Nonlinear Oscillations, Vol. 4, No. 4, 2001

ON SOLVABILITY OF SINGULAR PERIODIC BOUNDARY-VALUE PROBLEMS*

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We present conditions ensuring the existence of a solution in the class $C^1([0,T])$ for the singular periodic boundary-value problem $(r(x'))' = H(p(t)+q(x))k(x')f(t,x,x')$, $x(0) = x(T)$, $x'(0) = x'(T)$. Here the function k is singular at $u = 0$ in the following sense: $\lim_{u\to 0} k(u) = \infty$. Since the derivative of any solution to our periodic boundary-value problem vanishes at least once on $[0,T]$, solutions of the considered problem pass through the singular point $x' = 0$ of the phase variable x'.

AMS Subject Classification: 34B16, 34C25

1. Introduction, Notation

Let $T > 0$ and $J = [0,T]$. Throughout the paper $\|x\| = \max\{|x(t)| : t \in J\}$ denotes the norm in $C^0(J)$ and $\|x\|_L = \int_0^T |x(t)|\,dt$ the norm in $L(J)$. Next we use the following sets.

$L_{\rm loc}(\mathbb{R})$ is the set of Lebesgue integrable functions on any compact interval $[a,b] \subset \mathbb{R}$.

$AC(J)$ is the set of absolutely continuous functions on J.

$AC_{\rm loc}(\mathbb{R})$ is the set of functions x such that $x \in AC([a,b])$ for any compact interval $[a,b] \subset \mathbb{R}$.

$AC^1(J)$ is the set of functions having absolutely continuous derivative on J.

$Car_{\rm loc}(J \times D)$ is the set of functions satisfying the local Carathéodory conditions on $J \times D$, where D is a subset of $\mathbb{R}^2$.

Consider singular differential equations of the form

$$(r(x'(t)))' = H(p(t) + q(x(t)))k(x'(t))f(t, x(t), x'(t)) \tag{1}$$

together with the periodic boundary conditions

$$x(0) = x(T), \;\; x'(0) = x'(T), \tag{2}$$

where r, H, p, and q are continuous, k is continuous on $\mathbb{R} \setminus \{0\}$ and singular at $u = 0$ in the following sense $\lim_{u\to 0} k(u) = \infty$ and $f \in Car_{\rm loc}(J \times D)$ or $f \in C^0(J \times D)$.

In the case of $f \in Car_{\rm loc}(J \times D)$, we say that x is a *solution of the periodic boundary-value problem* (PBVP) (1), (2) if $x \in C^1(J)$, $r(x') \in AC(J)$, the set $\mathcal{A} = \{t : t \in J,\; x'(t) = 0\}$ is finite, x satisfies the periodic conditions (2) and (1) holds a.e. on J.

* Supported by grant no. 201/01/1451 of the Grant Agency of Czech Republic and by the Council of Czech Government J14/98:153100011.

In the case of $f \in C^0(J \times D)$, a function $x \in C^1(J)$ is said to be a *solution of PBVP* (1), (2) if the set $\mathcal{A} = \{t : t \in J,\ x'(t) = 0\}$ is finite, $r(x') \in C^1(J \setminus \mathcal{A})$, x satisfies the periodic conditions (2) and (1) holds on $J \setminus \mathcal{A}$.

There are many papers which deal with singular PBVPs where the considered second order differential equations have singularities in the phase variable x at the point $x = 0$ (see, e.g., [1–7] and the references therein). Common in these papers is the fact that the considered solutions are either positive or negative, that is, solutions do not pass through the singular point $x = 0$. This situation has been partially overcomed in [8] where differential equations of the form

$$(r(x(t))x'(t))' = \mu q(t) f_*(t, x(t))$$

with a positive parameter μ were studied. Here f_* may be singular at $x = 0$ and $x = A > 0$. If $\lim_{u \to 0^+} r(u) = \infty$ and some assumptions on r, q and f_* are satisfied, then there exists $\mu_* > 0$ such that for $\mu \in (0, \mu_*)$ the above differential equations have a solution x in the class $C^1(J)$ satisfying the periodic boundary conditions $x(0) = x'(0) = 0$, $x(T) = x'(T) = 0$ and $0 < x(t) < A$ for $t \in (0, T)$ (see [8], Theorem 2).

In this paper we present conditions ensuring the existence of a solution to the singular PBVP (1), (2) where the differential equation (1) is singular at the point $x' = 0$ of the phase variable x'. Since for any solution x of PBVP (1), (2) the derivative x' has at least one zero on J, we see that solutions of our PBVP pass through the singularity of (1). The proofs of existence results are based on a trick in [9] by which we transform the singularity on the right-hand side of (1) to its left-hand side. The obtained differential equation has now regular right-hand side, and so we can apply, to the transformed PBVP, existence results given in [10]. The solvability of PBVP (1), (2) is presented in Theorem 1 under the assumption that f satisfies the local Carathéodory conditions and for continuous f in Theorem 2.

From now on, we assume that the functions r, H, k, p, q, and f in (1) satisfy the following assumptions:

(H_1) $r \in AC_{\mathrm{loc}}(\mathbb{R})$ is increasing and maps $\mathbb{R}$ onto $\mathbb{R}$, $r(0) = 0$;

(H_2) $k \in C^0(\mathbb{R}) \setminus \{0\})$ is positive, $\lim_{u \to 0} k(u) = \infty$, and $\liminf_{|u| \to \infty} k(u) > 1$;

(H_3) $\displaystyle\int_0 \frac{1}{k(r^{-1}(s))}\, ds < \infty, \quad \int^0 \frac{1}{k(r^{-1}(s))}\, ds < \infty,$

$$\int_{-\infty}^0 \frac{1}{k(r^{-1}(s))}\, ds = \infty, \int_0 t\infty \frac{1}{k(r^{-1}(s))}\, ds = \infty;$$

(H_4) The inverse z^{-1} to $z : \mathbb{R} \to \mathbb{R}$, $z(t) = \displaystyle\int_0^u \frac{1}{k(r^{-1}(s))}\, ds$, is a locally Lipschitzian function on $\mathbb{R}$;

(H_5) $H \in C^0(\mathbb{R})$ is increasing on $\mathbb{R}$ and $H(0) = 0$;

(H_6) $p \in C^0(J)$ and there exist α, $\beta \in \mathbb{R}$, $\alpha < \beta$, such that $q \in C^0([\alpha, \beta])$ is increasing on $[\alpha, \beta]$,

$$p(t) + q(\alpha) \le 0 \le p(t) + q(\beta) \quad \text{for } t \in J,$$

$q^{-1}(-p)$ is differentiable on J and $(q^{-1}(-p(t)))' \neq 0$ for $t \in J$, where q^{-1} is the inverse to q and either

(H_7) $f \in Car_{\text{loc}}(J \times [\alpha, \beta] \times \mathbb{R})$ and

$$\chi(t) \leq f(t,x,y) \leq (h(t) + |y|)\omega(|y|) \quad \text{for } (t,x,y) \in J \times [\alpha,\beta] \times \mathbb{R},$$

where $\chi,\ h \in L(J)$ are positive on J and $\omega \in C^0([0,\infty))$ is positive and nondecreasing on $[0,\infty)$ or

(H_8) $f \in C^0(J \times [\alpha, \beta] \times \mathbb{R})$ and

$$0 < f(t,x,y) \leq (h(t) + |y|)\omega(|y|) \quad \text{for } (t,x,y) \in J \times [\alpha,\beta] \times \mathbb{R},$$

where $h \in C^0(J)$ is positive on J and $\omega \in C^0([0,\infty))$ is positive and nondecreasing on $[0,\infty)$.

Remark 1. The condition $\liminf\limits_{|u|\to\infty} k(u) > 1$ in (H_2) can be replaced by the weaker one $\liminf\limits_{|u|\to\infty} k(u) > 0$. Indeed, if $\gamma = \liminf\limits_{|u|\to\infty} k(u) \leq 1$, we use in (1) for example the functions $2k/\gamma$ and $\gamma f/2$ instead of k and f, respectively.

Remark 2. If we set

$$\varrho(u) = \begin{cases} \dfrac{1}{k(u)} & \text{for} \quad u \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{for} \quad u = 0, \end{cases}$$

then ϱ is continuous and bounded on $\mathbb{R}$.

Let $w \in AC_{\text{loc}}(\mathbb{R})$ and for each $n \in \mathbb{N}$ the functions $[1/k]_n \in C^0(\mathbb{R})$ and $w_n \in AC_{\text{loc}}(\mathbb{R})$ be defined by the formulas

$$w(u) = \int_0^{r(u)} \frac{1}{k(r^{-1}(s))}\, ds, \tag{3}$$

$$\left[\frac{1}{k}\right]_n (u) = \begin{cases} \dfrac{1}{k(u)} & \text{if} \quad k(u) \geq \dfrac{1}{n}, \\ n & \text{if} \quad k(u) < \dfrac{1}{n}, \\ 0 & \text{if} \quad u = 0, \end{cases} \tag{4}$$

$$w_n(u) = \int_0^{r(u)} \left[\frac{1}{k}\right]_n (r^{-1}(s))\, ds. \tag{5}$$

Then $[1/k]_n(u) \leq [1/k]_{n+1}(u)$ for $u \in \mathbb{R}$ and $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} [1/k]_n(u) = 1/k(u)$ for $u \in \mathbb{R} \setminus \{0\}$ and

$$\begin{aligned} 0 \leq w_n(u) \leq w_{n+1}(u) \quad & \text{for } u \in [0,\infty),\ n \in \mathbb{N}, \\ 0 \geq w_n(u) \geq w_{n+1}(u) \quad & \text{for } u \in (-\infty,0],\ n \in \mathbb{N}. \end{aligned} \tag{6}$$

By (H_2), there is a $c \in (0,\infty)$ such that $k(u) \geq 1$ for $|u| \geq c$, and so $[1/k]_n(u) = 1/k(u)$ for $|u| \geq c$ and $n \in \mathbb{N}$. Hence, by (H_1) and (H_3),

$$\lim_{u\to\pm\infty} w_n(u) = \pm\infty \quad \text{for } n \in \mathbb{N}. \tag{7}$$

According to Levi's theorem

$$\lim_{n\to\infty} w_n(u) = \int_0^{r(u)} \frac{1}{k(r^{-1}(s))}\,ds \quad \text{for } u \in \mathbb{R} \tag{8}$$

and combining (6), (8) and Dini's theorem we conclude that $\lim_{n\to\infty} w_n(u) = w(u)$ locally uniformly on $\mathbb{R}$ and

$$\begin{aligned} 0 \leq w_n(u) \leq w(u) \quad &\text{for } u \in [0,\infty),\ n \in \mathbb{N},\\ 0 \geq w_n(u) \geq w(u) \quad &\text{for } u \in (-\infty,0],\ n \in \mathbb{N}. \end{aligned} \tag{9}$$

For the inverse w_n^{-1} to w_n we have (cf. (6) and (7))

$$\begin{aligned} w_n^{-1}(u) \geq w_{n+1}^{-1}(u) \geq 0 \quad &\text{for } u \in [0,\infty),\ n \in \mathbb{N},\\ w_n^{-1}(u) \leq w_{n+1}^{-1}(u) \leq 0 \quad &\text{for } u \in (-\infty,0],\ n \in \mathbb{N} \end{aligned} \tag{10}$$

and since $\lim_{n\to\infty} w_n^{-1}(u) = w^{-1}(u)$ for $u \in \mathbb{R}$, where w^{-1} is the inverse to w, Dini's theorem gives that the last convergence is locally uniform on $\mathbb{R}$.

2. Auxiliary Regular Periodic Boundary-Value Problems

Consider the family of regular differential equations

$$(w_n(x'(t)))' = H(p(t) + q(x(t)))f(t, x(t), x'(t)) \tag{11}$$

depending on $n \in \mathbb{N}$. Here w_n is defined by (5).

If $f \in Car_{\text{loc}}(J \times D)$, a function x is said to be a *solution of PBVP* $(11)_n$, (2) if $x \in C^1(J)$, $w_n(x') \in AC(J)$, x satisfies the periodic conditions (2) and $(11)_n$ holds a.e. on the interval J.

Lemma 1. *Let $n \in \mathbb{N}$ and let assumptions $(H_1)-(H_7)$ and*

$$\int_{-\infty}^{0} \frac{1}{\omega(|w_1^{-1}(s)|)}\,ds = \int_0^{\infty} \frac{1}{\omega(w_1^{-1}(s))}\,ds = \infty \tag{12}$$

be satisfied with w_1^{-1} the inverse to w_1 given by (5). *Then there exists a solution of PBVP* $(11)_n$, (2) *such that*

$$\alpha \leq x(t) \leq \beta, \quad |x'(t)| \leq P \qquad \textit{for } t \in J, \tag{13}$$

where P is a positive constant satisfying the inequality

$$\min\left\{ \int_{w_1(-P)}^{0} \frac{1}{\omega(|w_1^{-1}(s)|)}\,ds,\ \int_0^{w_1(P)} \frac{1}{\omega(w_1^{-1}(s))}\,ds \right\} > 2L(\|h\|_L + 2\max\{|\alpha|,|\beta|\}) \tag{14}$$

with $L = \max\{|H(u)| : |u| \leq \|p\| + \max\{|q(\alpha)|,\ |q(\beta)|\}\}$.

Proof. By (H_6) and (H_7),

$$H(p(t)+q(\alpha))f(t,\alpha,0) \le 0 \le H(p(t)+q(\beta))f(t,\beta,0)$$

for a.e. $t \in J$, and so we see that the constant functions α and β are lower and upper functions of PBVP $(11)_n$, (2) (for the definition of lower and upper functions of PBVP $(11)_n$, (2), see [10]. Then, by [10], there exists a solution x of PBVP $(11)_n$, (2) such that

$$\alpha \le x(t) \le \beta \quad \text{for } t \in J \tag{15}$$

and from (15) and (H_7) it follows

$$(w_n(x'(t)))' \le |H(p(t)+q(x(t)))|(h(t)+|x'(t)|)\omega(|x'(t)|)$$

$$\le L(h(t)+|x'(t)|)\omega(|x'(t)|)$$

for a.e. $t \in J$ and applying Lemma 1 in [10] to the above inequality we have $\|x'\| \le P_n$, where P_n is a positive constant satisfying the inequality

$$\min\left\{\int_{w_n(-P_n)}^{0} \frac{1}{\omega(|w_n^{-1}(s)|)}\,ds,\ \int_{0}^{w_n(P_n)} \frac{1}{\omega(w_n^{-1}(s))}\,ds\right\} > 2L(\|h\|_L + 2\max\{|\alpha|,|\beta|\}). \tag{16}$$

We are going to show that P_n can be selected such that $P_n = P$. First assume that $u \in (-\infty, 0]$. Since $0 \ge w_1(u) \ge w_n(u)$, $w_1^{-1}(u) \le w_n^{-1}(u) \le 0$ by (6) and (10), and ω is positive and nondecreasing on $[0,\infty)$ by (H_7), we see that $\omega(|w_1^{-1}(u)|) \ge \omega(|w_n^{-1}(u)|)$ and

$$\int_{w_n(u)}^{0} \frac{1}{\omega(|w_n^{-1}(s)|)}\,ds \ge \int_{w_1(u)}^{0} \frac{1}{\omega(|w_1^{-1}(s)|)}\,ds \quad \text{for } u \in (-\infty,0]. \tag{17}$$

Similarly we can verify that

$$\int_{0}^{w_n(v)} \frac{1}{\omega(w_n^{-1}(s))}\,ds \ge \int_{0}^{w_1(v)} \frac{1}{\omega(w_1^{-1}(s))}\,ds \quad \text{for } v \in [0,\infty). \tag{18}$$

Set

$$\Delta_j(v) = \min\left\{\int_{w_j(-v)}^{0} \frac{1}{\omega(|w_j^{-1}(s)|)}\,ds,\ \int_{0}^{w_j(v)} \frac{1}{\omega(w_j^{-1}(s))}\,ds\right\}$$

for $v \in [0,\infty)$ and $j \in \{1,n\}$. Then Δ_1, Δ_n are continuous and increasing on $[0,\infty)$ and $\Delta_1 \le \Delta_n$ on $[0,\infty)$ by (17) and (18). In addition, $\lim_{v\to\infty} \Delta_1(v) = \infty$ by assumption (12). Since $\Delta_1(P) > 2L(\|h\|_L + 2\max\{|\alpha|,|\beta|\})$ by (14), we deduce that (16) is satisfied with $P_n = P$. We have proved that $\|x'\| \le P$.

Remark 3. If the function k in (1) satisfies $k(u) \geq 1$ for $u \in \mathbb{R} \setminus \{0\}$ then $[1/k]_1(u) = 1/k(u)$ for $u \in \mathbb{R} \setminus \{0\}$ and so (cf. (5)) $w_1(u) = w(u)$ on $\mathbb{R}$. Hence conditions (12) and (14) can be written in the form

$$\int_{-\infty}^{0} \frac{1}{\omega(|w^{-1}(s)|)}\, ds = \int_{0}^{\infty} \frac{1}{\omega(w^{-1}(s))}\, ds = \infty$$

and

$$\min\left\{ \int_{w(-P)}^{0} \frac{1}{\omega(|w^{-1}(s)|)}\, ds,\ \int_{0}^{w(P)} \frac{1}{\omega(w^{-1}(s))}\, ds \right\} > 2L(\|h\|_L + 2\max\{|\alpha|, |\beta|\}),$$

respectively.

3. Existence Results

Theorem 1. *Let assumptions* (H_1)–(H_7) *and* (12) *be satisfied. Then there exists a solution* x *of PBVP* (1), (2) *satisfying* (13)*, where* P *is a positive constant for which* (14) *holds.*

Proof. By Lemma 1, for each $n \in \mathbb{N}$, there exists a solution x_n of PBVP $(11)_n$, (2) such that

$$\alpha \leq x_n(t) \leq \beta, \quad |x_n'(t)| \leq P \qquad \text{for } t \in J,\ n \in \mathbb{N}. \tag{19}$$

By (19), $\{x_n\}$ is bounded in $C^1(J)$. We now verify that $\{x_n'(t)\}$ is equicontinuous on J. First we show that $\{w_n(x_n'(t))\}$ is equicontinuous on J. Since $f \in Car_{\text{loc}}(J \times [\alpha, \beta] \times \mathbb{R})$, there is a $\nu \in L(J)$ such that $0 \leq f(t, x_n(t), x_n'(t)) \leq \nu(t)$ for a.e. $t \in J$ and $n \in \mathbb{N}$. Then

$$|w_n(x_n'(t_1)) - w_n(x_n'(t_2))| \leq L \left| \int_{t_1}^{t_2} f(t, x_n(t), x_n'(t))\, dt \right| \leq L \left| \int_{t_1}^{t_2} \nu(t)\, dt \right|$$

for $t_1, t_2 \in J$ and $n \in \mathbb{N}$, where L is defined in Lemma 1. Consequently, $\{w_n(x'(t))\}$ is equicontinuous on J. Assume, on the contrary, that $\{x_n'(t)\}$ is not equicontinuous on J. Then there exist $\varepsilon_0 > 0$, a subsequence $\{k_n\}$ of $\mathbb{N}$, and sequences $\{\hat{t}_n\}, \{\bar{t}_n\} \subset J$ such that $\lim_{n\to\infty} (\hat{t}_n - \bar{t}_n) = 0$ and

$$|x_{k_n}'(\hat{t}_n) - x_{k_n}'(\bar{t}_n)| \geq \varepsilon_0 \quad \text{for } n \in \mathbb{N}. \tag{20}$$

From the boundedness of $\{\hat{t}_n\}$ and $\{\bar{t}_n\}$ it follows that we can assume their convergence and, with respect to $\lim_{n\to\infty} (\hat{t}_n - \bar{t}_n) = 0$, we then have

$$\lim_{n\to\infty} \hat{t}_n = \lim_{n\to\infty} \bar{t}_n = t_*. \tag{21}$$

We claim that there is a $\varrho > 0$ such that

$$\int_{r(u)}^{r(v)} \frac{1}{k(r^{-1}(s))}\, ds \geq \varrho \quad \text{whenever } u, v \in [-P, P] \text{ and } v - u \geq \varepsilon_0. \tag{22}$$

If not, there are sequences $\{u_n\}$, $\{v_n\} \subset [-P, P]$, $v_n - u_n \geq \varepsilon_0$ for which

$$\lim_{n\to\infty} \int\limits_{r(u_n)}^{r(v_n)} \frac{1}{k(r^{-1}(s))}\, ds = 0.$$

Without loss of generality we may assume that $\{u_n\}$, $\{v_n\}$ are convergent, say $\lim\limits_{n\to\infty} u_n = u_*$, $\lim\limits_{n\to\infty} v_n = v_*$. Of course, $v_* - u_* \geq \varepsilon_0$. Then

$$0 = \lim_{n\to\infty} \int\limits_{r(u_n)}^{r(v_n)} \frac{1}{k(r^{-1}(s))}\, ds = \int\limits_{r(u_*)}^{r(v_*)} \frac{1}{k(r^{-1}(s))}\, ds,$$

contrary to $r(v_*) > r(u_*)$ and $1/k(r^{-1}(s)) > 0$ on $\mathbb{R} \setminus \{0\}$. Hence (20) and (22) yield

$$\left| \int\limits_{r(x'_{k_n}(\hat{t}_n))}^{r(x'_{k_n}(\bar{t}_n))} \frac{1}{k(r^{-1}(s))}\, ds \right| \geq \varrho \quad \text{for } n \in \mathbb{N}. \tag{23}$$

We know that $\lim\limits_{n\to\infty} w_n(t) = w(t)$ uniformly on $[-P, P]$ and $\{w_n(x'_n(t))\}$ is equicontinuous on J. Therefore there exist $\mu \in (0, \infty)$ and $n_* \in \mathbb{N}$ such that

$$|w_{k_n}(x'_{k_n}(t_1)) - w_{k_n}(x'_{k_n}(t_2))| < \frac{\varrho}{6} \quad \text{for } n \in \mathbb{N} \text{ and } t_1, t_2 \in J,\ |t_1 - t_2| < \mu, \tag{24}$$

$$|w_{k_n}(u) - w(u)| < \frac{\varrho}{6} \quad \text{for } u \in [-P, P],\ n \geq n_*, \tag{25}$$

and

$$|\hat{t}_n - \bar{t}_n| < \mu \quad \text{for } n \geq n_*. \tag{26}$$

By (24) – (26),

$$|w_{k_n}(x'_{k_n}(\hat{t}_n)) - w(x'_{k_n}(\hat{t}_n))| < \frac{\varrho}{6},\ \ |w_{k_n}(x'_{k_n}(\bar{t}_n)) - w(x'_{k_n}(\bar{t}_n))| < \frac{\varrho}{6}$$

and

$$|w_{k_n}(x'_{k_n}(\hat{t}_n)) - w_{k_n}(x'_{k_n}(\bar{t}_n))| < \frac{\varrho}{6}$$

for $n \geq n_*$. Hence,

$$|w(x'_{k_n}(\bar{t}_n)) - w(x'_{k_n}(\hat{t}_n))| \leq |w(x'_{k_n}(\bar{t}_n)) - w_{k_n}(x'_{k_n}(\bar{t}_n))|$$
$$+ |w_{k_n}(x'_{k_n}(\bar{t}_n)) - w_{k_n}(x'_{k_n}(\hat{t}_n))| + |w_{k_n}(x'_{k_n}(\hat{t}_n)) - w(x'_{k_n}(\hat{t}_n))| < \frac{\varrho}{2},$$

and consequently

$$\left| \int\limits_{r(x'_{k_n}(\hat{t}_n))}^{r(x'_{k_n}(\bar{t}_n))} \frac{1}{k(r^{-1}(s))}\, ds \right| = |w(x'_{k_n}(\bar{t}_n)) - w(x'_{k_n}(\hat{t}_n))| < \frac{\varrho}{2}$$

for $n \geq n_*$, contrary to (23). Therefore $\{x'(t)\}$ is equicontinuous on J.

Applying the Arzelà – Ascoli theorem we can assume without loss of generality that $\{x_n\}$ is convergent in $C^1(J)$, $\lim_{n\to\infty} x_n = x$. Then $x \in C^1(J)$ satisfies the periodic conditions (2) and inequalities (13). Since $\lim_{n\to\infty} w_n(t) = w(t)$ uniformly on $[P, -P]$ and $\lim_{n\to\infty} x_n^{(j)}(t) = x^{(j)}(t)$ uniformly on J for $j = 0, 1$, we have $\lim_{n\to\infty} w_n(x_n'(t)) = w(x'(t))$, $\lim_{n\to\infty} q(x_n(t)) = q(x(t))$ uniformly on J and, by the Lebesgue dominated theorem,

$$\lim_{n\to\infty} \int_0^t H(p(s) + q(x_n(s)))f(s, x_n(s), x_n'(s))\, ds$$

$$= \int_0^t H(p(s) + q(x(s)))f(s, x(s), x'(s))\, ds$$

for $t \in J$. Taking the limit as $n \to \infty$ in the equalities

$$w_n(x_n'(t)) = w_n(x_n'(0)) + \int_0^t H(p(s) + q(x_n(s)))f(s, x_n(s), x_n'(s))\, ds$$

for $t \in J$ and $n \in \mathbb{N}$, we have

$$w(x'(t)) = w(x'(0)) + \int_0^t H(p(s) + q(x(s)))f(s, x(s), x'(s))\, ds \quad \text{for } t \in J. \tag{27}$$

Set $\mathcal{A} = \{t : t \in J, x'(t) = 0\}$. On the contrary, suppose that $\mathcal{A}$ is an infinite set. Then there exists a sequence $\{t_n\} \subset \mathcal{A}$, $t_i \neq t_j$ for $i \neq j$ and we can assume that $\{t_n\}$ is convergent, $\lim_{n\to\infty} t_n = t_0$. Clearly, $t_0 \in \mathcal{A}$. Now from (27) it follows

$$\int_{t_0}^{t_n} H(p(s) + q(x(s)))f(s, x(s), x'(s))\, ds = 0, \quad n \in \mathbb{N}.$$

By (H_7), $f(t, x(t), x'(t)) \geq \chi(t)$ for a.e. $t \in J$ with $\chi \in L^1(J)$ positive on J and consequently, by the mean value theorem, there exists a sequence $\{\xi_n\}$, where ξ_n lies in the open interval having the end points t_0 and t_n, such that $p(\xi_n) + q(x(\xi_n)) = 0$. Then from the continuity of p, q and from $\lim_{n\to\infty} \xi_n = t_0$ we deduce that $p(t_0) + q(x(t_0)) = 0$. Since $x(\xi_n) = q^{-1}(-p(\xi_n))$ and $x(t_0) = q^{-1}(-p(t_0))$, we have

$$\frac{x(\xi_n) - x(t_0)}{\xi_n - t_0} = \frac{q^{-1}(-p(\xi_n)) - q^{-1}(-p(t_0))}{\xi_n - t_0}, \quad n \in \mathbb{N}.$$

Letting $n \to \infty$ yields $x'(t_0) = (q^{-1}(-p(t)))'_{t=t_0}$. But $t_0 \in \mathcal{A}$ gives $x'(t_0) = 0$, contrary to $(q^{-1}(-p(t)))'_{t=t_0} \neq 0$ by (H_6). Hence $\mathcal{A}$ is a finite set. Now from (27) we deduce that

$w(x') \in AC(J)$ and since $w(x'(t)) = \int_0^{r(x'(t))} \frac{1}{k(r^{-1}(s))}\, ds$, we have $r(x'(t)) = z^{-1}(w(x'(t)))$ for $t \in J$, where z^{-1} is the inverse to z given in (H_4). By (H_4), z^{-1} is locally Lipschitzian on $\mathbb{R}$, and so $r(x') \in AC(J)$. We know that $w(x'(t)) = 0$ if and only if $t \in \mathcal{A}$, where $\mathcal{A}$ is a finite set and $k(u) > 0$ for $u \in \mathbb{R} \setminus \{0\}$. Hence

$$(w(x'(t)))' = \frac{(r(x'(t)))'}{k(x'(t))} \quad \text{for a.e. } t \in J$$

and then (27) implies

$$(r(x'(t)))' = H(p(t) + q(x(t)))k(x'(t))f(t, x(t), x'(t)) \quad \text{for a.e. } t \in J.$$

Hence x is a solution of PBVP (1), (2).

Theorem 2. *Let assumptions* $(H_1) - (H_6)$, (H_8), *and* (12) *be satisfied. Then there exists a solution* x *of PBVP* (1), (2) *satisfying* (13) *with a positive constant* P *for which* (14) *holds.*

Proof. As in the proof of Theorem 1, let $\{x_n\}$ be a sequence of solutions of PBVPs $(11)_n$, (2) satisfying inequalities (19). Since now f is continuous by (H_8), we have x_n', $w_n(x_n') \in C^1(J)$ and $(11)_n$ with $x = x_n$ holds for $t \in J$. Arguing as in the proof of Theorem 1 with

$$\nu(t) = \nu = \max\{f(t, x, y) : (t, x, y) \in J \times [\alpha, \beta] \times [-P, P]\}$$

we show that without loss of generality $\{x_n\}$ is convergent in $C^1(J)$, $\lim_{n\to\infty} x_n = x$ and (13) and (27) hold. Therefore $w(x') \in C^1(J)$. By (H_8),

$$\min\{f(t, x(t), x'(t)) : t \in J\} = \varepsilon > 0$$

and setting $\chi(t) = \varepsilon$ in the proof of Theorem 1, we verify that $\mathcal{A} = \{t : t \in J, x'(t) = 0\}$ is a finite set. Now from the equality $r(x'(t)) = z^{-1}(w(x'(t)))$, $t \in J$, we deduce that $r(x') \in C^1(J \setminus \mathcal{A})$, $(w(x'(t)))' = (r(x'(t)))'/k(x'(t))$ for $t \in J \setminus \mathcal{A}$, and so (27) yields

$$(r(x'(t)))' = H(p(t) + q(x(t)))k(x'(t))f(t, x(t), x'(t)) \quad \text{for } t \in J \setminus \mathcal{A}.$$

Hence x is a solution of PBVP (1), (2).

Example 1. Let $n \in \mathbb{N}$, $\gamma \in (0, \infty)$, and $a \in (1, \infty)$. Consider the differential equation

$$x'' = (\sin t + x)^{2n+1} \left(\frac{1}{|x'|^\gamma} + a \right) f_1(t, x), \tag{28}$$

where $f_1 \in Car(J \times [-1, 1])$ and $\chi(t) \le f_1(t, x)$ for $(t, x) \in J \times [-1, 1]$ with a positive function $\chi \in L(J)$. Then (28) satisfies assumptions $(H_1) - (H_7)$ and (12) with $r(u) = u$, $k(u) = 1/|u|^\gamma + a$, $H(u) = u^{2n+1}$, $p(t) = \sin t$, $q(x) = x$, $\omega(u) = 1$ and $\alpha = -1$, $\beta = 1$. Hence Theorem 1 can be applied to PBVP (28), (2).

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Received 24.09.2001