UDC 517 . 9

PRESERVING OF ONE-SIDED INVARIANCE IN R^n WITH RESPECT TO SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS*
ЗБЕРЕЖЕННЯ ОДНОБІЧНОЇ ІНВАРІАНТНОСТІ В R^n ВІДНОСНО СИСТЕМ ЗВИЧАЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

S.V. Yanchuk

Inst. Math. Nat. Acad. Sci. Ukraine
Tereshchenkivs'ka St., 3, Kyiv, 01601, Ukraine
e-mail: yanchuk@imath.kiev.ua

The one-sided invariance of sets with respect to systems of ordinary differential equations in R^n is investigated. We present a general class of sets which preserve such an invariance for a linear combination of differential equations. As an application of these results, we consider the case of two coupled identical systems, which is important for the synchronization problem.

Вивчаються множини,однобічно інваріантні відносно систем звичайних диференціальних рівнянь у R^n. Наведено загальний клас множин, що зберігають таку інваріантність для лінійної комбінації диференціальних рівнянь. Як застосування отриманих результатів розглядається випадок двох зв'язаних ідентичних систем, що важливо для проблеми синхронізації.

1. Introduction. The purpose of the present paper is to extend the analysis of invariant sets to systems of ordinary differential equations. In the study of dynamical systems, the theory of invariant manifolds has proved to be an important tool [1 – 8]. Here we discuss an approach which allows to obtain new conditions for the invariance of submanifolds of R^n with boundary. In order to outline the idea of this paper, consider the following example: let some smooth submanifold M of R^n be invariant under the action of flows generated by systems of ordinary differential equations $z' = g_1(z)$ and $z' = g_2(z)$, $z \in R^n$, respectively. This means that both vectors $g_i(x)$ belong to the tangent space T_xM of the manifold at any point $x \in M$. It is obvious that the flow which is generated by the system $z' = \alpha g_1(z) + \beta g_2(z)$, where α and β are some constants, will also be tangent to M at the point x. Hence, M is invariant with respect to the latter system as well. Using this idea, we indicate a general class of sets which possesses the similar property in the case of one-sided invariance. This class will incorporate submanifolds of R^n with boundary as a special case.

As an example, we consider the system of two coupled equations of the form

$$\frac{dx}{dt} = f(x) + g_1(x, y), \quad \frac{dy}{dt} = f(y) + g_2(x, y), \quad x, y \in R^n. \tag{1}$$

We found out a perturbation g_i which preserves the invariance of a "square" $\mathcal{A} \times \mathcal{A}, \mathcal{A} \in R^n$, assuming that $\mathcal{A}$ satisfies some additional conditions. To our knowledge, this result is useful in the framework of the synchronization problem [9, 10]. In more details this question will be dealt with in Section 4. Main results are formulated in Section 2. The proof of the main theorem is given in Section 3.

* The work supported by INTAS (grant N° YSF 01-084) and Ukrainian SFFI (grant N° 01.07100109).

Recall the definition of an invariant set [5].

Definition 1. *A set $\mathcal{A} \in R^n$ is invariant with respect to some flow $\varphi(t, z) \in R^n, z \in R^n$, if $z_0 \in \mathcal{A}$ implies $\varphi(t, z_0) \in \mathcal{A}$ for all $t \in R$.*

Respectively, the notion of one-sided invariance is introduced.

Definition 2. *A set $\mathcal{A}$ consisting of positive semi-trajectories is called a positively invariant set of this system. Similarly, a set, consisting of negative semi-trajectories of a system is called a negatively invariant set of this system.*

2. Main results. We first define the admissible subsets of R^n which are the objects of this study. Denote

$$H_k^m = \{x \in R^m : \ x_i \in R, i = 1, \dots, k,$$

$$x_i \in R^+ = [0, +\infty), \, i = k+1, \dots, m\}. \tag{2}$$

Definition 3. *We will say that a set $U \in R^n$ satisfies condition A if U can be represented as a union $U = \bigcup_q U_q$, where U_q satisfies the following conditions:*

1) U_q are open subsets of the set U (with respect to the relative topology);

2) for all q there exist homeomorphisms $f_q : V_q \to U_q$, $f_q \in C^1(V_q)$ and $\operatorname{rank}\dfrac{\partial f(\alpha)}{\partial \alpha} = m \ \forall \alpha \in V_q, \ m \leq n$. V_q is an open subset of H_k^m with some $k(q) \leq m$.

Remark 1. If $k(q) = 0$ for all q we obtain the definition of an $m-$dimensional differentiable submanifold of R^n as a special case. If $k(q) = 0$ or $k(q) = 1$ then Definition 3 includes differentiable submanifolds of R^n with boundary. Generally, the defined sets can be considered as differentiable m-dimensional submanifolds of R^n with boundary where the boundary can not be smoothly embedded into R^{m-1}. A representative example is a closed square in R^2, where a neighborhood of a "corner" point is diffeomorphic to H_2^2. We will prove later (Lemma 1) that direct product of sets which satisfy condition A will also satisfy condition A.

The following theorem establishes some property of the introduced sets:

Theorem 1. *Suppose that a set W, $W \subset R^n$, $m \leq n$, is positively (negatively) invariant with respect to the systems $z' = g_1(z)$ and $z' = g_2(z)$, $z \in R^n$, respectively, and, additionally, satisfies condition A. Then W is positively (negatively) invariant with respect to the system $z' = \gamma_1 g_1(z) + \gamma_2 g_2(z)$ for all nonnegative constants γ_1 and γ_2.*

The following corollary is straightforward.

Corollary. *Any linear combination of the form*

$$z' = \sum_i \alpha_i g_i(z)$$

will preserve the positive (negative) invariance of a set, if this set is positively (resp. negatively) invariant with respect to each system $z' = g_i(z), \ i \in \Gamma$, and satisfies condition A.

Finally, as an application of Theorem 1, we will show the invariance of the "square" $\mathcal{A} \times \mathcal{A}$ with respect to some symmetric coupled system.

Theorem 2. *Suppose that a set $\mathcal{A} \subset R^n$ is linearly convex and satisfies condition A. Let also the set $\mathcal{A}$ be positively (negatively) invariant with respect to some system $u' = f(u), \quad u \in R^n$. Then the set $\mathcal{A} \times \mathcal{A}$ is also linearly convex set and satisfies condition A. Moreover, it is positively (resp. negatively) invariant with respect to the following coupled system:*

$$x' = f(x) + \alpha(y - x),$$
$$y' = f(y) + \alpha(x - y), \tag{3}$$

where α is any nonnegative (resp. nonpositive) constant.

3. Proof of Theorem 1. For the sake of convenience we split the proof in a number of lemmas. We will use the notion of locally invariant sets [5].

Definition 4. *A set $\mathcal{A} \in R^n$ is said to be locally invariant with respect to the flow $\varphi(t, z) \in \in R^n, z \in R^n$, if $z_0 \in \mathcal{A}$ implies $\varphi(t, z_0) \in \mathcal{A}$ for $t \in (-s, s)$ with some $s > 0$.*

The notion of positive and negative local invariance is introduced similarly.

Definition 5. *A set $\mathcal{A} \in R^n$ is said to be locally positively (negatively) invariant with respect to the flow $\varphi(t, z) \in R^n, \ z \in R^n$, if $z_0 \in \mathcal{A}$ implies $\varphi(t, z_0) \in \mathcal{A}$ for $t \in (0, s)$ ($t \in (-s, 0)$, respectively) with some $s > 0$.*

Remark, that in those cases when the right-hand side $g(z)$ of a system of ordinary differential equations is only continuous, i.e., in general, it does not satisfy the conditions of the uniqueness theorem, we will also accept Definitions 3 – 5 with the only difference that an existence of solution with the required properties is supposed. For example, in the case of a positively invariant set (Definition 2), we have to require that for any initial point $\alpha_0 \in U$ there exists a solution $\alpha(t, \alpha_0)$ of the system which satisfies $\alpha(t, \alpha_0) \in U$ for $t \geq 0$.

Lemma 1. *Assume the sets U_1, U_2, $U_1 \subset R^m$, $U_2 \subset R^n$ satisfy condition A. Then $W = = U_1 \times U_2 \subset R^{n+m}$ satisfies condition A.*

Proof. Suppose $z \in W$. Then z can be represented as $z = (u_1, u_2)$ where $u_1 \in U_1, u_2 \in U_2$. According to Definition 3, there exist sets U_1^1 and U_2^1 such that $u_1 \in U_1^1$, $u_2 \in U_2^1$. Here U_1^1 is diffeomorphic to some open subset of $H_{k_1}^m$, $k_1 \leq m$, and U_1^2 is diffeomorphic to some open subset of $H_{k_2}^n$, $k_2 \leq n$. It is easy to see that $U_1^1 \times U_2^1$ is a neighborhood of z in W which is diffeomorphic to some open subset of $H_{k_1}^m \times H_{k_2}^n$, which can be represented as H_k^{m+n} with some $k \leq n + m$. Finally, note that an admissible covering can be chosen as $\bigcup\limits_{q_1, q_2} U_1^{q_1} \times U_2^{q_2}$ for the set W to satisfy Definition 3, where $\{U_1^{q_1}\}$ and $\{U_2^{q_2}\}$ are corresponding coverings of U_1 and U_2.

Lemma 2. 1. *Suppose a closed set $W \subset R^n$ is invariant (positively, negatively invariant) with respect to some flow $\varphi(t, z)$. Then any open subset V of the set W is locally invariant (resp. positively, negatively invariant).*

2. *If $\overline{W} = \bigcup\limits_q V_q$, where V_q are locally invariant (positively, negatively invariant) sets, then $\overline{W}$ is invariant (resp. positively, negatively invariant) with respect to the same system.*

Proof. Let us conduct the proof for the case of positive invariance. The other cases can be treated similarly.

1. Given any open subset V and a point $z \in V$, from the positive invariance of W we have $\varphi(t, z) \in W$ for $t \geq 0$. Because the set V is open in W, there exists an open neighborhood $U_z \subset$

$\subset R^n$ such that $U_z \cap W = V$. Therefore, there exists some $s > 0$ such that $\varphi(0 \le t < s, z) \in$ $\in U_z$, which implies the necessary inclusion $\varphi(0 \le t < s, z) \in V$.

2. We argue by contradiction. We assume that there exists $z \in \overline{W}$ such that $\varphi(0, z) \in \overline{W}$ and $\varphi(t, z) \notin \overline{W}$ for $t \in (0, s)$, $s > 0$. But for the point z there exists $V_q \ni z$ which is locally positively invariant. We arrive at the contradiction for the point z to be an "escape" point.

Lemma 3. *A set $H_k^m \subset R^m$ (or some open subset $V \subset H_k^m$) is positively invariant (resp. locally positively invariant) with respect to the flow, generated by system $z' = h(z)$, $z \in R^n$, if and only if the following inequalities hold:*

$$h_{k+1}(\bar{z}) \ge 0, \; \dots \; , \; h_m(\bar{z}) \ge 0$$
$$\forall \bar{z} = (z_1, \dots, z_k, 0, \dots, 0)^T, \quad z_i \in R, \quad i = \overline{1, k}. \tag{4}$$

(In the case of an open subset $V \subset H_k^m$, we have to additionally require $\bar{z} \in V$ in Eq. (4).)

Proof. We shall only note the case of an open subset $V \subset H_k^m$, because the case of H_k^m can be proved similarly. First, for any interior point $z \in V \setminus \partial H_k^m$, $\varphi(t, z) \in V$ for all $t \ge 0$ small enough. Next, consider a boundary point $z \in V \cap \partial H_k^m$. This point can be represented in the form $z = (z_1, \dots, z_k, 0, \dots, 0)^T$ with some real z_i.

We argue by contradiction. Assume that there exists some $k + 1 \le j \le m$ such that the j-component of the flow $\varphi_j(t, z_0) < 0$ for small $t \ge 0$. Let us represent the function $\varphi_j(t, z_0)$ in the form $\varphi_j(t, z_0) = h_j(z_0)t + \psi(t, z_0)$, where $\psi(t, z_0) = o(t^2)$. If $h_j(z_0) \ne 0$, than it follows that $h_j(z_0) < 0$ and we arrive at a contradiction. If $h_j(z_0) = 0$, then for all points $z \in V \cap \partial H_k^m$ close enough to z_0, we have $\varphi_j(t, z) = h_j(z)t + \psi(t, z) < 0$. For small enough t, sign of $\varphi_j(t, z)$ coincides with the sign of $h_j(z)$ unless $h_j(z) = 0$. Thus, we obtain the following conclusion: there exists $z \in V \cap \partial H_k^m$ such that $h_j(z) < 0$ or $h_j(z) \equiv 0$ in some neighborhood of z_0. In the latter case the flow is tangent to the plane $z_j = 0$, and, therefore, it can not escape this plane in the considered neighborhood. This again leads to contradiction and shows that Eq. (4) is a sufficient condition for local positive invariance of V. Let us show that Eq. (4) provides also necessary conditions. Assuming that for some $k + 1 \le j \le m$ and $z \in V$ $h_j(z) < 0$, we get $\varphi_j(t, z) = h_j(z)t + \psi(t, z) < 0$ for t small enough, which implies $\varphi_j(t, z) \notin V$ for these values of t. Thus we again argue by contradiction.

Lemma 4. *Let a set $U \subset R^m$ be positively invariant (locally positively invariant) with respect to the system*

$$\alpha' = \lambda(\alpha), \quad \alpha \in R^m. \tag{5}$$

Assume also that there exists a C^1 map $f : U \longrightarrow R^n$, which satisfies

$$g(f(\alpha)) = \frac{\partial f(\alpha)}{\partial \alpha} \lambda(\alpha) \quad \forall \alpha \in U. \tag{6}$$

Then $f(U) \subset R^n$ is positively invariant (resp. locally positively invariant) with respect to the system

$$z' = g(z), \quad z \in R^n. \tag{7}$$

(Note, that the same assertion is valid for invariant and negatively invariant sets.)

Proof. Consider an arbitrary point $z_0 \in f(U)$. Then there exists $\alpha_0 \in U$ such that $z_0 = f(\alpha_0)$. Denote the solution of Eq. (5) $\alpha(t, \alpha_0)$ with by initial condition $\alpha(0, \alpha_0) = \alpha_0$. We want to show that $\varphi(t, z_0) = f(\alpha(t, \alpha_0))$ is a solution of $z' = g(z)$ with the initial point z_0. Indeed, we have $\varphi(0, z_0) = f(\alpha(0, \alpha_0)) = f(\alpha_0) = z_0$ and

$$\frac{d\varphi}{dt} = \frac{\partial f}{\partial \alpha}\frac{d\alpha}{dt} = \frac{\partial f}{\partial \alpha}(\alpha(t, \alpha_0))\lambda(\alpha(t, \alpha_0)) = g(f(\alpha(t, \alpha_0))) = g(\varphi).$$

The positive invariance of U with respect to Eq. (5) implies $\alpha(t, \alpha_0) \in U$ for $t \geq 0$ ($0 \leq t \leq t_0$ in the case of local invariance of U). Hence, $\varphi(t, z_0) = f(\alpha(t, \alpha_0)) \in f(U)$ for these values of t. This completes the proof.

In a certain sense, the inverse statement is represented by the following lemma which is formulated for positively invariant sets. The same assertion holds true for invariant and negatively invariant sets.

Lemma 5. *Suppose that the set W is positively invariant (locally positively invariant) with respect to system (7), there exists a homeomorphism $f : U \longrightarrow W$, $f \in C^1(U)$, $\operatorname{rank}\frac{\partial f}{\partial \alpha}(\alpha) = m \ \forall \alpha \in U, W = f(U)$. Then there exists a continuous mapping $\lambda(\alpha) : U \longrightarrow R^m$ such that (6) holds. Moreover, U is positively invariant (resp. locally positively invariant) with respect to Eq. (5).*

Proof. The case of locally invariant sets is similar, therefore we consider only the case when W is positively invariant. Let us take an arbitrary $\alpha_0 \in U$. Then there exists $z_0 \in W$ such that $z_0 = f(\alpha_0)$. Denote a solution of Eq. (7) with the initial condition z_0 by $z(t, z_0)$. Consider the function $\alpha(t, \alpha_0) = f^{-1}(z(t, f(\alpha_0)))$. Because $z(t, z_0) \in W$ for $t \geq 0$, we have $\alpha(t, \alpha_0) \in U$ for $t \geq 0$, and it is not difficult to check that $\alpha(0, \alpha_0) = \alpha_0$.

We also have

$$\frac{d}{dt}z(t, z_0) = g(z(t, z_0)) = g(f(\alpha(t, \alpha_0))) =$$

$$= \frac{d}{dt}f(\alpha(t, \alpha_0)) = \frac{\partial f}{\partial \alpha}(\alpha(t, \alpha_0))\frac{d\alpha(t, \alpha_0)}{dt}.$$

Hence,

$$g(f(\alpha(t))) = \frac{\partial f}{\partial \alpha}(\alpha(t))\frac{d\alpha(t)}{dt}. \tag{8}$$

Denote $\lambda(t) = \frac{d\alpha(t)}{dt}$. Equation (8) implies that $\lambda(t)$ depends on $\alpha(t)$, i.e., we can rewrite $\lambda(\alpha(t))$. Expression (6) follows from this and Eq. (8). Moreover, $\alpha(t, \alpha_0)$ is a solution of Eq. (5) belonging to U for $t \geq 0$. This completes the proof.

Lemma 6. *Suppose W is a locally positively invariant set for Eq. (7); there exists a homeomorphism $f : V \to W$, where $f \in C^1(V)$, $\operatorname{rank}\frac{\partial f}{\partial \alpha} = m \ \forall \alpha \in V$, $m \leq n$; V is an open subset of*

H_k^m, $k \le m$. Then there exists a continuous mapping $\lambda(\alpha) : R^m \to R^m$ such that (6) holds. Moreover, $\lambda(\alpha)$ satisfies the following inequalities:

$$\lambda_{k+1}(\alpha) \ge 0, \dots, \lambda_m(\alpha) \ge 0 \qquad \forall \alpha = (\alpha_1, \dots, \alpha_k, 0, \dots, 0)^T \in V, \tag{9}$$

i.e., for all $\alpha \in V \cap \partial H_k^m$.

Proof. Note that the conditions of Lemma 5 are satisfied if $U = V$ is an open subset of H_k^m. Hence, the set V is locally positively invariant with respect to Eq. (5). The proof will then follow from Lemma 3.

The next lemma is converse to the previous one.

Lemma 7. *Assume that $W \subset R^n$, $V \subset R^m$, and there exists a homeomorphism $f : V \to W$, where $f \in C^1(V)$, $\operatorname{rank}\dfrac{\partial f}{\partial \alpha} = m \; \forall \alpha \in V$, $m \le n, V$ is an open subset of H_k^m, $k \le m$. Assume also that there exists a continuous mapping $\lambda(\alpha) : R^m \to R^m$ such that Eqs. (6) and (9) are fulfilled. Then the set W is locally positively invariant with respect to Eq. (7).*

Proof. In view of Lemma 3, V is locally invariant with respect to Eq. (5). Now by Lemma 4, $W = f(V)$ is locally positively invariant with respect to Eq. (7).

Consider two systems in R^n:

$$z' = g_1(z), \quad z' = g_2(z). \tag{10}$$

Lemma 8. *Suppose a set $W \subset R^n$ is given and there exists a homeomorphism $f : V \to W$, where $f \in C^1(V)$, $\operatorname{rank}\dfrac{\partial f(\alpha)}{\partial \alpha} = m \; \forall \alpha \in V$, $m \le n, V$ is an open subset of H_k^m, $k \le m$. If, in addition, W is locally positively invariant with respect to each system (10), then W is locally positively invariant with respect to the system*

$$z' = \gamma_1 g_1(z) + \gamma_2 g_2(z), \tag{11}$$

where γ_1 and γ_2 are nonnegative constants.

Proof. In view of Lemma 6, there exists a function $\lambda_{g_1}(\alpha)$ such that $g_1(f(\alpha)) = \left(\dfrac{\partial f}{\partial \alpha}\right) \lambda_{g_1}(\alpha)$ for all $\alpha \in V$ and $\lambda_{g_1}(\alpha)$ satisfies conditions (9). Similarly, there exists $\lambda_{g_2}(\alpha)$ with the same properties. Consider the function $\lambda_{\gamma_1 g_1 + \gamma_2 g_2}(\alpha) = \gamma_1 \lambda_{g_1}(\alpha) + \gamma_2 \lambda_{g_2}(\alpha)$. Note first that this function satisfies conditions (9) when the constants γ_1 and γ_2 are nonnegative. Also, it is easy to check that $(\gamma_1 g_1 + \gamma_2 g_2)(f(\alpha)) = \left(\dfrac{\partial f}{\partial \alpha}\right) \lambda_{\gamma_1 g_1 + \gamma_2 g_2}(\alpha)$. Thus the conditions of Lemma 7 are fulfilled for system (11). The result now follows from Lemma 7.

Proof of Theorem 1. We consider the case of positive invariance, because the case of negatively invariant sets can be proved in much the same way. According to Definition 3, $W = \bigcup_q U_q$ and there exist mappings f_q with the corresponding properties. For any q the set U_q is locally positively invariant with respect to both systems $z' = g_1(z)$ and $z' = g_2(z)$ in virtue of Lemma 2. Now, Lemma 8 implies that U_q is locally positively invariant with respect to Eq. (11). Using the second statement of Lemma 2, we obtain that W is positively invariant with respect to Eq. (11).

4. Linearly coupled system. The goal of this section is to consider the case of coupled systems of the form (3), in order to apply Theorem 1 and to obtain conditions for positive and negative invariance of the direct product $\mathcal{A} \times \mathcal{A}$. We begin with the proof of Theorem 2, then discuss the obtained result in the framework of the synchronization problem.

In order to simplify the notations, let us introduce the following notations: $z = (x, y)^T$, $x \in R^n$, $g(z) = (f(x), f(y))^T$, $n(z) = (\alpha(y-x), \alpha(x-y))^T$. Using this notations, coupled system (3) will have the form

$$z' = g(z) + n(z). \tag{12}$$

Here $n(z)$ represents the "perturbation" term which vanishes for $\alpha = 0$.

Proof of Theorem 2. The following lemmas are necessary for the proof.

Lemma 9. *If some set U is invariant (positively, negatively invariant) with respect to the system $u' = f(u)$, then $W = U \times U$ is invariant (positively, negatively invariant) with respect to the system $z' = g(z)$.*

Proof is evident, taking into account that the system under consideration is uncoupled.

Lemma 10. *Any set $W \subset R^{2n}$ which can be represented in the form $W = U \times U$, where U is a linearly convex set, is positively (negatively) invariant with respect to the system*

$$z' = n(z), \tag{13}$$

if the constant α is nonnegative (nonpositive).

Proof. System (13) is linear. The general solution of this system has the form

$$\begin{aligned} x(t) &= \frac{1}{2}\left[e^{-2\alpha t}(x_0 - y_0) + (x_0 + y_0)\right] = \kappa_1(t)x_0 + \kappa_2(t)y_0, \\ y(t) &= \frac{1}{2}\left[(x_0 + y_0) - e^{-2\alpha t}(x_0 - y_0)\right] = \kappa_2(t)x_0 + \kappa_1(t)y_0, \end{aligned} \tag{14}$$

where $\kappa_1(t) = \dfrac{1}{2} + \dfrac{1}{2}e^{-2\alpha t}$, $\kappa_2(t) = \dfrac{1}{2} - \dfrac{1}{2}e^{-2\alpha t}$, x_0 and y_0 are initial values. Note that $\kappa_1(t) + {}+\kappa_2(t) = 1$ and $\kappa_1(t), \kappa_2(t) \in [0,1]$ for all $t \geq 0$ for $\alpha \geq 0$ (for all $t \leq 0$ for $\alpha \leq 0$). Suppose (x_0, y_0) belongs to W. Hence, $x_0 \in U$ and $y_0 \in U$. Equation (14) and convexity of U implies that $x(t)$ and $y(t)$ also belong to U for any $t \geq 0$ if $\alpha \geq 0$ ($t \leq 0$ if $\alpha \leq 0$), i.e., $(x(t), y(t)) \in W$ in the corresponding time interval and the set W satisfies the definition of positive (negative) invariance.

Now the assertion of Theorem 2 is a straightforward consequence of Lemmas 9, 10, and Theorem 1.

Let us discuss now the application of Theorem 2 to the synchronization problem for two coupled systems [9]. Lately, the synchronization problem attracts a lot of attention in the scientific community, see references in [9], in connection with a wide variety of phenomena in physics, biology, and economics. In the biological sciences, for instance, one of the most interesting problems is to understand how a group of cells or functional units, each displaying complicated nonlinear dynamic behavior, can interact with one another to produce different forms of coordinated function on a higher organizational level. Similarly, different sectors of the economy may adjust their behavior relative to one another via the exchange of commodities and capital or via aggregate signals in the form of interest rates and prices of common raw materials. At the same time, application of chaotic synchronization for monitoring and control of nonlinear dynamic systems and for new types of communication are vigorously being pursued by numerous investigators. Therefore it becomes extremely important to obtain some analytical general results in this direction.

First, we briefly describe the mathematical core of this problem (see, for example, [9, 10]). Suppose that some system $u' = f(u)$, $u \in R^n$ possess an attractor A with some attracting neighborhood U. In terms of invariant sets we can describe this situation as existing of some positively invariant set U which contains an invariant set $A \subset U \subset R^n$. Consider a coupled system of the form

$$x' = f(x) + \alpha(y - x), \quad y' = f(y) + \alpha(x - y). \tag{15}$$

Then the subject of the synchronization problem is the symmetric set $A \times A \subset R^{2n}$ which is invariant in the phase space of system [9]. In the case when this set is asymptotically stable, the *synchronization* takes place. Another important question concerns the mechanisms of desynchronization, i.e., the stability loss of the set $A \times A$. It is clear that for $\alpha = 0$ this set is not asymptotically stable in general (it is enough to consider an example when A is a stable periodic orbit), while for α large enough this set seems to be asymptotically stable [9]. Therefore, for some $\alpha \in (0, \alpha_0)$ the symmetric set $A \times A$ is not stable. In this case, according to Theorem 2, there still exists some positively invariant set $U \times U$ around $A \times A$ for these parameter values. It implies that, after stability loss of $A \times A$, trajectories of the system are attracted to another attractor in a neighborhood and never escape to infinity.

1. *Bogoliubov N., Mitropolsky Yu.* The methods of integral manifolds in nonlinear mechanics // Contrib. Different. Equat. — 1963. — **2**. — P. 123 – 167.
2. *Diliberto S.* Perturbation theorem for periodic surfaces I, II // Rend. Circ. mat. Palermo. — 1960-1961. — **9**, № 10. — P. 265 – 299, 111 – 161.
3. *Hadamard J.* Sur l'itération et les solutions asymptotiques des équations différentielles // Bull. Soc. math. France. — 1901. — **29**. — P. 224 – 228.
4. *Hale J.* Integrale manifolds of perturbed differential systems // Ann. Math. — 1961. — **73**. — P. 469 – 531.
5. *Samoilenko A.M.* Elements of the mathematical theory of multi-frequency oscillations. — Dordrecht: Kluwer Acad. Publ., 1991. — 314 p.
6. *Poincaré H.* Les methodes nouvelles de la mécanique celeste. — Paris, 1892 – 1899.
7. *Hirsch M.W., Pugh C.C., Shub M.* Invariant manifolds // Lect. Notes Math. — 1977. — **583**.
8. *Levi-Civita T., Amaldi U.* Lezioni di meccanica razionale. — Bologna: Zanichelli, 1952. — Vol. 2.
9. *Pecora L.M., Carroll T.L., Johnson G.A., Mar D.J., Heagy J.F.* Fundamentals of synchronization in chaotic systems, concepts, and applications // Chaos. — 1997. — **7**. — P. 520 – 543.
10. *Yanchuk S., Maistrenko Yu., Lading B., Mosekilde E.* Effects of a parameter mismatch on the synchronization of two coupled chaotic oscillators // Int. J. Bifurcation and Chaos. — 2000. — **10**, № 3.

Received 20.12.2001