R}^m$, owing to the smoothness of the mapping $\xi \colon M \to \Phi$ with respect to the corresponding norms in the Hilbert spaces M and Φ, we derive that coherent vector (3.6) is differentiable with respect to the evolution parameter $t \in [0, T)$. Thus, one can easily find [20, 26] that

$$\frac{d}{dt}|u\rangle = \hat{K}[a^+, a]|u\rangle, \tag{3.7}$$

where

$$|u\rangle|_{t=+0} = |u_0\rangle \tag{3.8}$$

and the mapping $\hat{K}[a^+, a] \colon \Phi \to \Phi$ is defined by the exact analytical expression

$$\hat{K}[a^+, a] := (a^+, K[a]).$$

As a result of the consideration above we obtain the following theorem.

Theorem 3.1. *Any smooth nonlinear dynamical system (3.3) in the Hilbert space $M := := L_2(\mathbb{R}^m; \mathbb{R}^s)$ is representable by means of the Fock space embedding isomorphism $\xi \colon M \to \Phi$ in the completely linear form (3.7).*

We now make some comments concerning the solution to the linear equation (3.7) under the Cauchy condition (3.8). Since any vector $|u\rangle \in \Phi$ allows for the series representation

$$|u\rangle = \bigoplus_{n := \sum_{j=1}^{s} n_j \in \mathbb{Z}_+} \frac{1}{(n_1! n_2! \ldots n_s!)^{1/2}} \int\limits_{(\mathbb{R}^m)^n} f^{(n)}_{n_1 n_2 \ldots n_s}(x_1^{(1)}, x_2^{(1)}, \ldots, x_{n_1}^{(1)};$$

$$x_1^{(2)}, x_2^{(2)}, \ldots, x_{n_2}^{(2)}; \ldots; x_1^{(s)}, x_2^{(s)}, \ldots, x_{n_s}^{(s)}) \prod_{j=1}^{s} \left(\prod_{k=1}^{n_j} dx_k^{(j)} a_j^+(x_k^{(j)}) \right) |\Omega\rangle \tag{3.9}$$

where, for any $n = \sum_{j=1}^{s} n_j \in \mathbb{Z}_+$, the functions

$$f^{(n)}_{n_1 n_2 \ldots n_s} \in \bigotimes_{j=1}^{s} L_{2,s}((\mathbb{R}^m)^{n_j}; \mathbb{C}) \simeq L_{2,s}(\mathbb{R}^{mn_1} \times \mathbb{R}^{mn_2} \times \ldots \mathbb{R}^{mn_s}; \mathbb{C}), \tag{3.10}$$

and the norm

$$\|u\|_\Phi^2 = \sum_{n = \sum_{j=1}^{s} n_j} \|f^{(n)}_{n_1 n_2 \ldots n_s}\|_2^2 = \exp(\|u\|^2).$$

Equation (3.7), by substituting (3.9), reduces to an infinite recurrent set of linear evolution partial differential equations on the coefficient functions (3.10). The latter can often be solved [26] step by step analytically in an exact form, thereby, making it possible to obtain owing to representation (3.2), the exact solution $u \in M$ to the Cauchy problem (3.4) for our nonlinear dynamical system in partial derivatives (3.3).

Remark 3.1. Concerning some applications of nonlinear dynamical systems like (3.1) in mathematical physics problems, it is very important to construct their so-called conservation laws or smooth invariant functionals $\gamma\colon M \rightarrow \mathbb{R}$ on M. Making use of the quantum mathematics technique described above one can suggest an effective algorithm for construction these conservation laws in an exact form.

Indeed, consider a vector $|\gamma\rangle \in \Phi$ satisfying the linear equation

$$\frac{\partial}{\partial t}|\gamma\rangle + \hat{K}^{*}[a^{+}, a]|\gamma\rangle = 0.$$

Then the following proposition [26] holds.

Proposition 3.1. *The functional*

$$\gamma := \langle u|\gamma\rangle$$

is a conservation law for dynamical system (3.1), that is,

$$\left.\frac{d\gamma}{dt}\right|_{K} = 0$$

along any orbit of the evolution mapping (3.5).

4. Conclusion. Within the scope of this work we have described main mathematical preliminaries and properties of the quantum mathematics techniques suitable for analytical studying the important linearization problem for a wide class of nonlinear dynamical systems in partial derivatives on Hilbert spaces. This problem was analyzed in many details using the Gelfand – Vilenkin representation theory [22] of infinite dimensional groups and the Goldin – Menikoff – Sharp theory [23, 24, 27] of generating Bogolubov type functionals, classifying these representations. The related problem of constructing Fock type space representations and retrieving their creation-annihilation generating structure still needs a deeper investigation within the approach devised. We here mention only that some aspects of this problem within the so-called Poissonian. White noise analysis were studied in a series of works [15, 28 – 30], based on some generalization of the Delsarte type characters technique. It is necessary to mention also the related results obtained in [19, 26, 31], devoted to application of the Fock space embedding method to finding conservation laws and the so called recursion operators for the well known Korteweg – de Vries type nonlinear dynamical systems. Concerning some of important applications of the methods devised in the work to concrete dynamical systems, we plan to devote a next investigation.

5. Acknowledgements. Two of the authors (N.B. and A.P.) are cordially thankful to the Abdus Salam International Center for Theoretical Physics in Trieste, Italy, for the hospitality during their research 2007-scholarships. A.P. is also grateful to Profs. Y. Kondratyev and Y. Kozicki for interesting discussions of the results during the Third International Conference on Infinite Dimensional Systems, held June 23 – 28, 2007 in Kazimierz Dolny, Poland.

1. *Dirac P. A. M.* The principles of quantum mechanics. — Oxford Univ. Press, 1932.
2. *Neumann J. von.* Mathematische Grundlagen der Quanten Mechanik. — Berlin: Springer, 1932.
3. *Fock V. A.* Konfigurationsraum und zweite Quantelung // Z. Phys. — 1932. — **75**. — P. 622–647.
4. *Weyl H.* The theory of groups and quantum mechanics. — New York: Dover, 1931.
5. *Faddeev L. D., Sklyanin E. K.* Quantum mechanical approach to completely integrable field theories // Proc. USSR Acad. Sci. (DAN). — 1978. — **243**. — P. 1430–1433 (in Russian).
6. *Drinfeld V. G.* Quantum groups // Proc. Int. Congr. Math. — 1986. — P. 798–820.
7. *Donaldson S. K.* An application of gauge theory to four dimansional topology // J. Different. Geom. — 1982. — **17**. — P. 279–315.
8. *Witten E.* Nonabelian bozonization in two dimensions // Communs Math. Phys. — 1984. — **92**. — P. 455–472.
9. *Manin Yu. I.* Computable and uncomputable. — Moscow: Sov. Radio, 1980 (in Russian).
10. *Feynman R.* Quantum mechanical computers // Found. Phys. — 1986. — **16**. — P. 507–531.
11. *Feynman R.* Simulating physics with computers // Int. J. Theor. Phys. — 1982. — **21**. — P. 467–488.
12. *Shor P. W.* Polynomial time algorithms for prime factorization and discrete logarithms on a quantum computer // SIAM J. Comput. — 1997. — **26**, № 5. — P. 1484–1509.
13. *Deutsch D.* Quantum theory, the Church–Turing principle and the universal quantum computer // Proc. Roy. Soc. London A. — 1985. — **400**. — P. 97–117.
14. *Grover L. K.* Quantum mechanics helps in searching for a needle in a haystack // Phys. Rev. Lett. — 1997. — **79**. — P. 325–328.
15. *Berezansky Y. M.* A generalization of white noice analysis by means of theory of hypergroups // Repts Math. Phys. — 1996. — **38**, № 3. — P. 289–300.
16. *Berezansky Y. M., Kondratiev Y. G.* Spectral methods in infinite dimensional analysis. — Kluwer, 1995. — Vols 1, 2.
17. *Bogolubov N. N., Bogolubov N. N. (jr.)* Introduction into quantum statistical mechanics. — New York: World Sci., 1986. — 384 p.
18. *Berezin F. A.* The second quantization method. — Moscow: Nauka, 1986 (in Russian).
19. *Prykarpatsky A. K., Taneri U., Bogolubov N. N. (jr.)* Quantum field theory and application to quantum nonlinear optics. — New York: World Sci., 2002.
20. *Kowalski K., Steeb W.-H.* Nonlinear dynamical systems and Carleman linearization. — World Sci., 1991.
21. *Dixmier J.* C^*-algebras. — Amsterdam: North-Holland, 1982.
22. *Gelfand I., Vilenkin N.* Generalized functions. — New York: Acad. Press, 1964.
23. *Goldin G. A.* Nonrelativistic current algebras as unitary representations of groups // J. Math. Phys. — 1971. — **12**, № 3. — P. 462–487.
24. *Goldin G. A., Grodnik J., Powers R. T., Sharp D.* Nonrelativistic current algebra in the N/V limit // Ibid. — 1974. — **15**. — P. 88–100.
25. *Bogoliubov N. N. (jr.), Prykarpatsky A. K.* Quantum method of generating Bogolubov functionals in statistical physics: current Lie algebras, their representations and functional equations // Phys. Element. Particles and Atom. Nucleus. — 1986. — **17**, № 4. — P. 791–827 (in Russian).
26. *Kowalski K.* Methods of Hilbert spaces in the theory of nonlinear dynamical systems. — World Sci., 1994.
27. *Goldin G. A., Menikoff R., Sharp F. H.* Diffeomorphism groups, gauge groups, and quantum theory // Phys. Rev. Lett. — 1983. — **51**. — P. 2246–2249.
28. *Albeverio S., Kondratiev Y. G., Streit L.* How to generalize white noice analysis to non-gaussian measures. — Preprint BiBoS, Bielefeld, 1992.
29. *Kondratiev Y. G., Streit L., Westerkamp W., Yan J.-A.* Generalized functions in infinite dimensional analysis. II. — AS preprint, 1995.
30. *Lytvynov E. W., Rebenko A. L., Shchepaniuk G. V.* Wick calculus on spaces of generalized functions compound Poisson white noise // Repts Math. Phys. — 1997. — **39**, № 2. — P. 219–247.
31. *Kowalski K., Steeb W.-H.* Symmetries and first integrals for nonlinear dynamical systems: Hilbert space approach. I, II // Progress Theor. Phys. — 1991. — **85**, № 4. — P. 713–722; 975–983.

Received 08.10.07