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OSCILLATION OF A CLASS OF NONLINEAR PARTIAL DIFFERENCE EQUATIONS WITH CONTINUOUS VARIABLES*

ОСЦИЛЯЦІЯ КЛАСУ НЕЛІНІЙНИХ ЧАСТКОВО РІЗНИЦЕВИХ РІВНЯНЬ З НЕПЕРЕРВНИМИ ЗМІННИМИ

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This paper is concerned with a class of nonlinear partial difference equations with continuous variables. Some oscillation criteria are obtained using an integral transformation and inequalities.

Розглянуто клас нелінійних частково різницевих рівнянь з неперервними змінними. Отримано деякі критерії осциляції з використанням інтегральних перетворень та нерівностей.

1. Introduction. Partial difference equations are difference equations which involve functions with two or more independent variables. Such equations arise in investigation of random walk problems, molecular structure problems [1], and numerical difference approximation problems [2], etc. Recently, oscillation problems for partial difference equations with invariable coefficients and discrete variables have been investigated in [3–8]. We can further investigate oscillation properties of nonlinear equations with variable coefficients and continuous variables and obtain some oscillation criteria.

In this paper, we consider a class of nonlinear partial difference equations with continuous variables,

$$p_1(x,y)A(x+a,y+b)+p_2(x,y)A(x+a,y)+p_3(x,y)A(x,y+b)-p_4(x,y)A(x,y)+ \\ +\sum_{i=1}^{m}h_i(x,y,A(x-\sigma_i,y-\tau_i))\,=\,0, \qquad (1)$$

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where $p_1(x,y) \in C(R^+ \times R^+, [0,\infty))$; $p_2(x,y), p_3(x,y), p_4(x,y) \in C(R^+ \times R^+, (0,\infty))$; a, b, σ_i, τ_i are negative and $h_i(x,y,u) \in C(R^+ \times R^+ \times R, R)$, $i = 1,\ldots,m$.

Let $\sigma = \max_{i=1,\ldots,m}\{\sigma_i\}$, $\tau = \max_{i=1,\ldots,m}\{\tau_i\}$. A solution of (1) is defined to be a continuous function $A(x,y)$, for all $x \geq -\sigma, y \geq -\tau$, which satisfies (1) on $R^+ \times R^+$. A solution $A(x,y)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Some oscillation criteria for a solution of (1) are obtained using integral transformation and inequalities. Our results extend some oscillation properties of nonlinear equations with invariable coefficients and discrete variables to nonlinear equations with variable coefficients and continuous variables.

2. Main lemmas. We assume that the following conditions are satisfied throughout this paper:

(I) $p_1(x,y) \geq p_1 \geq 0$, $p_2(x,y) \geq p_2 > 0$, $p_3(x,y) \geq p_3 > 0$, $0 < p_4(x,y) \leq p_4$, and p_i, $i = 1,2,3,4$, are constants and also satisfy $p_2, p_3 \geq p_4$;

(II) $\tau_i = k_i a + \theta_i$, $\sigma = l_i b + \xi_i$, $i = 1,\ldots,m$, where k_i, l_i are nonnegative integers and $\theta_i \in (a,0]$, $\xi_i \in (b,0]$.

Lemma 1. *Assume that*

(i) $h_i \in C(R^+ \times R^+ \times R, R)$, $uh_i(x,y,u) > 0$ *for* $u \neq 0$, *and* $h_i(x,y,u)$, $i = 1,\ldots,m$, *is a nondecreasing function in* u;

(ii) $h_i(x,y,u)$, $i = 1,\ldots,m$, *is convex in* u *for* $u \geq 0$.

Let $A(x,y)$ *be an eventually positive solution of (1), then there exists a positive function* $Z(x,y) = \dfrac{1}{ab}\displaystyle\int_{x+a}^{x}\int_{y+b}^{y} A(u,v)\,du\,dv$ *eventually satisfying the following results:*

(1) if $\min_{i=1,\ldots,m}\{k_i\} \neq 0$ *and* $\min_{i=1,\ldots,m}\{l_i\} \neq 0$, *then*

$$p_1 Z(x+a,y+b) + p_2 Z(x+a,y) + p_3 Z(x,y+b) - p_4 Z(x,y) + \\ + \sum_{i=1}^{m} h_i(x,y,Z(x-\sigma_i, y-\tau_i)) \leq 0; \tag{2}$$

(2) if $\min_{i=1,\ldots,m}\{k_i\} = 0$ *and* $min_{i=1,\ldots,m}\{l_i\} = 0$, *then*

$$p_1 Z(x+a,y+b) + p_2 Z(x+a,y) + p_3 Z(x,y+b) - p_4 Z(x,y) + \\ + \sum_{i=1}^{m} h_i(x,y,Z(x-k_i a, y-l_i b)) \leq 0; \tag{3}$$

(3) if $\min_{i=1,\ldots,m}\{k_i\} = 0$ *and* $\min_{i=1,\ldots,m}\{l_i\} \neq 0$, *then*

$$p_1 Z(x+a,y+b) + p_2 Z(x+a,y) + p_3 Z(x,y+b) - p_4 Z(x,y) + \\ + \sum_{i=1}^{m} h_i(x,y,Z(x-\sigma_i, y-l_i b)) \leq 0; \tag{4}$$

(4) if $\min_{i=1,\ldots,m}\{k_i\} \neq 0$ *and* $\min_{i=1,\ldots,m}\{l_i\} = 0$, *then*

$$p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + p_3 Z(x, y+b) - p_4 Z(x,y) + \\ + \sum_{i=1}^{m} h_i(x, y, Z(x - k_i a, y - \tau_i)) \le 0. \tag{5}$$

Proof. From (I), we have the following inequality:

$$p_4(A(x+a,y) + A(x,y+b) - A(x,y)) \le \\ \le p_1 A(x+a,y+b) + p_2 A(x+a,y) + p_3 A(x,y+b) - p_4 A(x,y) \le \\ \le p_1(x,y)A(x+a,y+b) + p_2(x,y)A(x+a,y) + \\ + p_3(x,y)A(x,y+b) - p_4(x,y)A(x,y) < 0$$

eventually.

Since

$$Z(x,y) = \frac{1}{ab} \int\limits_{x+a}^{x} \int\limits_{y+b}^{y} A(u,v)\, du\, dv, \tag{6}$$

we have

$$\frac{\partial Z(x,y)}{\partial x} = \frac{1}{ab} \int\limits_{y+b}^{y} (A(x,v) - A(x+a,v)) dv > 0 \tag{7}$$

and

$$\frac{\partial Z(x,y)}{\partial y} = \frac{1}{ab} \int\limits_{x+a}^{x} (A(u,y) - A(u,y+b)) du > 0. \tag{8}$$

From the above, we have $Z(x,y)$ is nondecreasing in x and y eventually.

Integrating (1), from (I) we have

$$p_1 Z(x+a,y+b) + p_2 Z(x+a,y) + p_3 Z(x,y+b) - p_4 Z(x,y) + \\ + \frac{1}{ab} \sum_{i=1}^{m} \int\limits_{x}^{x+a} \int\limits_{y}^{y+b} h_i(u, v, A(u - \sigma_i, v - \tau_i))\, dv\, du \le 0.$$

By (i), (ii), and Jensen's inequality, we obtain the following inequality:

$$p_1 Z(x+a,y+b) + p_2 Z(x+a,y) + p_3 Z(x,y+b) - p_4 Z(x,y) + \sum_{i=1}^{m} h_i(x, y, Z(x - \sigma_i, y - \tau_i)) \le 0.$$

Thus (2) holds.

Since a, b, τ_i, σ_i are negative real numbers, there exist nonnegative integers k_i and l_i satisfying $\sigma_i = k_i a + \theta_i$, $\tau_i = l_i b + \xi_i$, where $a < \theta_i \le 0$, $b < \xi_i \le 0$, $i = 1, 2, \ldots, m$. From (7) and (8), we obtain $Z(x, y)$ is nondecreasing eventually. So if

$$\min_{i=1,\ldots,m}\{k_i\} \ne 0, \quad \min_{i=1,\ldots,m}\{l_i\} \ne 0,$$

we have $Z(x-\sigma_i, y-\tau_i) \ge Z(x-k_i a, y-l_i b)$, $i = 1, 2, \ldots, m$. Since $h_i(x, y, u)$, $i = 1, 2, \ldots, m$, is nondecreasing in u, we have

$$p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + p_3 Z(x, y+b) - p_4 Z(x, y) + \sum_{i=1}^{m} h_i(x, y, Z(x-k_i a, y-l_i b)) \le 0.$$

Hence, (3) holds.

Similarly if $\min_{i=1,\ldots,m}\{k_i\} = 0$, $\min_{i=1,\ldots,m}\{l_i\} \ne 0$, $Z(x, y)$ is nondecreasing in x and y eventually, we have $Z(x-\sigma_i, y-\tau_i) \ge Z(x-\sigma_i, y-l_i b)$. Since $h_i(x, y, u)$, $i = 1, 2, \ldots, m$, is nondecreasing in u eventually, we have the following inequality:

$$p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + p_3 Z(x, y+b) - p_4 Z(x, y) + \sum_{i=1}^{m} h_i(x, y, Z(x-\sigma_i, y-l_i b)) \le 0,$$

implying (4).

Similarly if $\min_{i=1,\ldots,m}\{k_i\} \ne 0$, $\min_{i=1,\ldots,m}\{l_i\} = 0$, $i = 1, 2, \ldots, m$, $Z(x, y)$ is nondecreasing in x and y eventually, we have $Z(x-\sigma_i, y-\tau_i) \ge Z(x-k_i a, y-\tau_i)$. Since $h_i(x, y, u)$ is nondecresing in u, we have the following inequality:

$$p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + p_3 Z(x, y+b) - p_4 Z(x, y) + \sum_{i=1}^{m} h_i(x, y, Z(x-k_i a, y-\tau_i)) \le 0.$$

Hence, (5) holds.

The proof is completed.

By a similar method, we can obtain properties of an eventually negative solution of (1).

Lemma 2. *Assume that*

(i) $h_i \in C(R^+ \times R^+ \times R, R)$, $u h_i(x, y, u) > 0$ *for* $u \ne 0$ *and* $h_i(x, y, u)$, $i = 1, \ldots, m$, *is a nondecreasing function in* u;

(ii) $h_i(x, y, u)$, $i = 1, \ldots, m$, *is concave in* u *for* $u \le 0$.

Let $A(x, y)$ *be an eventually negative solution of (1), then there exists a negative function* $Z(x, y) = \dfrac{1}{ab}\displaystyle\int_{x+a}^{x}\int_{y+b}^{y} A(u, v)\,du\,dv$ *eventually satisfying the following results:*

(1) if $\min_{i=1,\ldots,m}\{k_i\} \ne 0$ *and* $\min_{i=1,\ldots,m}\{l_i\} \ne 0$, *then*

$$p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + p_3 Z(x, y+b) - p_4 Z(x, y) + \sum_{i=1}^{m} h_i(x, y, Z(x-\sigma_i, y-\tau_i)) \ge 0; \quad (9)$$

(2) if $\min_{i=1,\ldots,m}\{k_i\} = 0$ *and* $\min_{i=1,\ldots,m}\{l_i\} = 0$*, then*

$$p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + p_3 Z(x, y+b) - p_4 Z(x,y) + \\ + \sum_{i=1}^{m} h_i(x, y, Z(x - k_i a, y - l_i b)) \geq 0; \tag{10}$$

(3) if $\min_{i=1,\ldots,m}\{k_i\} = 0$ *and* $\min_{i=1,\ldots,m}\{l_i\} \neq 0$*, then*

$$p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + p_3 Z(x, y+b) - p_4 Z(x,y) + \\ + \sum_{i=1}^{m} h_i(x, y, Z(x - \sigma_i, y - l_i b)) \geq 0; \tag{11}$$

(4) if $\min_{i=1,\ldots,m}\{k_i\} \neq 0$ *and* $\min_{i=1,\ldots,m}\{l_i\} = 0$*, then*

$$p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + p_3 Z(x, y+b) - p_4 Z(x,y) + \\ + \sum_{i=1}^{m} h_i(x, y, Z(x - k_i a, y - \tau_i)) \geq 0. \tag{12}$$

3. Main results. In the following, we investigate oscillatory properties of a solution of (1) and obtain the main results of this paper.

Theorem 1. *Assume that*

(i) $h_i(x, y, u) \in C(R^+ \times R^+ \times R, R)$ *is nondecreasing in* u *and* $u h_i(x, y, u) > 0$, $i = 1, 2, \ldots, m$, *for all* $u \neq 0$,

(ii) $\liminf_{x,y\to+\infty, u\to 0} h_i(x, y, u)/u = S_i \geq 0$, $\sum_{i=1}^{m} S_i > 0$, $i = 1, 2, \ldots, m$,

(iii) $h_i(x, y, u)$ *is convex in* u *for* $u > 0$, $h_i(x, y, u)$, $i = 1, \ldots, m$, *is concave in* u *for* $u < 0$,

(iv) one of the following conditions holds:

$$\sum_{i=1}^{m} S_i \frac{(\eta_i + 1)^{\eta_i + 1}}{\eta_i^{\eta_i}} \frac{(p_1 + p_2 + p_3)^{\eta_i}}{p_4^{(\eta_i + 1)}} > 1, \quad \eta_i = \min\{k_i, l_i\} > 0, \quad i = 1, \ldots, m, \tag{13}$$

$$\sum_{i=1}^{m} S_i \frac{k_i^{k_i}}{(k_i - 1)^{k_i - 1}} \frac{p_2^{k_i - 1}}{p_4^{k_i}} > 1, \quad \min_{i=1,\ldots,m}\{k_i\} > 0, \quad \min_{i=1,\ldots,m}\{l_i\} = 0, \tag{14}$$

$$\sum_{i=1}^{m} S_i \frac{l_i^{l_i}}{(l_i - 1)^{l_i - 1}} \frac{p_3^{l_i - 1}}{p_4^{l_i}} > 1, \quad \min_{i=1,\ldots,m}\{k_i\} = 0, \quad \min_{i=1,\ldots,m}\{l_i\} > 0, \tag{15}$$

$$\frac{1}{p_4} \sum_{i=1}^{m} S_i > 1, \quad \min_{i=1,\ldots,m}\{k_i\} = \min_{i=1,\ldots,m}\{l_i\} = 0. \tag{16}$$

Then every solution of (1) is oscillatory.

Proof. Assume the contrary. Let $A(x,y)$ be an eventually positive solution of (1), and $Z(x,y)$ be defined by (6). Then by Lemma 1, we obtain $\lim_{x,y\to+\infty} Z(x,y) = \zeta \geq 0$. In the following, we claim that $\zeta = 0$. Otherwise, let $\zeta > 0$. By Lemma 1, we know that (2) holds. From (2) and condition (I), we have

$$p_1 Z(x+a,y+b) + p_4(Z(x+a,y) + Z(x,y+b) - Z(x,y)) \leq$$
$$\leq p_1 Z(x+a,y+b) + p_2 Z(x+a,y) + p_3 Z(x,y+b) - p_4 Z(x,y) \leq 0.$$

So

$$Z(x+a,y) + Z(x,y+b) - Z(x,y) \leq 0. \tag{17}$$

Taking the limit on both side of (17), we have $\zeta \leq 0$. Consider $\zeta \geq 0$. Then we have $\zeta = 0$.

If $\min_{i=1,\ldots,m}\{k_i\} > 0$, $\min_{i=1,\ldots,m}\{l_i\} > 0$, in the view of (2), we have

$$\frac{(p_1+p_2+p_3)Z(x+a,y+b)}{Z(x,y)} - p_4 \leq \frac{p_1 Z(x+a,y+b) + p_2 Z(x+a,y) + p_3 Z(x,y+b)}{Z(x,y)} - p_4 \leq$$
$$\leq -\sum_{i=1}^{m} \frac{h_i(x,y,Z(x-\sigma_i,y-\tau_i))}{Z(x,y)}. \tag{18}$$

Since $Z(x,y)$ is nondecreasing eventually, from (18) for all large x and y we have

$$\frac{(p_1+p_2+p_3)Z(x+a,y+b)}{Z(x,y)} - p_4 \leq -\sum_{i=1}^{m} \frac{h_i(x,y,Z(x-\eta_i a,y-\eta_i b))}{Z(x,y)} =$$
$$= -\sum_{i=1}^{m} \frac{h_i(x,y,Z(x-\eta_i a,y-\eta_i b))}{Z(x-\eta_i a,y-\eta_i b)} \prod_{j=1}^{\eta_i} \frac{Z(x-ja,y-jb)}{Z(x-(j-1)a,y-(j-1)b)}, \tag{19}$$

where $\eta_i = \min\{k_i, l_i\}$, $i = 1,\ldots,m$.

Let $\alpha(x,y) = Z(x,y)/Z(x+a,y+b)$. Then $\alpha(x,y) > 1$ for all large x and y. From (19), we have

$$\frac{p_1+p_2+p_3}{\alpha(x,y)} + \sum_{i=1}^{m} \frac{h_i(x,y,Z(x-\eta_i a,y-\eta_i b))}{Z(x-\eta_i a,y-\eta_i b)} \prod_{j=1}^{\eta_i} \alpha(x-ja,y-jb) \leq p_4,$$

i.e.,

$$(p_1+p_2+p_3) + \sum_{i=1}^{m} \frac{h_i(x,y,Z(x-\eta_i a,y-\eta_i b))}{Z(x-\eta_i a,y-\eta_i b)} \prod_{j=1}^{\eta_i} \alpha(x-ja,y-jb)\alpha(x,y) \leq p_4\alpha(x,y). \tag{20}$$

By (ii), (20) implies that $\alpha(x,y)$ is bounded.

Let $\liminf\limits_{x,y\to+\infty} \alpha(x,y) = \beta$. Taking the limit inferior on both sides of (20), we obtain

$$(p_1+p_2+p_3) + \sum_{i=1}^{m} S_i \beta^{\eta_i+1} \leq p_4\beta,$$

i.e.,

$$\frac{p_1+p_2+p_3}{\beta} \le p_4 - \sum_{i=1}^{m} S_i \beta^{\eta_i} < p_4. \tag{21}$$

Hence, $\beta > \dfrac{p_1+p_2+p_3}{p_4}$. Since $\sum_{i=1}^m S_i\beta^{\eta_i+1}/(p_4\beta-(p_1+p_2+p_3)) \le 1$, computing the minimum of the function $f(x) = x^{\eta_i+1}/(p_4x-(p_1+p_2+p_3))$ as $x > \dfrac{p_1+p_2+p_3}{p_4}$ we obtain

$$\min_{\beta>\frac{p_1+p_2+p_3}{p_4}} \frac{\beta^{\eta_i+1}}{p_4\beta-(p_1+p_2+p_3)} = \frac{(\eta_i+1)^{\eta_i+1}}{\eta_i^{\eta_i}} \frac{(p_1+p_2+p_3)^{\eta_i}}{p_4^{\eta_i+1}}.$$

So we have $\sum_{i=1}^m S_i \dfrac{(\eta_i+1)^{\eta_i+1}}{\eta_i^{\eta_i}} \dfrac{(p_1+p_2+p_3)^{\eta_i}}{p_4^{\eta_i+1}} \le 1$ which is contrary to (13). Therefore if (13) holds we can obtain that every solution of (1) is oscillatory.

If $\min_{i=1,\ldots,m}\{k_i\} > 0$, $\min_{i=1,\ldots,m}\{l_i\} = 0$, by Lemma 1, we obtain

$$p_1Z(x+a,y+b)+p_2Z(x+a,y)+p_3Z(x,y+b)-p_4Z(x,y)+\sum_{i=1}^{m} h_i(x,y,Z(x-k_ia,y-\tau_i)) \le 0.$$

Then we have

$$\frac{p_2Z(x+a,y)}{Z(x,y)} - p_4 \le -\sum_{i=1}^{m} \frac{h_i(x,y,Z(x-k_ia,y-\tau_i))}{Z(x,y)} =$$

$$= -\sum_{i=1}^{m} \frac{h_i(x,y,Z(x-k_ia,y-\tau_i))}{Z(x-k_ia,y-\tau_i)} \frac{Z(x-a,y-\tau_i)}{Z(x,y)} \prod_{j=2}^{k_i} \frac{Z(x-ja,y-\tau_i)}{Z(x-(j-1)a,y-\tau_i)}. \tag{22}$$

Since $Z(x,y)$ is nondecreasing in x, y eventually, we have $Z(x-a,y-\tau_i)/Z(x,y) > 1$ for all large x and y. From (22), we have

$$\frac{p_2Z(x+a,y)}{Z(x,y)} + \sum_{i=1}^{m} \frac{h_i(x,y,Z(x-k_ia,y-\tau_i))}{Z(x-k_ia,y-\tau_i)} \prod_{j=2}^{k_i} \frac{Z(x-ja,y-\tau_i)}{Z(x-(j-1)a,y-\tau_i)} \le p_4. \tag{23}$$

Let $\alpha(x,y) = Z(x,y)/Z(x+a,y) > 1$. From (23), we have

$$\frac{p_2}{\alpha(x,y)} + \sum_{i=1}^{m} \frac{h_i(x,y,Z(x-k_ia,y-\tau_i))}{Z(x-k_ia,y-\tau_i)} \prod_{j=2}^{k_i} \alpha(x-ja,y-\tau_i) \le p_4. \tag{24}$$

By condition (ii) the above inequality implies that $\alpha(x,y)$ is bounded. Let $\liminf\limits_{x,y\to+\infty} \alpha(x,y) = \beta$. From (22), we can obtain

$$p_2 + \sum_{i=1}^{m} \frac{h_i(x,y,Z(x-k_ia,y-\tau_i))}{Z(x-k_ia,y-\tau_i)} \prod_{j=1}^{k_i} \alpha(x-ja,y-\tau_i)\alpha(x,y) \le p_4\alpha(x,y). \tag{25}$$

Taking the limit inferior on both sides of (25), we have $p_2 + \sum_{i=1}^{m} S_i \beta^{k_i} \le p_4 \beta$.

Hence we have $\dfrac{p_2}{\beta} + \sum_{i=1}^{m} S_i \beta^{k_i - 1} \le p_4$, i.e., $\dfrac{p_2}{\beta} \le p_4 - \sum_{i=1}^{m} S_i \beta^{k_i - 1} < p_4$.

Then we obtain $\beta > \dfrac{p_2}{p_4}$ and $\sum_{i=1}^{m} S_i \beta^{k_i - 1}/(p_4\beta - p_2) \le 1$.

Since $\min_{\beta > \frac{p_2}{p_4}} \dfrac{\beta^{k_i}}{p_4\beta - p_2} = \dfrac{p_2^{k_i-1}}{p_4^{k_i}} \dfrac{k_i^{k_i}}{(k_i-1)^{k_i-1}}$, we have $\sum_{i=1}^{m} S_i \dfrac{p_2^{k_i-1}}{p_4^{k_i}} \dfrac{k_i^{k_i}}{(k_i-1)^{k_i-1}} \le 1$, which contradicfs (14). So if (14) holds we can obtain that every solution of (1) is oscillatory.

Similarly, we can prove that if (15) holds then we can also obtain every solution of (1) is oscillatory.

If $\min_{i=1,\ldots,m}\{k_i\} = \min_{i=1,\ldots,m}\{l_i\} = 0$, from Lemma 1, we know that (3) holds.

Hence we have

$$p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + p_3 Z(x, y+b) - p_4 Z(x,y) + \\ + \sum_{i=1}^{m} h_i(x, y, Z(x,y)) \le p_1 Z(x+a, y+b) + p_2 Z(x+a, y) + \\ + p_3 Z(x, y+b) - p_4 Z(x,y) + \sum_{i=1}^{m} h_i(x, y, Z(x-\sigma_i, y-\tau_i)) \le 0.$$

Then

$$\sum_{i=1}^{m} \frac{h_i(x, y, Z(x,y))}{Z(x,y)} - p_4 \le 0. \tag{26}$$

Taking the limit inferior on both sides of (26), we have $\sum_{i=1}^{m} S_i \le p_4$, which is contrary to (16). So if (16) holds we can obtain that every solution of (1) is oscillatory.

If $A(x,y)$ is the eventually negative solution of (1), we can obtain a contradiction by assuming that $A(x,y)$ is an eventually negative solution of equation (1). Therefore we know the result is correct.

The proof is over.

The results indicate that there are some criteria of oscillatory properties of solutions of some partial difference equations with forward front difference. In some sense, the results play some roles in investigating properties of solutions of advanced partial differential equations.

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