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ON SOME PROPERTIES OF THE SET OF LYAPUNOV'S FUNCTIONS IN THE THEORY OF LINEAR EXTENSIONS OF DYNAMICAL SYSTEMS ON THE TORUS

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A form of some sets of quadratic forms having a sign-fixed derivative by virtue of the linear extension of the dynamical system on a torus is proposed. The problem of comparison of different sets with each other is investigated.

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Let us consider a system of differential equations

$$\frac{d\varphi}{dp} = a(\varphi), \;\; \frac{dx}{dt} = A(\varphi)x, \quad \varphi \in T_m, \;\; x \in R^n, \;\; a(\varphi) \in C_{\mathrm{Lip}}(T_m), \;\; A(\varphi) \in C^0(T_m). \tag{1}$$

And suppose that it has the Green – Samoilenko function

$$G_0(\tau, \varphi) = \begin{cases} \Omega_\tau^0(\varphi) C(\varphi_\tau(\varphi)), \tau \leq 0; \\ \Omega_\tau^0(\varphi)\big[C(\varphi_\tau(\varphi)) - I_n\big], \tau > 0, \end{cases} \tag{2}$$

which satisfies the following exponential estimate:

$$\|G_0(\tau, \varphi)\| \leq K e^{-\gamma|\tau|}, \quad K, \gamma = \text{const} > 0.$$

Hereafter we use definitions and notation from [1].

Let us suppose that system (1) has a unique function of the form (2). Then the following conditions (see [2])

$$C^2(\varphi) \equiv C(\varphi), \quad C(\varphi_t(\varphi)) \equiv \Omega_0^t(\varphi) C(\varphi) \Omega_t^0(\varphi) \quad \forall\, \varphi \in T_m, \;\; \forall t \in R \tag{3}$$

hold true necessarily. Consider now the matrix-valued functions

$$S_1(\varphi) = \int\limits_{-\infty}^{0} \big\{\Omega_0^\sigma(\varphi)[C(\varphi) - I_n]\big\}^T \big\{\Omega_0^\sigma(\varphi)[C(\varphi) - I_n]\big\} d\sigma, \tag{4}$$

$$S_2(\varphi) = \int\limits_{0}^{+\infty} \big\{\Omega_0^\sigma(\varphi) C(\varphi)\big\}^T \big\{\Omega_0^\sigma(\varphi) C(\varphi)\big\} d\sigma. \tag{5}$$

It is easy to verify that $S_j(\varphi) \in C'(T_m; a)$ and the following equations

$$\dot{S}_j(\varphi) + S_j(\varphi)A(\varphi) + A^T(\varphi)S_j(\varphi) = \begin{cases} [C(\varphi) - I_n]^T[C(\varphi) - I_n], & j = 1; \\ C^T(\varphi)C(\varphi), & j = 2, \end{cases}$$

take place. It is follows that the derivative of the quadratic form

$$V(\varphi; x) = \langle [S_1(\varphi) - S_2(\varphi)]x, x \rangle, \tag{6}$$

by virtue of system (1), is positive definite,

$$\dot{V}(\varphi; x) = \|[C(\varphi) - I_n]x\|^2 + \|C(\varphi)x\|^2 \geq \frac{1}{2}\|x\|^2.$$

It is worth noticing that for some systems of the form (1) the representation of the matrix-valued functions $S_j(\varphi)$ as integrals (4), (5) does not always lead to a desirable form of the quadratic form (6). This is immediate from the following example:

$$\frac{d\varphi}{dt} = 1, \quad \frac{dx}{dt} = (1 + 2\sin\varphi)x. \tag{7}$$

Obviously, in this case, $C(\varphi) \equiv 0$ and by (4) the quadratic form (6) has the following representation:

$$V = \left(\int_{-\infty}^{0} e^{2\sigma - 4\cos(\sigma+\varphi) + 4\cos\varphi} d\sigma \right) x^2. \tag{8}$$

On the other hand, it is easy to see that the function

$$V = e^{4\cos\varphi} x^2 \tag{9}$$

has positive definite derivative by virtue of the system (7). If we now introduce an auxiliary function $h(\varphi) \in C^0(T_1)$, $h(\varphi) > 0$, under the integral sign in (8), namely, we consider the set of quadratic forms

$$V = \left(\int_{-\infty}^{0} e^{2\sigma - 4\cos(\sigma+\varphi) + 4\cos\varphi} h(\sigma + \varphi) d\sigma \right) x^2,$$

then, putting here $h(\varphi) = 2e^{4\cos\varphi}$, we obtain the quadratic form (9).

So, it is important to study the generalized matrix-valued functions (4), (5), namely,

$$S_1(\varphi; H_1) = \int_{-\infty}^{0} \{\Omega_0^\sigma(\varphi)[C(\varphi) - I_n]\}^T H_1(\varphi_\sigma(\varphi)) \{\Omega_0^\sigma(\varphi)[C(\varphi) - I_n]\} d\sigma, \tag{10}$$

$$S_2(\varphi; H_2) = \int_{0}^{+\infty} \{\Omega_0^\sigma(\varphi)C(\varphi)\}^T H_2(\varphi_\sigma(\varphi)) \{\Omega_0^\sigma(\varphi)C(\varphi)\} d\sigma, \tag{11}$$

where both matrices H_1, H_2 are positive definite,

$$\langle H_i(\varphi)x, x\rangle \geq \varepsilon_i \|x\|^2, \quad \varepsilon_i = \text{const} > 0. \tag{12}$$

The present article is devoted to studying the set of quadratic forms with matrix-valued coefficients (10),(11).

By a direct verification with the use of (12) we obtain that the derivative of the quadratic form

$$V(\varphi; x) = \langle [S_1(\varphi, H_1) - S_2(\varphi, H_2)]x, x\rangle, \tag{13}$$

in virtue of the system (1), is positive definite,

$$\dot{V}(\varphi, x) = \langle H_1(\varphi)[C(\varphi) - I_n]x, [C(\varphi) - I_n]x\rangle$$

$$+ \langle H_2(\varphi)C(\varphi)x, C(\varphi)x\rangle \geq \frac{1}{2}\min\{\varepsilon_1, \varepsilon_2\}\,\|x\|^2.$$

It should be mentioned that for any two matrices $H_1(\varphi), H_2(\varphi)$ there always exists a matrix $H(\varphi)$ such that

$$S_1(\varphi, H_1) - S_2(\varphi, H_2) \equiv S_1(\varphi, H) - S_2(\varphi, H).$$

As $H(\varphi)$ we can take, for example, the following one:

$$H(\varphi) = [C(\varphi) - I_n]^T H_1(\varphi)[C(\varphi) - I_n] + C^T(\varphi)H_2(\varphi)C(\varphi).$$

Obviously, if the matrices H_1, H_2 are positive definite, then such is $H(\varphi)$.

Remark. Due to (3) the relation

$$\det\,[S_1(\varphi, H) - S_2(\varphi, H)] \neq 0$$

holds true for any positive definite matrices $H(\varphi)$.

Let us now consider the following quadratic form:

$$W = -\langle [S_1(\varphi, H) - S_2(\varphi, H)]^{-1}y, y\rangle. \tag{14}$$

By a direct calculation we can check that the derivative of this form by virtue of the system that is conjugate to (1),

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dy}{dt} = -A^T(\varphi)y, \tag{15}$$

is positive definite,

$$\dot{W} \geq \varepsilon_0\|y\|^2, \quad \varepsilon_0 = \text{const} > 0.$$

On the other hand, if one considers the following two matrices:

$$\overline{S}_1(\varphi, \widetilde{H}) = \int_{-\infty}^{0} C(\varphi)\Omega_\sigma^0(\varphi)\widetilde{H}(\varphi_\sigma(\varphi))\{C(\varphi)\Omega_\sigma^0(\varphi)\}^T d\sigma, \tag{16}$$

$$\overline{S}_2(\varphi,\widetilde{H}) = \int\limits_0^{+\infty} [C(\varphi)-I_n]\,\Omega_\sigma^0(\varphi)\widetilde{H}(\varphi_\sigma(\varphi))\big\{[C(\varphi)-I_n]\,\Omega_\sigma^0(\varphi)\big\}^T d\sigma, \tag{17}$$

where the matrix $\tilde{H}(\varphi)$ is positive definite, then the derivative of the quadratic form

$$\big\langle[\overline{S}_1(\varphi,\widetilde{H})-\overline{S}_2(\varphi,\widetilde{H})]y,y\big\rangle = \overline{W}, \tag{18}$$

by virtue of the system (15), is positive definite. Because the quadratic form (14) is also positive definite, there is the problem of comparison of two quadratic forms (18) and (14).

The following proposition takes place.

Theorem. *For any positive definite symmetric matrix $H(\varphi) \in C^0(T_m)$ there exists a positive definite matrix $\tilde{H}(\varphi) \in C^0(T_m)$ such that the equality*

$$-[S_1(\varphi,H)-S_2(\varphi,H)]^{-1} = \overline{S}_1(\varphi,\widetilde{H})-\overline{S}_2(\varphi,\widetilde{H}) \tag{19}$$

holds true. And conversely for any positive definite symmetric matrix $\tilde{H}(\varphi)$ there exists a positive definite matrix $\tilde{H}(\varphi) \in C^0(T_m)$ such that the equality (19) takes place.

Proof. Fix any symmetric matrix $H(\varphi) \in C^0(T_m)$ and put

$$X(\varphi) = -[S_1(\varphi,H)-S_2(\varphi,H)]^{-1}. \tag{20}$$

Using notation (16), (17) , we can write the equality (19) in the following form:

$$\int\limits_{-\infty}^{0} C(\varphi)\Omega_\sigma^0(\varphi)\widetilde{H}(\varphi_\sigma(\varphi))\left\{C(\varphi)\Omega_\sigma^0(\varphi)\right\}^T d\sigma$$

$$-\int\limits_0^{+\infty} [C(\varphi)-I_n]\,\Omega_\sigma^0(\varphi)\widetilde{H}(\varphi_\sigma(\varphi))\left\{[C(\varphi)-I_n]\,\Omega_\sigma^0(\varphi)\right\}^T d\sigma = X(\varphi). \tag{21}$$

Substituting here $\varphi \to \varphi_t(\varphi)$ 0and differentiating both sides of the equality (21) with respect to t gives

$$C(\varphi)\widetilde{H}(\varphi)C(\varphi)+[I_n-C(\varphi)]\,\widetilde{H}(\varphi)[I_n-C(\varphi)]^T$$

$$=\dot{X}(\varphi)-X(\varphi)A^T(\varphi)-A(\varphi)X(\varphi). \tag{22}$$

Let us denote the right-hand side of the latter equality by $\hat{H}(\varphi)$.

Obviously, if $C(\varphi) \equiv 0$, or $C(\varphi) \equiv 1$ we can take the matrix $\hat{X}(\varphi)$ as $\widetilde{H}(\varphi)$.

In order to take $\widetilde{H}(\varphi) = \hat{X}(\varphi)$ in the general case, we have to prove that the matrix $X(\varphi)$ can be represented in the form

$$\hat{X}(\varphi) = C(\varphi)F_1(\varphi)C(\varphi)+[I_n-C(\varphi)]\,F_2(\varphi)\,[I_n-C(\varphi)]^T \tag{23}$$

with some matrices $F_i(\varphi)$. Having this in mind, we use the notation (20) and formulate (10), (11) to get

$$X^{-1}(\varphi) = C^T(\varphi)X^{-1}(\varphi)C(\varphi) + [I_n - C(\varphi)]^T X^{-1}(\varphi)\,[I_n - C(\varphi)]. \tag{24}$$

Let us verify that this equality yields

$$X(\varphi) = C(\varphi)X(\varphi)C^T(\varphi) + [I_n - C(\varphi)]\,X(\varphi)\,[I_n - C(\varphi)]^T. \tag{25}$$

Indeed, multiplying the right-hand sides of the equalities (24), (25) and taking into consideration the properties of the projection matrix (3) and the equality $CX = XC^T$, we have

$$C^TX^{-1}CXC^T + (I_n - C)^TX^{-1}(I_n - C)X(I_n - C)^T = C^T + (I_n - C)^T = I_n,$$

which leads to (25).

Let us represent the matrix $\hat{X}(\varphi)$ in the form

$$\hat{X}(\varphi) = \dot{X} - XA^T - AX = -X[-X^{-1}\dot{X}X^{-1} + A^TX^{-1} + X^{-1}A]\,X$$

$$= -X[-C^THC - (I_n - C)^TH(I_n - C)]X.$$

Substituting X from (25) into this equality gives

$$\hat{X}(\varphi) = CXC^THCXC^T + (I_n - C)X(I_n - C)^TH(I_n - C)X(I_n - C)$$

and, hence, the representation (23) holds true, which means that the matrix $\hat{X}(\varphi)$ can be taken as $\widetilde{H}(\varphi)$ and, at the same time, the equality (22) holds true which implies the equality (19). Similar arguments yield that for any fixed symmetric matrix $\widetilde{H}(\varphi) \in C^0(T_m)$ there exists a symmetric matrix $H(\varphi)$ which satisfies the equality (19).

Remark. Considering the case where the matrix $C(\varphi) \equiv I_n$ we can easily see that the equation $\int\limits_{-\infty}^{0} \Omega_\tau^0(\varphi)X(\varphi_\tau(\varphi))\{\Omega_\tau^0(\varphi)\}^T d\tau = M(\varphi)$ has the unique solution $X(\varphi) = \dot{M}(\varphi) - M(\varphi)A^T(\varphi) - A(\varphi)M(\varphi)$ for any fixed $n \times n$-matrix $M(\varphi) \in C'(T_m; a)$.

As an example of system (1) for which there exits a nondegenerate quadratic form (13) possessing positive definite derivative, we consider the following one:

$$\frac{d\varphi_i}{dt} = \omega_i, \quad \frac{dx_j}{dt} = \left[\lambda_j + \sum_{i=1}^{m}\sum_{k=1}^{N}(a_{ijk}\cos k\varphi_i + b_{ijk}\sin k\varphi_i)\right]x_j,$$

$$\omega_i = \text{const} \neq 0, \quad i = \overline{1,m}, \quad \lambda_j = \text{const} \neq 0, \quad j = \overline{1,n}, \quad a_{ijk}, b_{ijk} = \text{const}.$$

Choosing a suitable function $H(\varphi)$ in the formulae (10), (11) we obtain the following quadratic form:

$$V = \sum_{j=1}^{n}\left\{\text{sign}\,(\lambda_j)\exp\left[\sum_{i=1}^{m}\sum_{k=1}^{N}\frac{-2a_{ijk}\sin k\varphi_i + 2b_{ijk}\cos k\varphi_i}{k\omega_i}\right]\right\}x_j^2,$$

and its derivative is positive definite by virtue of the system in question.

Let us consider the case where the system (1) has two different Green – Samoilenko functions of the form (2). Then, obviously, there exists an infinite number of such functions and the conditions (3) fail to hold for each of them. In such a case one can always write the following quadratic form:

$$W = \langle [S_1(\varphi; H_1) - S_2(\varphi; H_2)]y, y\rangle, \tag{18'}$$

which has positive definite derivative by virtue of the system (15), with the matrix of coefficients having the form $S(\varphi; H_1, H_2) = S_1(\varphi; H_1) - S_2(\varphi, H_2)$, where

$$S_1(\varphi; H_1) = \int\limits_{-\infty}^{0} \Omega_\sigma^0(\varphi) C(\varphi_\sigma(\varphi)) H_1(\varphi_\sigma(\varphi)) \{\Omega_\sigma^0(\varphi) C(\varphi_\sigma(\varphi))\}^T d\sigma, \tag{26}$$

$$S_2(\varphi; H_2) = \int\limits_{0}^{+\infty} \Omega_\sigma^0(\varphi)\,[C(\varphi_\sigma(\varphi)) - I_n]\, H_2(\varphi_\sigma(\varphi)) \{\Omega_\sigma^0(\varphi)[C(\varphi_\sigma(\varphi)) - I_n]\,\}^T d\sigma. \tag{27}$$

It is worth noticing that the matrices (26), (27) can not be represented in the forms (16), (17), because the conditions (3) fail.

A question arises: whether there exists a unique positive definite matrix $H(\varphi) \equiv H^T(\varphi)$ such that

$$S(\varphi; H_1, H_2) = S(\varphi; H, H) \tag{28}$$

for any fixed symmetric and positive definite matrices $H_1(\varphi)$, $H_2(\varphi) \in C^0(T_m)$?

It turns out that, in general, this is not the case. We show this by the following example:

$$\frac{d\varphi}{dt} = \sin\varphi, \quad \frac{dx}{dt} = (\cos\varphi)x.$$

Obviously, this system has an infinite number of different Green – Samoilenko functions. For example, some of them have the form

$$G_0^{(n)}(\tau,\varphi) = \begin{cases} \dfrac{e^{n\tau}\sin^{2n}\dfrac{\varphi}{2}}{\left(e^{-\tau}\cos^2\dfrac{\varphi}{2} + e^{\tau}\sin^2\dfrac{\varphi}{2}\right)^{n-1}}, & \tau \le 0; \\[4ex] \left(e^{-\tau}\cos^2\dfrac{\varphi}{2} + e^{\tau}\sin^2\dfrac{\varphi}{2}\right)\left[\dfrac{e^{n\tau}\sin^{2n}\dfrac{\varphi}{2}}{\left(e^{-\tau}\cos^2\dfrac{\varphi}{2} + e^{\tau}\sin^2\dfrac{\varphi}{2}\right)^{n}} - 1\right], & \tau > 0, \end{cases}$$

where $n = 1, 2, 3, \ldots$. Putting here $n = 1$, we write down

$$S(\varphi; H_1, H_2) = \int\limits_{-\infty}^{0} \left[e^{\tau}\sin^2\frac{\varphi}{2}\right]^2 H_1(\varphi_\tau(\varphi))d\tau - \int\limits_{0}^{+\infty} \left[e^{-\tau}\cos^2\frac{\varphi}{2}\right]^2 H_2(\varphi_\tau(\varphi))d\tau,$$

where H_i are positive scalar functions.

Let us suppose that there exists a unique function $H(\varphi) > 0$ for which the equality (28) holds true, namely,

$$\sin^4\frac{\varphi}{2}\int\limits_{-\infty}^{0} e^{2\tau}H_1(\varphi_\tau(\varphi))d\tau - \cos^4\frac{\varphi}{2}\int\limits_{0}^{+\infty} e^{-2\tau}H_2(\varphi_\tau(\varphi))d\tau$$

$$= \sin^4\frac{\varphi}{2}\int\limits_{-\infty}^{0} e^{2\tau}H(\varphi_\tau(\varphi))d\tau - \cos^4\frac{\varphi}{2}\int\limits_{0}^{+\infty} e^{-2\tau}H(\varphi_\tau(\varphi))d\tau. \quad (29)$$

Substituting here $\varphi \to \varphi_t(\varphi)$ gives

$$\sin^4\frac{\varphi}{2}\int\limits_{-\infty}^{t} e^{2\sigma}H_1(\varphi_\sigma(\varphi))d\sigma - \cos^4\frac{\varphi}{2}\int\limits_{t}^{+\infty} e^{-2\sigma}H_2(\varphi_\sigma(\varphi))d\sigma$$

$$= \sin^4\frac{\varphi}{2}\int\limits_{-\infty}^{t} e^{2\sigma}H(\varphi_\sigma(\varphi))d\sigma - \cos^4\frac{\varphi}{2}\int\limits_{t}^{+\infty} e^{-2\sigma}H(\varphi_\sigma(\varphi))d\sigma.$$

Differentiating both sides of this equality with respect to t and putting $t = 0$ gives to

$$H(\varphi) = \frac{H_1(\varphi)\sin^4\frac{\varphi}{2} + H_2(\varphi)\cos^4\frac{\varphi}{2}}{\sin^4\frac{\varphi}{2} + \cos^4\frac{\varphi}{2}}. \quad (30)$$

So, if a function $H(\varphi)$ for which the equality (29) holds true exists, this function must have the form (30). Putting now the function (30) into the right-hand side of (29) we obtain

$$\sin^4\frac{\varphi}{2}\int\limits_{-\infty}^{0} e^{2\tau}\frac{H_1(\varphi_\tau(\varphi))e^{2\tau}\sin^4\frac{\varphi}{2} + H_2(\varphi_\tau(\varphi))e^{-2\tau}\cos^4\frac{\varphi}{2}}{e^{2\tau}\sin^4\frac{\varphi}{2} + e^{-2\tau}\cos^4\frac{\varphi}{2}}d\tau$$

$$- \cos^4\frac{\varphi}{2}\int\limits_{0}^{+\infty} e^{-2\tau}\frac{H_1(\varphi_\tau(\varphi))e^{2\tau}\sin^4\frac{\varphi}{2} + H_2(\varphi_\tau(\varphi))e^{-2\tau}\cos^4\frac{\varphi}{2}}{e^{2\tau}\sin^4\frac{\varphi}{2} + e^{-2\tau}\cos^4\frac{\varphi}{2}}d\tau$$

$$= \sin^4\frac{\varphi}{2}\left\{\int\limits_{-\infty}^{0} e^{2\tau}H_1(\varphi_\tau(\varphi))d\tau + \int\limits_{-\infty}^{0}\frac{[H_2(\varphi_\tau(\varphi)) - H_1(\varphi_\tau(\varphi))]\cos^4\frac{\varphi}{2}}{e^{2\tau}\sin^4\frac{\varphi}{2} + e^{-2\tau}\cos^4\frac{\varphi}{2}}d\tau\right\}$$

$$- \cos^4\frac{\varphi}{2}\left\{\int\limits_{0}^{+\infty} e^{-2\tau}H_2(\varphi_\tau(\varphi))d\tau + \int\limits_{0}^{+\infty}\frac{[H_1(\varphi_\tau(\varphi)) - H_2(\varphi_\tau(\varphi))]\sin^4\frac{\varphi}{2}}{e^{2\tau}\sin^4\frac{\varphi}{2} + e^{-2\tau}\cos^4\frac{\varphi}{2}}d\tau\right\}.$$

It follows that the equality (29) holds true if and only if

$$\int\limits_{-\infty}^{+\infty} \frac{[H_1(\varphi_\tau(\varphi)) - H_2(\varphi_\tau(\varphi))] \sin^4 \dfrac{\varphi}{2} \cos^4 \dfrac{\varphi}{2}}{e^{2\tau} \sin^4 \dfrac{\varphi}{2} + e^{-2\tau} \cos^4 \dfrac{\varphi}{2}} d\tau \equiv 0.$$

Obviously, the last identity holds true not for any functions H_1, H_2.

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