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STABILITY OF PERIODIC CLUSTERS IN GLOBALLY COUPLED MAPS

СТІЙКІСТЬ ПЕРІОДИЧНИХ КЛАСТЕРІВ У СИСТЕМІ ГЛОБАЛЬНО ЗВ'ЯЗАНИХ ВІДОБРАЖЕНЬ

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The phenomenon of partial synchronization, — or clustering, — in a system of globally coupled C^1-smooth maps is analyzed. We prove stability of equally populated K-clustered states with period-n temporal dynamics, referred to as P_nC_K-states. For this, we first obtain formulas giving relation between longitudinal and transverse nultipliers of the in-cluster periodic orbits and then, using these formulas, find exact parameter intervals for the transverse stability. We conclude that typically, for the symmetric P_nC_K-states, in-cluster stability implies transverse stability. Moreover, transverse stability can take place even if the in-cluster dynamics is unstable.

Проводиться аналіз явища часткової синхронізації, або кластеризації, в системі глобально зв'язаних відображень гладкості C^1. Розглядаються K-кластерні стани з n-періодичною динамікою, які називаються P_nC_K-станами, і доводиться їх стійкість. Для цього спочатку отримано формули, що пов'язують поздовжні та трансверсальні мультиплікатори кластеризованих періодичних орбіт, а потім з допомогою цих формул знайдено точні межі інтервалів для трансверсальної стійкості. Зроблено висновок, що для симетричних P_nC_K-станів із стійкості всередині кластера випливає стійкість трансверсальна. Більше того, навіть у випадку, коли динаміка всередині кластера нестійка, трансверсальна стійкість може мати місце.

1. Introduction. The highly complex nature of different dynamical systems in various areas of science, such as physics, biology, and other natural sciences, has attracted recently a growing interest. In this context, new interesting phenomena, such as partial synchronization, have been discovered. This new properties are investigated from both practical and theoretical viewpoints [1–9]. The effects of synchronization and partial synchronization could be observed in a great variety of applied problems, such as pattern formation, Josephson junction arrays, multimode lasers, charge-density waves, insulin secretion, oscillatory neuronal systems, and so on [10–13].

The paper presented has an aim to investigate the stability of partially synchronized states in a system of globally coupled maps. Together with the "ordinary" stability, the notion of transverse stability is also of great importance for partially synchronized systems [1, 2]. An attractor of such a clustered system could be represented as a stable set that belongs to a manifold having lower dimension than the whole initial space. Transverse stability means that the attractor is stable not only inside the manifold, but also from outside, i.e., along all of the basic vectors of the whole space. And such an attractor loses its stability when at least one of the clusters has split up, i.e., when the attractor spreads over the manifold of higher dimension.

One of the main results of the work presented is the analytical proof of the fact that the

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transverse stability results from the longitudinal stability for absolutely symmetric clustered periodic states of different periods.

2. The model. In this paper we study the dynamics of a system of N globally coupled one-dimensional maps, which initially was introduced by K. Kaneko [14, 15],

$$F_\varepsilon : x_i^t \mapsto (1-\varepsilon) f\left(x_i^t\right) + \frac{\varepsilon}{N} \sum_{j=1}^{N} f\left(x_j^t\right), \quad i = \overline{1, N}, \tag{1}$$

where $\mathbf{x}^t = \{x_i^t \in \mathbb{R}\}_{i=1}^N$ is an N-dimensional state vector, $t = 0, 1, \ldots$ represents a discrete time index, and $f : \mathbb{R} \mapsto \mathbb{R}$ is a C^1-smooth one-dimensional map. In the numerical experiments the one-dimensional map f is taken in the form of the logistic map $f(x) = ax(1-x)$. Thus, the behaviour of the system (1) is ruled by its two parameters, the nonlinearity parameter $a \in [0, 4]$ and the coupling parameter $\varepsilon \in [0, 1]$. When $\varepsilon = 0$ the system (1) represents N noncoupled one-dimensional maps. Every coordinate of the state vector $\mathbf{x}^t = (x_1^t, x_2^t, \ldots, x_N^t)$ moves independently according to the map f. When $\varepsilon = 1$, after the first iteration all coordinates x_i^t, $i = \overline{1, N}$, become identical and continue to move synchronously according to the map f.

For the other values of $\varepsilon \in (0, 1)$, in particular, the system (1) can fall into the state of so-called *partial synchronization* or *clustering* [1–5], when the coordinates of the state vector $\mathbf{x}^t$ unite into several groups, called *clusters*, producing the same behaviour within each group. Then, each cluster can be considered as one dynamical element [1, 3, 4],

$$\begin{gathered} x_1^t = x_2^t = \ldots = x_{N_1}^t \overset{\text{df}}{=} y_1^t, \\ x_{N_1+1}^t = x_{N_1+2}^t = \ldots = x_{N_1+N_2}^t \overset{\text{df}}{=} y_2^t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{N_1+\ldots+N_{K-1}+1}^t = x_{N_1+\ldots+N_{K-1}+2}^t = \ldots = x_N^t \overset{\text{df}}{=} y_K^t, \end{gathered} \tag{2}$$

where N_i is the number of elements in the i-th cluster, $N_1 + N_2 + \ldots + N_K = N$.

Relations (2) define a K-dimensional manifold $M^{(K)} \subset \mathbb{R}^{(N)}$ of the form

$$\begin{aligned} M^{(K)} = \big\{ (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N :\ & x_1 = x_2 = \ldots = x_{N_1}, \\ & x_{N_1+1} = x_{N_1+2} = \ldots = x_{N_1+N_2}, \ldots, x_{N_1+\ldots+N_{k-1}+1} = \ldots = x_N; \\ & N_1 + \ldots + N_K = N \big\}. \end{aligned} \tag{3}$$

It is easy to see, that, by virtue of the complete symmetry of the system (1), for any set $\{N_i \in \in \mathbb{N} : \sum_{i=1}^K N_i = N\}$ the manifold $M^{(K)}$ is invariant with respect to the map F_ε, i.e., $\forall\, \mathbf{x} \in \in M^{(K)}$ its image $F_\varepsilon(\mathbf{x}) \in M^{(K)}$. So it is possible to consider the restriction $G_\varepsilon \overset{\text{df}}{=} F_\varepsilon\big|_{M^{(K)}}$ as a dynamical system on $M^{(K)}$:

$$G_\varepsilon : y_i^t \mapsto (1-\varepsilon) f(y_i^t) + \varepsilon \sum_{j=1}^{K} p_j f(y_j^t), \quad i = \overline{1, K}. \tag{4}$$

Here $\mathbf{y}^t = \{y_i^t\}_{i=1}^K$ is a K-dimensional state vector, and $p_i \overset{\text{df}}{=} N_i/N$, $\sum_{i=1}^K p_i = 1$, is referred to as *population* of the i-th cluster given by Eq. (2).

Note that system (4) has also the form of a globally coupled map system but, in contrast with system (1), different coupling weights p_i as a measure for the contribution of the i-th coordinate y_i to the global coupling term. Therefore system (4) is asymmetric; it becomes symmetric only when $p_1 = p_2 = \ldots = p_K = 1/K$ (then the dimension N of the initial system (1) must be divisible by the dimension K of the system (4), i.e., $N = k_1 K$ for some $k_1 \in \mathbb{N}$).

Suppose that the K-dimensional map G_ε has an n-periodic cycle $P_n^{(K)} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ such that

$$\begin{aligned} \mathbf{y}_{i+1} &= G_\varepsilon(\mathbf{y}_i), \quad i = \overline{1, n-1}, \\ \mathbf{y}_1 &= G_\varepsilon(\mathbf{y}_n), \end{aligned} \tag{5}$$

where $\mathbf{y}_j = (y_{1j}, y_{2j}, \ldots, y_{Kj}) \in \mathbb{R}^K, j = \overline{1, n}$. Then the original N-dimensional system (1) has the corresponding n -periodic cycle $P_n^{(N)} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of the form

$$\begin{aligned} \mathbf{x}_{i+1} &= F_\varepsilon(\mathbf{x}_i), \quad i = \overline{1, n-1}, \\ \mathbf{x}_1 &= F_\varepsilon(\mathbf{x}_n), \end{aligned} \tag{6}$$

which belongs to the manifold $M^{(K)} \subset \mathbb{R}$,

$$\mathbf{x}_j = \Big(\underbrace{y_{1j}, \ldots, y_{1j}}_{N_1}, \underbrace{y_{2j}, \ldots, y_{2j}}_{N_2}, \ldots, \underbrace{y_{Kj}, \ldots, y_{Kj}}_{N_K}\Big) \in \mathbb{R}^N, \quad j = \overline{1, n}. \tag{7}$$

For system (1) a cycle $P_n^{(N)}$ of the form (7) is referred to as *n-periodic K-clustered state*, or simply *$P_n C_K$-state*. Our goal is to investigate the stability of these periodic states in the whole N -dimensional space $\mathbb{R}^N$.

Definition 1. *Consider the $P_n C_K$-state of system (1), i.e., a period-n cycle $P_n^{(N)} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of the map F_ε of the form (7). Let $\nu_i^{(N)}$, $i = \overline{1, N}$, be eigenvalues of the Jacobian matrix $DF_\varepsilon^n(\mathbf{x}_1) = DF_\varepsilon(\mathbf{x}_n) DF_\varepsilon(\mathbf{x}_{n-1}) \ldots DF_\varepsilon(\mathbf{x}_1)$, where F_ε^n is the n-th iteration of the map F_ε. Then, the values $\mu_i^{(N)} = \sqrt[n]{\nu_i^{(N)}}$, $i = \overline{1, N}$, are called multipliers of $P_n C_K$-state, i.e., of the cycle $P_n^{(N)}$.*

Definition 2. *The $P_n C_K$-state of the system (1), i.e., the cycle $P_n^{(N)} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of the map F_ε is called Lyapunov stable if all its multipliers $\{\mu_i^{(N)}\}_{i=1}^N$ lie inside the unit circle, i.e., $|\mu_i^{(N)}| < 1$, $i = \overline{1, N}$.*

Any $P_n C_K$-state has N multipliers. K of them, $\{\mu_{\|,i}^{(N)}\}_{i=1}^K$, correspond to the eigenvectors of the matrix DF_ε^n lying in the K-clustered manifold $M^{(K)}$ of the form (3). Hence, they control stability of the $P_n C_K$-state inside $M^{(K)}$. They coincide with the multipliers $\{\mu_i^{(K)}\}_{i=1}^K$

of the corresponding cycle $P_n^{(K)}$ of the form (5) for the K-dimensional system (4). We will call them *longitudinal multipliers* of the P_nC_K-state. The other $N-K$ multipliers correspond to the eigenvectors of DF_ε^n transverse to the manifold $M^{(K)}$ (i.e., lying in the supplement $\mathbb{R}^N \backslash M^{(K)}$ of $M^{(K)}$). They will be referred to as *transverse multipliers of* P_nC_K.

As it was shown [1], if all the numbers $N_i > 1,\ i = \overline{1,K}$ (see (3)), then there are K distinct transverse multipliers $\{\mu_{\perp,i}^{(N)}\}_{i=1}^{K}$ each having multiplicity $N_i - 1$. Otherwise, the number K_1 of distinct transverse multipliers $\{\mu_{\perp,i}^{(N)}\}_{i=1}^{K_1}$ equals the number of clusters having more than 1 element, i.e., $K_1 = \mathrm{card}\{i = \overline{1,K} : N_i > 1\}$. The transverse multipliers control *out-of-cluster* stability of the P_nC_K-state in $\mathbb{R}^N$ space, i.e., stability with respect to small perturbations beyond the clustered manifold $M^{(K)}$. To ensure the transverse stability of P_nC_K we demand the transverse multipliers to lie in the unit circle,

$$\left|\mu_{\perp,i}^{(N)}\right| < 1, \qquad i = \overline{1,K_1}, \quad 0 \le K_1 \le K.$$

Lemma 1. *The transverse multipliers* $\{\mu_{\perp,i}^{(N)}\}_{i=1}^{K_1}$ *for the* P_nC_K*-state of the form (6), (7) are equal to*

$$\mu_{\perp,i}^{(N)} = (1-\varepsilon)\left(\prod_{j=1}^{n} f'(y_{ij})\right)^{\frac{1}{n}}, \quad i = \overline{1,K_1}. \tag{8}$$

Proof. The proof follows from formula (7) in the paper [1].

Remark 1. As it is easy to see, both longitudinal and transverse multipliers $\mu_{\|,i}^{(N)}$, and $\mu_{\perp,i}^{(N)}$, of the P_nC_K-state do not depend on the space dimension N. Therefore we can omit the upper index $^{(N)}$ writing simply $\mu_{\|,i}$ and $\mu_{\perp,i}$.

3. Relations between the transverse and longitudinal multipliers. Let the coupling weights $\{p_i\}_{i=1}^K$ in the map $G_\varepsilon : \mathbb{R}^K \mapsto \mathbb{R}^K$ of the form (4) be equal, $p_1 = p_2 = \ldots = p_K = 1/K$, and the map $f \in C^1$. Consider a period-n cycle $P_n^{(K)}$ of the map G_ε. For any $N = k_1 K$ ($k_1 > 1$ is an integer), this cycle generates a P_nC_K-state of system (1) in the N-dimensional phase space (in accordance with formulas (6) and (7), where $N_i = k_1,\ i = \overline{1,K}$). The P_nC_K-state has K longitudinal multipliers $\{\mu_{\|,i}\}_{i=1}^K$ (which coincide with the multipliers $\{\mu_i^{(K)}\}_{i=1}^K$ of the cycle $P_n^{(K)}$) and K transverse multipliers $\{\mu_{\perp,i}\}_{i=1}^K$ (which are given by formula (7) of the Lemma 1).

Theorem 1. *For the transverse and longitudinal multipliers of the* P_nC_K*-state as above the following relation holds:*

$$\prod_{i=1}^{K} \mu_{\perp,i} = (1-\varepsilon)\prod_{i=1}^{K} \mu_{\|,i}. \tag{9}$$

To prove the theorem, the following lemma is needed.

Lemma 2. *The determinant of the Jacobian matrix* DG_ε *can be represented in the form*

$$\det DG_\varepsilon(\mathbf{y}) = (1-\varepsilon)^{K-1}\prod_{i=1}^{K} f'(y_i) \tag{10}$$

for any $\mathbf{y} = (y_1, y_2, \ldots, y_K) \in \mathbb{R}^K$.

Proof. The Jacobian matrix of the map G_ε is

$$DG_\varepsilon(\mathbf{y}) = \begin{bmatrix} \left(1-\varepsilon+\dfrac{\varepsilon}{K}\right) f'(y_1) & \dfrac{\varepsilon}{K} f'(y_2) & \ldots & \dfrac{\varepsilon}{K} f'(y_K) \\ \dfrac{\varepsilon}{K} f'(y_1) & \left(1-\varepsilon+\dfrac{\varepsilon}{K}\right) f'(y_2) & \ldots & \dfrac{\varepsilon}{K} f'(y_K) \\ \vdots & \vdots & \ldots & \vdots \\ \dfrac{\varepsilon}{K} f'(y_1) & \dfrac{\varepsilon}{K} f'(y_2) & \ldots & \left(1-\varepsilon+\dfrac{\varepsilon}{K}\right) f'(y_K) \end{bmatrix}.$$

For any $j = 1, \ldots, K$ consider the $(j\times j)$-matrix

$$A_j = \begin{bmatrix} (1-\varepsilon) f'(y_{K-j+1}) & 0 & \ldots & -(1-\varepsilon) f'(y_K) \\ 0 & (1-\varepsilon) f'(y_{K-j+2}) & \ldots & -(1-\varepsilon) f'(y_K) \\ \vdots & \vdots & \ldots & \vdots \\ \dfrac{\varepsilon}{K} f'(y_{k_j+1}) & \dfrac{\varepsilon}{K} f'(y_{k_j+2}) & \ldots & \left(1-\varepsilon+\dfrac{\varepsilon}{K}\right) f'(y_K) \end{bmatrix}. \tag{11}$$

It is easy to show that

$$\det DG_\varepsilon(\mathbf{y}) = \det A_K. \tag{12}$$

The determinant of A_j can be represented recursively as

$$\det A_j = (1-\varepsilon) f'(y_{K-j+1}) \det A_{j-1} + (-1)^{j+1}\frac{\varepsilon}{K} f'(y_{K-j+1}) \det B_{j-1}, \tag{13}$$

where B_{j-1} is a $(j-1)\times(j-1)$-matrix of the form

$$B_{j-1} = \begin{bmatrix} 0 & 0 & \ldots & 0 & -(1-\varepsilon) f'(y_K) \\ (1-\varepsilon) f'(y_{K-j+2}) & 0 & \ldots & 0 & -(1-\varepsilon) f'(y_K) \\ 0 & (1-\varepsilon) f'(y_{K-j+3}) & \ldots & 0 & -(1-\varepsilon) f'(y_K) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & (1-\varepsilon) f'(y_{K-1}) & -(1-\varepsilon) f'(y_K) \end{bmatrix}.$$

Its determinant equals

$$\det B_{j-1} = -(-1)^j(1-\varepsilon)f'(y_K)(1-\varepsilon)^{j-2}\prod_{i=K-j+2}^{K-1} f'(y_i) = (-1)^{j+1}(1-\varepsilon)^{j-1}\prod_{i=K-j+2}^{K} f'(y_i).$$

Then, as it follows from (13),

$$\det A_j = (1-\varepsilon)f'(y_{K-j+1})\det A_{K-1} +$$

$$+ (-1)^{j+1}\frac{\varepsilon}{K}f'(y_{K-j+1})(-1)^{j+1}(1-\varepsilon)^{j-1}\prod_{i=K-j+2}^{K} f'(y_i) =$$

$$= (1-\varepsilon)f'(y_1)\det A_{K-1} + \frac{\varepsilon}{K}(1-\varepsilon)^{j-1}\prod_{i=K-j+1}^{K} f'(y_i). \tag{14}$$

Let us prove that

$$\det A_j = (1-\varepsilon)^{j-1}\left(1-\varepsilon+\frac{j\varepsilon}{K}\right)\prod_{i=K-j+1}^{K} f'(y_i)$$

using the method of mathematical induction. For $j = 2$ it is easy to see that

$$\det A_2 = \det\begin{bmatrix} (1-\varepsilon)f'(y_{K-1}) & -(1-\varepsilon)f'(y_K) \\ \frac{\varepsilon}{K}f'(y_{K-1}) & \left(1-\varepsilon+\frac{\varepsilon}{K}\right)f'(y_K) \end{bmatrix} = (1-\varepsilon)\left(1-\varepsilon+\frac{2\varepsilon}{K}\right)f'(y_{K-1})f'(y_K).$$

Suppose that $\det A_{j-1} = (1-\varepsilon)^{j-2}\left(1-\varepsilon+\frac{(j-1)\varepsilon}{K}\right)\prod_{i=K-j+2}^{K} f'(y_i)$. Substituting the latter expression into Eq. (14) we obtain

$$\det A_j = (1-\varepsilon)f'(y_{K-j+1})(1-\varepsilon)^{j-2}\left(1-\varepsilon+\frac{(j-1)\varepsilon}{K}\right)\prod_{i=K-j+2}^{K} f'(y_i)+$$

$$+\frac{\varepsilon}{K}(1-\varepsilon)^{j-1}\prod_{i=K-j+1}^{K} f'(y_i) =$$

$$= (1-\varepsilon)^{j-1}\prod_{i=K-j+1}^{K} f'(y_i)\left(1-\varepsilon+\frac{(j-1)\varepsilon}{K}+\frac{\varepsilon}{K}\right) =$$

$$= (1-\varepsilon)^{j-1}\left(1-\varepsilon+\frac{(j-1)\varepsilon}{K}\right)\prod_{i=K-j+1}^{K} f'(y_i).$$

From Eq. (12) it follows that

$$\det DG_\varepsilon(\mathbf{y}) = \det A_K = (1-\varepsilon)^{K-1}\left(1-\varepsilon+\frac{K\varepsilon}{K}\right)\prod_{i=1}^{K} f'(y_i) = (1-\varepsilon)^{K-1}\prod_{i=1}^{K} f'(y_i).$$

The lemma is proved.

Proof of Theorem 1. The longitudinal multipliers $\{\mu_{\|,i}\}_{i=1}^{K}$ of the P_nC_K-state equal to the multipliers of the cycle $P_n^{(K)}$ of the K-dimensional map G_ε of the form (4). They can be obtained as the n-th roots of the eigenvalues of the Jacobian matrix for the n-th iteration of the map G_ε,

$$DG_\varepsilon^n(\mathbf{y}_1) = DG_\varepsilon(\mathbf{y}_n)DG_\varepsilon(\mathbf{y}_2)\ldots DG_\varepsilon(\mathbf{y}_1).$$

By Vieta's theorem, the product of all eigenvalues of a $K\times K$ matrix equals its determinant, i.e.,

$$\prod_{i=1}^{K}\mu_{\|,i} = \left(\det DG_\varepsilon^n(\mathbf{y}_1)\right)^{\frac{1}{n}} = \left(\det DG_\varepsilon(\mathbf{y}_n)\ldots\det DG_\varepsilon(\mathbf{y}_1)\right)^{\frac{1}{n}}.$$

Then using Lemma 2 we obtain

$$\prod_{i=1}^{K}\mu_{\|,i} = (1-\varepsilon)^{K-1}\left(\prod_{j=1}^{n}\prod_{i=1}^{K} f'(y_{ij})\right)^{\frac{1}{n}}. \tag{15}$$

On the other hand, due to Lemma 1, the transverse multipliers $\mu_{\perp,i},\ i=\overline{1,K}$, of the P_nC_K-state can be represented in the form (8). Therefore, it follows that the product of the multipliers is equal to

$$\prod_{i=1}^{K}\mu_{\perp,i} = (1-\varepsilon)^{K}\left(\prod_{i=1}^{K}\prod_{j=1}^{n} f'(y_{ij})\right)^{\frac{1}{n}}. \tag{16}$$

Finally, Eqs. (15) and (16) imply Eq. (9).

4. Cyclicity condition. For the map G_ε of the form (4) with the equal populations $p_i = 1/K,\ i=\overline{1,K}$, consider the cycle $P_{mK}^{(K)}$ of the period mK for some $m\ge 1$. Put $\varepsilon=0$. Then the coupling term vanishes and the K-dimensional map G_0 is a direct product of K one-dimensional maps $f: x\mapsto f(x),\ x\in\mathbb{R}$. Therefore, the dynamics of G_0 is defined by the dynamics of f. In particular, if the one-dimensional map f has a period-mK cycle

$$P_{mK} = \{y_1, y_2, \ldots, y_{mK}\},$$

then for the map G_0 there exists a corresponding period-mK cycle $P_{mK}^{(K)}$ of the form

$$P_{mK}^{(K)} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{mK}\} \tag{17}$$

such that

$$\mathbf{y}_j = \big(y_j, y_{m+j}, \ldots, y_{m(K-1)+j}\big), \quad j = \overline{1,m},$$
$$\mathbf{y}_{im+j} = \pi_K^i(\mathbf{y}_j), \quad i = \overline{1,K-1}, \quad j = \overline{1,m}, \tag{18}$$

where π_K is a cyclical permutation, e.i.,

$$\pi_K(s_1, s_2, \ldots, s_K) = (s_2, s_3, \ldots, s_K, s_1)$$

for any set of K real numbers s_i, $i = \overline{1,K}$, and π_K^i is the i-th iteration of π_K. We will call relations (18) the *cyclicity condition* for the cycle $P_{mK}^{(K)}$. Obviously, for $m = 1$ the cyclicity condition for the cycle

$$P_K^{(K)} = \big\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_K\big\} \tag{17$'$}$$

becomes

$$\mathbf{y}_1 = \big(y_1, y_2, \ldots, y_K\big),$$
$$\mathbf{y}_{i+1} = \pi_K^i(\mathbf{y}_1), \quad l = \overline{1,K-1}. \tag{18$'$}$$

Due to the symmetry and smooth dependence of the map G_ε, we expect that the cycles $P_{mK}^{(K)}(\varepsilon) = \big\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{mK}\big\}$ satisfying the cyclicity condition can also exist for $\varepsilon > 0$.

Indeed, due to the implicit function theorem, if $\det DG_0^{(n)}(\mathbf{y}_1) \neq 0$ there exists $\varepsilon_0 > > 0$ such that $\forall \varepsilon \in [0, \varepsilon_0]$ for the map G_ε, there exists a cycle $P_{mK}^{(K)} = P_{mK}^{(K)}(\varepsilon)$ that is a continuation of the cycle $P_{mK}^{(K)}(0)$, i.e.,

$$\lim_{\varepsilon \to 0} P_{mK}^{(K)}(\varepsilon) = P_{mK}^{(K)}(0).$$

Sufficient conditions for the cycle $P_{mK}^{(K)} = P_{mK}^{(K)}(\varepsilon)$ to satisfy the cyclicity condition (18) (or (18$'$) for the case $m = 1$) are given by the following Lemmas 3 and 4.

Lemma 3. *Suppose that the one-dimensional map*

$$g : x \mapsto (1-\varepsilon) f(x) + \varepsilon h, \quad h \equiv \text{const}, \tag{19}$$

has a period-K cycle $P_K = \{y_1, y_2, \ldots, y_K\}$ such that

$$\frac{1}{K} \sum_{j=1}^{K} y_j = h.$$

Then the K-dimensional map G_ε has a period-K cycle $P_K^{(K)}$ satisfying the cyclicity condition (18$'$).

Proof. Consider any point $\mathbf{y}_1 = (y_1, y_2, \ldots, y_K) \in \mathbb{R}^K$. The i-th coordinate of its G_ε image is equal to

$$(G_\varepsilon(\mathbf{y}_1))_i = (1-\varepsilon) f\left((\mathbf{y}_1)_i\right) + \frac{\varepsilon}{K} \sum_{j=1}^{K} f\left((\mathbf{y}_1)_j\right), \quad i = \overline{1, K}.$$

Since P_K is a cycle for the map g of the form (19), we have

$$\begin{aligned} y_{i+1} &= (1-\varepsilon) f(y_i) + \varepsilon h, \quad i = \overline{1, K-1}, \\ y_1 &= (1-\varepsilon) f(y_K) + \varepsilon h. \end{aligned}$$

Adding all these K equations and then dividing the result by K we obtain

$$\begin{aligned} \frac{1}{K} \sum_{j=1}^{K} y_j &= (1-\varepsilon) \frac{1}{K} \sum_{j=1}^{K} f(y_j) + \varepsilon h \Leftrightarrow \\ \Leftrightarrow h(1-\varepsilon) &= (1-\varepsilon) \frac{1}{K} \sum_{j=1}^{K} f(y_j) \Leftrightarrow \\ \Leftrightarrow h &= \frac{1}{K} \sum_{j=1}^{K} f(y_j). \end{aligned}$$

This implies

$$(G_\varepsilon(\mathbf{y}_1))_i = (1-\varepsilon) f(y_i) + \frac{\varepsilon}{K} \sum_{j=1}^{K} y_j = (1-\varepsilon) f(y_i) + \varepsilon h, \quad i = \overline{1, K},$$

and, therefore,

$$\begin{aligned} (G_\varepsilon(\mathbf{y}_1))_i &= y_{i+1}, \quad i = \overline{1, K-1}, \\ (G_\varepsilon(\mathbf{y}_1))_K &= y_1, \end{aligned}$$

i.e., $G_\varepsilon(\mathbf{y}_1) = \pi_K(\mathbf{y}_1)$. Obviously, applying the map G_ε K times we obtain

$$G_\varepsilon^K(\mathbf{y}_1) = \mathbf{y}_1,$$

which means that $P_K^{(K)} = \left\{\mathbf{y}_1, G_\varepsilon(\mathbf{y}_1), G_\varepsilon^2(\mathbf{y}_1), \ldots, G_\varepsilon^{K-1}(\mathbf{y}_1)\right\}$ is a K-cycle of G_ε, satisfying the cyclicity condition.

Lemma 4. *Let* $m > 1$ *and let the* m*-dimensional map*

$$g : x_i \mapsto (1-\varepsilon) f(x_i) + \varepsilon h_i, \quad h_i = \text{const}, \quad i = \overline{1, m}, \tag{20}$$

have a period-mK cycle $P_{mK} = \{y_1, y_2, \ldots, y_{mK}\}$ such that

$$\frac{1}{K}\sum_{j=1}^{K} y_{(j-1)m+i} = h_i, \quad i = \overline{1, m}.$$

Then the map G_ε has a period-mK cycle $P_{mK}^{(K)}$ satisfying the cyclicity condition (17), (18).

The proof of Lemma 4 is analogous to the one of the Lemma 3.

5. Stability of P_nC_K-states. Formula (9) obtained in Section 3 enables to prove the transverse stability of P_nC_K-states, in the case $n = mK$, in terms of stability of the cycle $P_n^{(K)}$ inside the cluster manifold $M^{(K)}$.

Consider the $P_{mK}C_K$-state in the N-dimensional phase space generated by a period- mK cycle $P_{mK}^{(K)}$ of the K-dimensional map G_ε ($m \in \mathbb{N}$ is any integer). Suppose the $P_{mK}C_K$-state to be symmetric, i.e.,

1) $N_i = k_1, \ k_1 > 1,$ for all $i = \overline{1, K}$ (see Eq. (7)),

2) the cycle $P_{mK}^{(K)}$ satisfies the cyclicity condition (17), (18) (or $(17')$, $(18')$ resp.).

Theorem 2. *Let the conditions 1 and 2 be satisfied. Then all transverse multipliers $\{\mu_{\perp,i}\}_{i=1}^K$ of the $P_{mK}C_K$-state are equal and can be represented as*

$$\mu_{\perp,i} = (1-\varepsilon)\sqrt[K]{\prod_{j=1}^{K} f'(y_j)} \overset{\mathrm{df}}{=} \mu_\perp, \quad i = \overline{1, K}.$$

Proof. Directly from the Eq. (8) we derive that the transverse multipliers for the $P_{mK}C_K$-state of the form (6), (7) are

$$\mu_{\perp,i} = (1-\varepsilon)\left(\prod_{j=1}^{K} f'(y_j)\right)^{\frac{1}{mK}}, \quad i = \overline{1, K}.$$

Theorem 2 results in the following corollaries.

Theorem 3. *If the cycle $P_n^{(K)}$, $n = mK$, satisfying conditions 1 and 2 above, is Lyapunov stable inside the manifold $M^{(K)}$, then the clustered $P_{mK}C_K$-state is Lyapunov stable for any $\varepsilon \in [0, 2]$.*

Let us prove the theorem only for the case $m = 1$. For $m > 1$ the proof is analogous.

Proof. Since the longitudinal multipliers $\{\mu_{\|,i}\}_{i=1}^K$ of the P_KC_K-state are the multipliers $\{\mu_i^{(K)}\}_{i=1}^K$ of the K-dimensional cycle $P_K^{(K)}$, we have

$$\left|\mu_{\|,i}\right| = \left|\mu_i^{(K)}\right| < 1, \quad i = \overline{1, K}.$$

Then, by Theorem 2,

$$\mu_{\perp,i} = \mu_{\perp}, \quad i = \overline{1,K},$$

which implies

$$\prod_{i=1}^{K} \mu_{\perp,i} = \mu_{\perp}^{K}.$$

Therefore, using Theorem 1,

$$|\mu_{\perp,i}|^K = \left|(1-\varepsilon)\prod_{i=1}^{K}\mu_{\|,i}\right| = |1-\varepsilon|\prod_{i=1}^{K}\left|\mu_{\|,i}\right| < 1$$

for any $\varepsilon \in [0,2]$. Finally, this gives

$$|\mu_{\perp}| < 1,$$

which means transverse stability of the $P_K C_K$-state. The theorem is proved.

Denote the product of the multipliers of the cycle $P_{mK}^{(K)}$ by

$$\sigma \stackrel{\mathrm{df}}{=} \prod_{i=1}^{K} \mu_i^{(K)}$$

and call σ *the generalized saddle value of the cycle* $P_{mK}^{(K)}$.

Theorem 4. *Let the cycle* $P_n^{(K)}$, $n = mK$, *satisfy the conditions 1 and 2. If the saddle value of the cycle* $P_n^{(K)}$ *lies inside the unit circle, i.e.,*

$$|\sigma| < 1,$$

then the corresponding $P_{mK}C_K$*-state is transversally stable in the whole* k_1K*-dimensional phase space, for any* $\varepsilon \in [0,2]$.

Proof. As in the previous proof, due to Theorems 1 and 2,

$$|\mu_{\perp}|^K = |\mu_{\perp,i}|^K = \left|(1-\varepsilon)\prod_{i=1}^{K}\mu_{\|,i}\right| = |1-\varepsilon|\left|\prod_{i=1}^{K}\mu_{\|,i}\right| = |1-\varepsilon||\sigma|.$$

Since $|\sigma| < 1$ and $\varepsilon \in [0,2]$, we have

$$|\mu_{\perp}|^K = |\mu_{\perp,i}|^K < 1.$$

Theorem 5. *Let the cycle* $P_n^{(K)}$, $n = mK$, *satisfy the conditions 1 and 2. Then the corresponding* $P_{mK}C_K$*-state is transversally stable for any*

$$\varepsilon \in \left[1 - \frac{1}{|\sigma|}, 1 + \frac{1}{|\sigma|}\right].$$

Proof. As in the previous proof, we have

$$|\mu_{\perp}|^K = |1-\varepsilon||\sigma|.$$

For the $P_{mK}C_K$-state to be transversally stable all the transverse multipliers must lie in the unit circle,

$$|\mu_{\perp}|^K < 1.$$

Therefore,

$$|1-\varepsilon||\sigma| < 1 \Leftrightarrow |\varepsilon - 1| < \frac{1}{|\sigma|} \Leftrightarrow 1 - \frac{1}{|\sigma|} < \varepsilon < 1 + \frac{1}{|\sigma|},$$

which completes the proof.

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