UDC 517.9

MODIFIED GINZBURG – LANDAU EQUATION AND BENJAMIN – FEIR INSTABILITY

МОДИФІКОВАНЕ РІВНЯННЯ ГІНЗБУРГА – ЛАНДАУ І НЕСТАБІЛЬНІСТЬ У СЕНСІ БЕНДЖАМІНА – ФЕЙРА

E. Kengne

Univ. Dschang
P.O. Box 173, Dschang, Cameroon
e-mail: ekengne6@yahoo.fr

In this paper the modulated wave train in nonlinear monoinductance LC circuit is studied. Using the method of multiple scales in general form, we establish that the evolution of nonlinear excitations is governed by what we called the Modified Ginzburg – Landau Equation (MGLE). Benjamin – Feir instability for the MGLE is analyzed.

Вивчається проходження модульованих хвиль у нелінійному моноіндуктивному LC-ланцюзі. З використанням методу кратних шкал отримано, що еволюція нелінійних збуджень описується за допомогою рівняння, яке ми називаємо модифікованим рівнянням Гінзбурга – Ландау (МРГЛ). Аналізується стабільність МРГЛ у сенсі Бенджаміна – Фейра.

1. Introduction. Considering nonlinear transmission line as a convenient tool to examine wave propagations in dispersive media, various physical systems have been studied [1 – 3]. Since the pioneering works of Hirota and Suzuki [4, 5] in order to stimulate the integrable Toda lattice [6] by electric circuits there has been an increased interest in the propagation of wave trains in nonlinear-dispersive transmission lines, involving the phenomena such as Benjamin – Feir instability [7 – 9], the formation of stationary localized waves, that is, the envelope solitons [10, 11] and the dark solitons [12, 13].

The Benjamin – Feir (or, as it is sometimes called, the modulational) instability is widespread and plays an important role in various nonlinear wave phenomena. Simply put, if dispersion and nonlinearity act against each other, monochromatic wave trains do not wish to remain monochromatic. The sidebands of the carrier wave can draw on its energy via a resonance mechanism with the result that the envelope becomes modulated. In one space dimension, this envelope modulation continues to grow until the soliton shape is reached. At this point, nonlinearity and dispersion are in exact balance and no further distortion occurs [14, 15].

It is well known that the self-modulation of one space dimension waves in nonlinear dispersive systems can be described by the so-called Ginzburg – Landau equation (GLE) [16 – 18],

$$iu_t + Pu_{xx} + Q\,|u|^2\,u = i\gamma, \tag{1.1}$$

where the subscripts t and x denote the partial differentiation with respect to t and x, respectively. If $PQ < 0$, a plane wave in this system is stable for the modulation and, otherwise, is unstable. Especially in the later case there exist special families of solutions, which are called envelope solitons and show various interesting phenomena [19, 20].

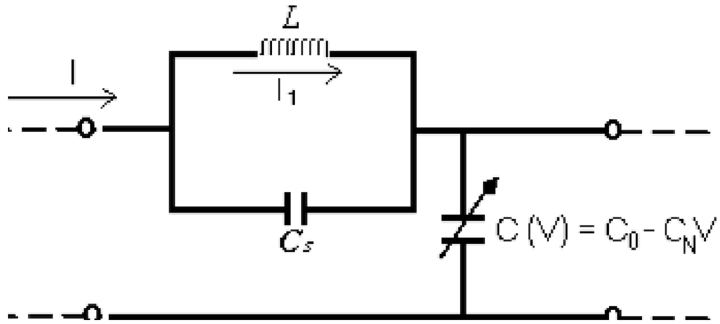

Fig. 1. Asection for a distributed nonlinear-dispersive transmission line.

Recently, there has been a progress towards a mathematical understanding of equation (1.1). Kirchgassner [21] and Mielke [22–24] restrict attention to steady-state equations and view the single unbounded spatial direction as an evolution variable.

In this paper, we give a rigorous derivation of the full time-dependent Modified Ginzburg–Landau Equation (MGLE). The Benjamin–Feir (modulational) instability for the obtained MGLE is investigated.

2. Basic equations. In this section we derive a nonlinear wave equation for the electromagnetic wave propagation in a nonlinear-dispersive transmission line shown in Fig. 1. By using the method of multiple scales, we derive a MGLE.

2.1. The model equations. In the considered transmission line, Fig. 1, C_N is a nonlinear capacitor such as a "VARICAP" or a reverse-biased $p-n$ junction diode, the capacitance of which depends on the voltage applied across it.

By applying the Kirchhoff's voltage theorem and the current theorem we obtain

$$\frac{\partial I}{\partial x}+\frac{\partial \rho(V)}{\partial t}=0, \qquad \frac{\partial V}{\partial x}+L\frac{\partial I_1}{\partial t}=0, \qquad \frac{\partial^2 V}{\partial x \partial t}+\frac{1}{C_s}\left(I-I_1\right)=0 \tag{2.1}$$

where the current through the nonlinear capacitor is given by $\partial\rho(V)/\partial t$. From equations (2.1) we can eliminate I and I_1 and write

$$C_s\frac{\partial^4 V}{\partial x^2 \partial t^2}+\frac{1}{L}\frac{\partial^2 V}{\partial x^2}-\frac{\partial^2 \rho}{\partial t^2}=0. \tag{2.2}$$

With no loss of generality, we may regard $\rho(0)=0$ and expand $\rho(V)$ to obtain $\rho(V)\approx\rho'(0)V+$ $+\frac{\rho''(0)}{2}V^2$. For bounded solutions, we must have $\rho'(0)\geq 0$. Hence we have

$$\rho(V)\approx C_0\left(V-\beta' V^2\right)=C_0 V-C_N V^2. \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain the following partial differential equation for the voltages:

$$C_0\frac{\partial^2 V}{\partial t^2}-\frac{1}{L}\frac{\partial^2 V}{\partial x^2}-C_S\frac{\partial^4 V}{\partial x^2 \partial t^2}-C_N\frac{\partial^2 V^2}{\partial t^2}=0. \tag{2.4}$$

2.2. Derivation of the generalized complex Ginzburg–Landau equation. If we introduce the notations

$$\alpha = -1/L,\ \beta = -C_N,\ \lambda = C_S,$$

equation (2.4) takes the form

$$C_0\frac{\partial^2 V}{\partial t^2} + \alpha\frac{\partial^2 V}{\partial x^2} - C_s\frac{\partial^4 V}{\partial x^2 \partial t^2} + \beta\frac{\partial^2 V^2}{\partial t^2} = 0. \tag{2.5}$$

We follow Taniuti and Yajima [25, 26] and seek a first-order uniform expansion by using the method of multiple scales in the form

$$\begin{aligned} V &= \varepsilon^{1/2} v_{11} \exp\left[i(kX_0 - \omega T_0)\right] + \varepsilon V_{22} \exp\left[2i(kX_0 - \omega T_0)\right] + \\ &+ \varepsilon^{3/2} v_{33} \exp\left[3i(kX_0 - \omega T_0)\right] + \varepsilon^2 \left[v_{42} \exp\left[2i(kX_0 - \omega T_0)\right] + \right. \\ &+ \left. v_{44} \exp\left[4i(kX_0 - \omega T_0)\right]\right] + cc + \ldots , \end{aligned} \tag{2.6}$$

where cc stands for the complex conjugate, ε is a small, dimensionless parameter related to the amplitudes $(0 < \varepsilon \ll 1)$, $v_{ij} = v_{ij}(X_1, T_1, T_2)$, $T_n = \varepsilon^n t$, and $X_n = \varepsilon^n x$.

Substituting (2.6) into (2.5) and equating coefficients of like powers of ε and $\exp[i\theta]$ (here $\theta = kX_0 - \omega T_0$), we obtain the following:

for order $\varepsilon^{1/2}$, $\exp[i\theta]$,

$$\left[C_0\omega^2 + \alpha k^2 + \lambda k^2 \omega^2\right] v_{11} = 0, \tag{2.7}$$

for order $\varepsilon^{3/2}$, $\exp[i\theta]$,

$$-2i\omega\left[C_0 + \lambda k^2\right]\frac{\partial v_{11}}{\partial T_1} + 2ik\left[\alpha + \lambda\omega^2\right]\frac{\partial v_{11}}{\partial X_1} - 2\beta\omega^2 v_{11}^* v_{22} = 0, \tag{2.8}$$

for order ε, $\exp[2i\theta]$,

$$-4\left(C_0\omega^2 + \alpha k^2 + 4\lambda k^2\omega^2\right) - 4\omega^2\beta v_{11}^2 = 0, \tag{2.9}$$

for order $\varepsilon^{3/2}$, $\exp[3i\theta]$,

$$-9\left(C_0\omega^2 + \alpha k^2 + 9\lambda k^2\omega^2\right) - 18\omega^2\beta v_{11} v_{22} = 0, \tag{2.10}$$

for order ε^2, $\exp[2i\theta]$,

$$\left[-4\left(C_0\omega^2 + \alpha k^2 + 4\lambda k^2\omega^2\right)\right] v_{42} - 8\beta\omega^2 v_{11}^* v_{33} = 0, \tag{2.11}$$

for order ε^2, $\exp[4i\theta]$,

$$\left[-16\left(C_0\omega^2 + \alpha k^2 + 16\lambda k^2\omega^2\right)\right] v_{44} - \beta\omega^2\left[32 v_{11} v_{33} + 16 v_{22}^2\right] = 0, \tag{2.12}$$

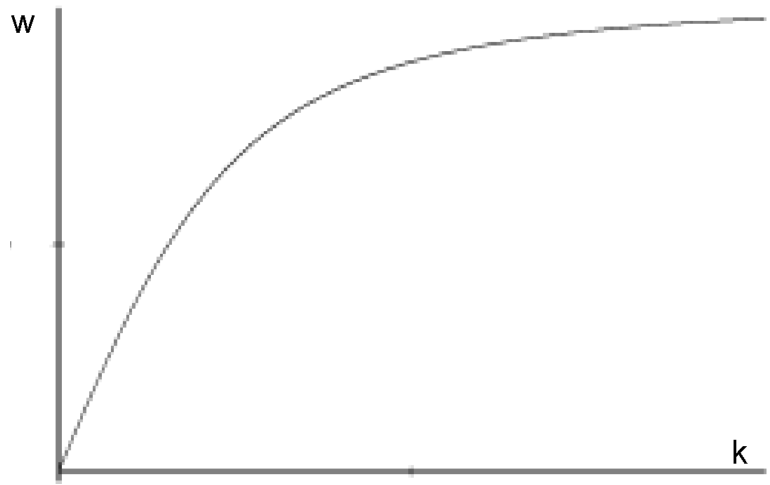

Fig. 2. The dispersion curve for the linearized version of the above transmission line.

for order $\varepsilon^{5/2}$, $\exp[i\theta]$,

$$C_0\left[\frac{\partial^2 v_{11}}{\partial T_1^2} - 2i\omega\frac{\partial v_{11}}{\partial T_2}\right] + \alpha\frac{\partial^2 v_{11}}{\partial X_1^2} - \lambda\left[-k^2\frac{\partial^2 v_{11}}{\partial T_1^2} + 2i\omega k^2\frac{\partial v_{11}}{\partial T_2} + 4k\omega\frac{\partial^2 v_{11}}{\partial T_1 \partial X_1} - \right.$$

$$\left. - \ \omega^2\frac{\partial^2 v_{11}}{\partial X_1^2}\right] + \beta\left[-4i\omega\frac{\partial v_{11}^* v_{22}}{\partial T_1} - 2\omega^2 v_{11}^* v_{42} - 2\omega^2 v_{22}^* v_{33}\right] = 0. \tag{2.13}$$

For the nontrivial solution we must have $v_{11} \neq 0$. Then (2.7) gives

$$C_0\omega^2 + \alpha k^2 + \lambda k^2\omega^2 = 0. \tag{2.14}$$

Equation (2.14) is the dispersion relation illustrated in Fig. 2 for the line parameters

$$C_s = 5C_0 = 1200pF,\ L = 14\mu H,\ C_N = 38,4pF,\ 0 \le k \le 1,58,\ \varepsilon = 0,1. \tag{2.15}$$

Using the dispersion relation (2.14), equations (2.9) – (2.12) give

$$v_{22} = -\frac{\beta}{3\lambda k^2}v_{11}^2, \tag{2.16}$$

$$v_{33} = \frac{\beta^2}{12\lambda^2 k^4}v_{11}^3, \tag{2.17}$$

$$v_{42} = -\frac{\beta^3}{108\lambda^2 k^8\omega^2}\left|v_{11}\right|^2 v_{11}^2, \tag{2.18}$$

$$v_{44} = -\frac{\beta^3}{54\lambda^3 k^6} v_{11}^4, \tag{2.19}$$

respectively.

Solving for $\partial v_{11}/\partial T_1$, from (2.9) and (2.16) we obtain

$$\frac{\partial v_{11}}{\partial T_1} = \frac{C_0}{\alpha}\frac{\omega^3}{k^3}\frac{\partial v_{11}}{\partial X_1} - \frac{i\beta\omega^3}{\alpha k^2} v_{11}^* v_{22},$$

where $V_g = -\frac{C_0}{\alpha}\left(\frac{\omega}{k}\right)^3 = \frac{C_0\sqrt{-\alpha}}{(C_0+\lambda k^2)^{3/2}}$ is the group velocity. Hence,

$$\begin{aligned} \frac{\partial^2 v_{11}}{\partial T_1^2} &= \frac{C_0^2}{\alpha^2}\frac{\omega^6}{k^6}\frac{\partial^2 v_{11}}{\partial X_1^2} - C_0\frac{i\beta\omega^6}{\alpha^2 k^5}\frac{\partial}{\partial X_1}\left(v_{11}^* v_{22}\right) - \frac{i\beta\omega^3}{\alpha k^2}\frac{\partial}{\partial T_1}\left(v_{11}^* v_{22}\right), \\ \frac{\partial^2 v_{11}}{\partial T_1 \partial X_1} &= \frac{C_0}{\alpha}\frac{\omega^3}{k^3}\frac{\partial^2 v_{11}}{\partial X_1^2} - \frac{i\beta\omega^3}{\alpha k^2}\frac{\partial}{\partial X_1}\left(v_{11}^* v_{22}\right). \end{aligned} \tag{2.20}$$

Combining (2.20) and (2.13), and using (2.16) – (2.18), we obtain, in terms of the original variables t and x,

$$i\frac{\partial v_{11}}{\partial t} + P\frac{\partial^2 v_{11}}{\partial x^2} + iQ_1\frac{\partial}{\partial t}\left(|v_{11}|^2 v_{11}\right) + iQ_2\frac{\partial}{\partial x}\left(|v_{11}|^2 v_{11}\right) + Q_3|v_{11}|^4 v_{11} = 0, \tag{2.21}$$

where

$$P = P(k) = -\frac{1}{2}\omega'' = -\frac{3C_0\lambda\sqrt{-\alpha}k}{2\left(C_0+\lambda k^2\right)^{5/2}}, \tag{2.22}$$

$$Q_1 = Q_1(k) = -\frac{\beta^3\varepsilon}{2\lambda k^2\left(C_0+\lambda k^2\right)}, \tag{2.23}$$

$$Q_2 = Q_2(k) = \frac{\beta^2\varepsilon\left(C_0+4\lambda k^2\right)}{3\lambda k^3\left(C_0+\lambda k^2\right)^2}, \tag{2.24}$$

$$Q_3 = Q_3(k) = -\frac{\beta^4\varepsilon^2\sqrt{-\alpha}}{12\lambda^3 k^5\left(C_0+\lambda k^2\right)^{3/2}}. \tag{2.25}$$

Using the transformation

$$\xi = x - Q_2 Q_1^{-1} t, \qquad \tau = t, \qquad v_{11}(\xi,\tau) = u(\xi,\tau)\exp\left[iQ_2 P^{-1} Q_1^{-1}\xi/2\right], \tag{2.26}$$

we write (2.21) in the final form

$$i\frac{\partial u}{\partial \tau} + P\frac{\partial^2 u}{\partial \xi^2} + iQ_1\frac{\partial}{\partial \tau}\left(|u|^2 u\right) - \gamma u + Q_3|v_{11}|^4 v_{11} = 0, \tag{2.27}$$

where $\gamma = -Q_2^2 P^{-1} Q_1^{-2}/4$.

Thus the resulting equation (2.27) that describes the evolution of a wavepacket is a complex envelope equation that involves higher order nonlinearities. We call this equation the MGLE.

For the line parameters (2.15) we plot the following coefficient of the spatial dispersion curve.

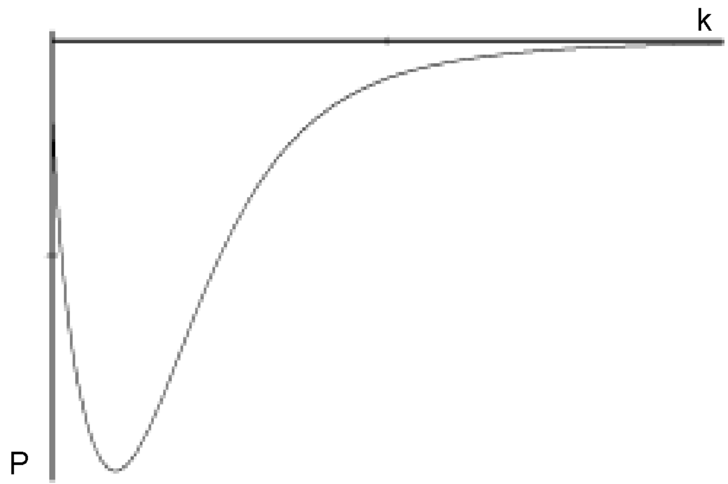

Fig. 3. The coefficient of the spatial dispersion curve.

It is seen from Fig. 3 that the coefficient of the spatial dispersion is always negative when $0 \leq$ $\leq k \leq 1,58$. In the next section we study the Benjamin–Feir instability of the monochromatic wave solutions.

3. The Benjamin–Feir instability. To study the Benjamin–Feir instability of the monochromatic wave solutions, we first express u in the polar form

$$u(\xi,\tau) = a(\xi,\tau)\exp\left[ib(\xi,\tau)\right]. \tag{3.1}$$

Substituting (3.1) into (2.27) and separating imaginary and real parts we obtain

$$\left(1+3Q_1a^2\right)\frac{\partial a}{\partial \tau} + P\left(2\frac{\partial a}{\partial \xi}\frac{\partial b}{\partial \xi} + a\frac{\partial^2 b}{\partial \xi^2}\right) = 0, \tag{3.2}$$

$$\left(a+Q_1a^3\right)\frac{\partial b}{\partial \tau} + P\left(a\left(\frac{\partial b}{\partial \xi}\right)^2 - \frac{\partial^2 a}{\partial \xi^2}\right) + \gamma a - Q_3a^5 = 0. \tag{3.3}$$

If the wave has a fixed, single wavenumber, then $P = -\dfrac{1}{2}\dfrac{d^2\omega}{dk^2} = 0$ and system (3.2), (3.3) reduces to

$$\left(1+3Q_1a^2\right)\frac{\partial a}{\partial \tau} = 0, \qquad \left(a+Q_1a^3\right)\frac{\partial b}{\partial \tau} + \gamma a - Q_3a^5 = 0$$

whose solutions are

$$a = a_0, \qquad b = \frac{Q_3 a_0^4 - \gamma}{1 + Q_1 a_0^2}\tau + \text{const}, \tag{3.4}$$

where a_0 is constant.

It natural to define the local wavenumber k as the ξ derivative of the total phase and the local frequency as the negative of the τ derivative of the total phase $\theta = k_0\xi - \omega_0\tau + b(\xi,\tau)$,

$$k = k_0 + b_\xi, \qquad \omega = \omega_0 - b_\tau.$$

Note that

$$k_\tau + \omega_\xi = b_{\xi\tau} - b_{\tau\xi} = 0, \tag{3.5}$$

which expresses the conservation of the number of waves. We will write the change in the wavenumber b_ξ as K. Now equation (3.2) gives

$$\frac{\partial}{\partial\tau}\left(2a^2 + 3Q_1a^4\right) + 4P\frac{\partial}{\xi}\left(a^2K\right) = 0, \tag{3.6}$$

which is an equation of conservation of the wave action. On the other hand, equation (3.3) gives

$$a\left(1 + Q_1a^2\right)b_\tau + P\left(aK^2 - a_{\xi\xi}\right) + \gamma a - Q_3a^5 = 0,$$

which when differentiating with respect to ξ gives

$$a^2(1+Q_1a^2)^2K_\tau + a_\xi\left(1+3Q_1a^2\right)\left[Q_3a^5 - \gamma a + P\left(a_{\xi\xi} - aK^2\right)\right] +$$

$$+\ Pa\left(1+Q_1a^2\right)\left(a_\xi K^2 + 2aKK_\xi - a_{\xi\xi\xi}\right) + a\left(1+Q_1a^2\right)\left(\gamma - 5Q_3a^4\right)a_\xi = 0. \tag{3.7}$$

Equation (3.7) is a relation for conservation of waves, since

$$\omega = \omega_0 + \frac{\gamma + P\left(K^2 - \dfrac{a_{\xi\xi}}{a}\right) - Q_3a^4}{1 + Q_3a^2}.$$

Next, the monochromatic wave solution (3.4) means that

$$k = k_0, \qquad \omega = \omega_0 - \frac{Q_3a_0^4 - \gamma}{1 + Q_1a_0^2}.$$

This is the Stokes wave. Test it linear stability by setting

$$a = a_0 + \tilde{a}, \qquad K = \tilde{K}, \tag{3.8}$$

where $\tilde{a}$ is assumed to be infinitesimal. Substituting (3.8) into (3.6) and (3.7) and keeping only linear terms in perturbation quantities, we obtain

$$\tilde{a}_\tau = -\frac{Pa_0}{1+3Q_1a_0^2}\tilde{K}_\xi,$$

$$K_\tau = \frac{P}{a_0\left(1+Q_1a_0^2\right)}\tilde{a}_{\xi\xi\xi} + \frac{2\left(Q_1Q_3a_0^4+2Q_3a_0^2+Q_1\gamma\right)a_0^2}{a_0\left(1+Q_1a_0^2\right)^2}\tilde{a}_\xi,$$

or

$$\tilde{a}_{\tau\tau} = -\frac{P\left(1+Q_1a_0^2\right)^{-2}}{\left(1+3Q_1a_0^2\right)}\left[P\left(1+Q_1a_0^2\right)\tilde{a}_{\xi\xi\xi\xi} + 2a_0^2\left(Q_1Q_3a_0^4+2Q_3a_0^2+Q_1\gamma\right)\tilde{a}_{\xi\xi}\right]. \qquad (3.9)$$

Therefor if $\tilde{a} \propto \exp\left[il\xi+\Omega\tau\right]$,

$$\Omega^2 = -\frac{\left(1+Q_1a_0^2\right)^{-2}k^2}{\left(1+3Q_1a_0^2\right)}\left[\left(1+Q_1a_0^2\right)P^2l^2 - 2a_0^2P\left(Q_1Q_3a_0^4+2Q_3a_0^2+Q_1\gamma\right)\right]. \qquad (3.10)$$

Because $\beta = -C_N < 0$, it follows from (2.22)–(2.25) that $P(k) < 0$, $Q_1(k) > 0$, $Q_3(k) < 0$, and $\gamma = -Q_2^2P^{-1}Q_1^{-2}/4 > 0$. Therefore we have the following results.

Theorem 3.1. *If*

$$Q_1Q_3a_0^4+2Q_3a_0^2+Q_1\gamma < 0, \qquad (3.11)$$

the monochromatic wave solution (3.4) will be unstable to long waves in the range

$$0 < l^2 < \frac{2a_0^2\left(Q_1Q_3a_0^4+2Q_3a_0^2+Q_1\gamma\right)}{P\left(1+Q_1a_0^2\right)}. \qquad (3.12)$$

Inequality (3.11) is the Benjamin–Feir instability criterion to the MGLE in the electrical monoinductance transmission line. This new result is different from the Lange and Newell criterion for the Stokes wave [27, 28] by the presence of the amplitude a_0 of the monochromatic wave.

For the line parameters (2.15) we plot $Q_1Q_3a_0^4+2Q_3a_0^2+Q_1\gamma$ as a function of the amplitude a_0 or/and a function of the wavenumber k.

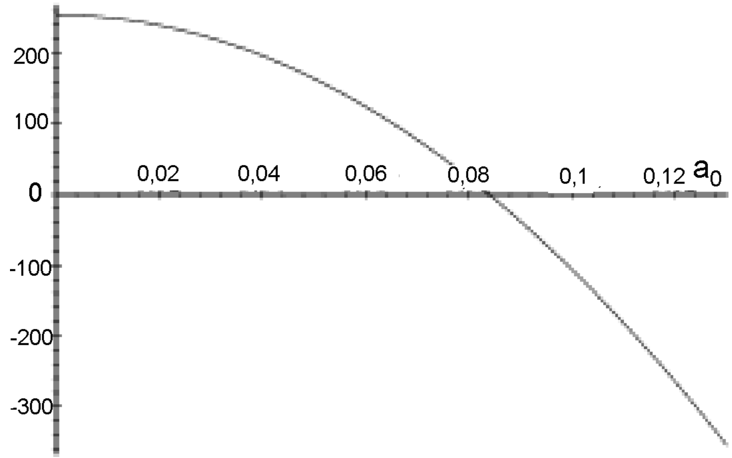

Fig. 4. The dependence of $Q_1Q_3a_0^4 + 2Q_3a_0^2 + Q_1\gamma$ on a_0 with $k = 0,1$.

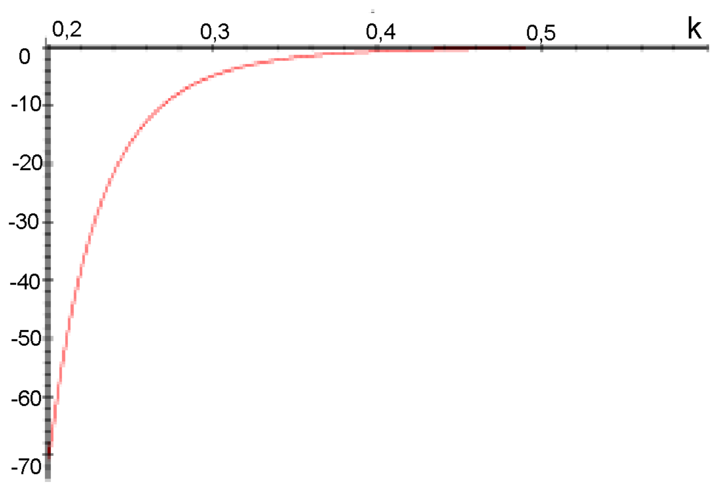

Fig. 5. The dependence of $Q_1Q_3a_0^4 + 2Q_3a_0^2 + Q_1\gamma$ on k, $0,09 \le k \le 0,624$, for $a_0 = 1000$.

Fig. 4 shows that for the wavenumber $k = 0,1$, condition (3.11) holds for all $a_0 > a_{0c} \simeq \simeq 0,084$. For these values of a_0, the monochromatic wave solutions corresponding to the fixed wavenumber $k = 0,1$ are modulational unstable. All the monochromatic wave solutions associated to $k = 0,1$ with any amplitude $a_0 < a_{0c}$ are stable.

It is seen from Figures 5, 6, and 7 that for any fixed wavenumber $0 < k < 0,624$, the corresponding monochromatic wave solution with the amplitude $a_0 = 1000$ is modulational unstable, while any monochromatic wave solution corresponding to the wavenumber $0,625 \le \le k \le 1,58$ with the amplitude $a_0 = 1000$ is modulational stable.

Figures 4–7 show that $Q_1Q_3a_0^4 + 2Q_3a_0^2 + Q_1\gamma$, as a function of k or/and a_0, changes its sign for particular value of k or/and a_0.

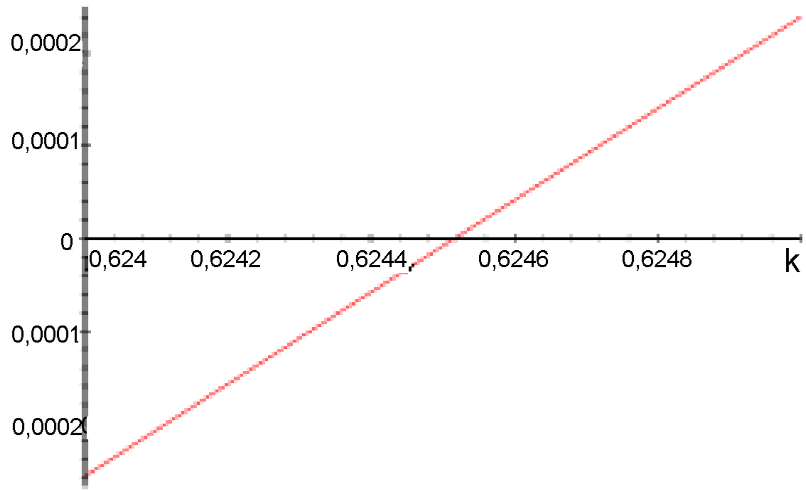

Рис. 6. The dependence of $Q_1Q_3a_0^4 + 2Q_3a_0^2 + Q_1\gamma$ on k,
$0,624 < k \leq 0,625$, for $a_0 = 1000$.

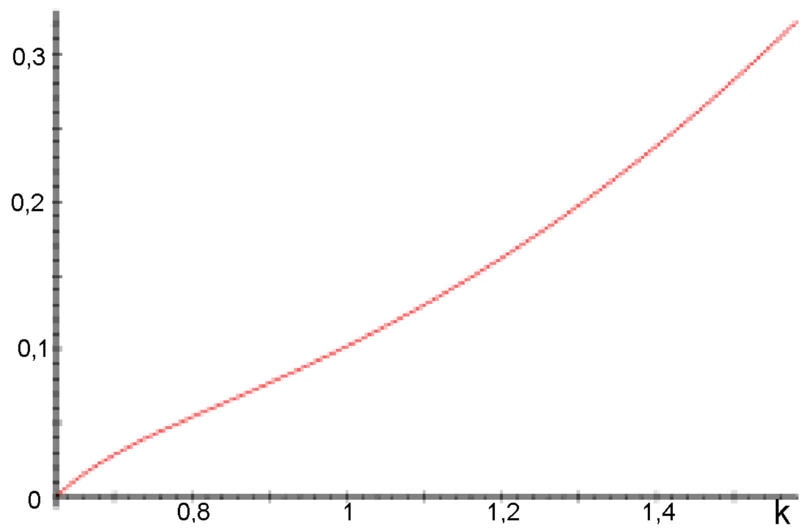

Рис. 7. The dependence of $Q_1Q_3a_0^4 + 2Q_3a_0^2 + Q_1\gamma$ on k,
$0,625 \leq k \leq 1,58$, for $a_0 = 1000$.

4. Conclusion. In this paper monoinductance LC circuit is considered and envelope modulation is reduced to the MGLE. Benjamin–Feir instability for the MGLE is analyzed. As far as we know there have been no such Stokes wave analysis related to LC circuit. As in most cases, the linear part of the modulation equation (the coefficient of the spatial dispersion) is fixed, that is, $P = \dfrac{-1}{2}\dfrac{d^2\omega}{dk^2}$. We also have that $Q_1Q_3a_0^4 + 2Q_3a_0^2 + Q_1\gamma < 0$ is a necessary but not sufficient condition for the instability. It should be noted that $Q_1Q_3a_0^4 + 2Q_3a_0^2 + Q_1\gamma > 0$ is a sufficient condition for the stability. In fact, if this last condition is satisfied then for every real l, Ω will always be pure imaginary and $\tilde{a} \propto \exp\left[ik\xi + \Omega\tau\right]$ will be bounded. In most cases the criterion of the instability does not depend on the amplitude of the monochromatic wave. But for our MGLE, the said criterion depends on the amplitude a_0. This fact allows us to construct an unstable monochromatic wave for a given wavenumber.

1. *Scott A.* Active and nonlinear wave propagation in electronics. — New York: Wiley-Intersci., 1970.
2. *Solitons* in action / Eds. K.E. Lonngren, A.C. Scott. — New York: Acad. Press, 1978.
3. *Ostrovski L.A., Gorshkov K.A., andPapko V.V.* Phys. scr. — 1979. — **20**. — P. 357.
4. *Hirota R., Suzuki K.* Direct methods in soliton theory // J. Phys. Soc. Jap. — 1970. — **28**. — P. 1366.
5. *Hirota R., Suzuki K.* Direct methods of finding exact solutions of nonlinear evolution equations // Backlund Transformation: Proc. IECE. — 1973. — **61**. — P. 1438.
6. *Toda M.* Theory of nonlinear Lattices // J. Phys. Soc. Jap. — 1967. — **22**. — P. 413; **23**. — P. 501.
7. *Benjamin T.B., Feir J.F.* The disintegration of wave trains on deep water // J. Fluid Mech. — 1967. — **27**. — P. 417 – 430.
8. *Lake B.M., Yuen H.C., Rungaldier H., and Ferguson W.E.* Nonlinear deep water waves: theory and experiment // Ibid. — 1977. — **83**. — P. 49 – 74.
9. *Kengne E.* Envelope modulational instability in a nonlinear dissipative transmission line // Nonlinear Oscillations. — 2002. — **5**, № 1. — P. 20 – 29.
10. *Kawahara T., Sakai J., and Kakutani T.* Weak ion-acoustic shock waves // J. Phys. Jap. — 1970. — **29**. — P. 1068 – 1073.
11. *Inoue H Y., Sahai J., and Kawata T.* Nonlinear wave modulation in dispersive media // Trans. IECE Jap. E. — 1976. — **60**. — P. 339.
12. *Saito H., Muraya K., and Watanabe S.* Nonlinear forced vibrations of a beam carrying concentrated mass under gravity // J. Sound. and Vibr. — 1976. — **46**. — P. 515 – 525.
13. *Nayfeh A.H.* Perturbation methods // Phys. scr. — 1985. — **31**. — P. 415.
14. *Whitham G.B.* Linear and nonlinear waves. — New York: Wiley-Intersci., 1974.
15. *Benney D.J., Roskes G.J.* Wave instabilities // Stud. Appl. Math. — 1969. — **48** — P. 377 – 385.
16. *Karpman V.I., Kadomtsev B.B.* Nonlinear waves // Sov. Phys. Uspekhi. — 1971. — **14**. — P. 40 – 60.
17. *Karpman V.I., Krushkal E.M.* Modulated waves in nonlinear dispersive media // Zh. Eksp. i Teor. Fiz. — 1968. — **55**. — P. 530.
18. *Taniuti T.* Reductive perturbation method and far fields of wave equations // Progr. Theor. Phys. Suppl. — 1974. — **55**. — P. 1 – 35.
19. *Asano N., Taniuti T., and Yajima N.* Reductive perturbation method for nonlinear wave propagation in inhomogeneous media. II // J. Phys. Soc. Jap. — 1970. — **29**. — P. 209 – 214.
20. *Yajima N., Asano N., and Taniuti T.* Perturbation method for a nonlinear wave modulation // J. Math. Phys. — 1969. — **10**. — P. 2020 – 2024.
21. *Kirchgassner K.* Wave solutions of reversible systems and applications // J. Different. Equat. — 1982. — **45**. — P. 113 – 127.
22. *Mielke A.* A reductible principle for nonautonomous systems in infinite dimensional spaces // Ibid. — 1986. — **65**. — P. 68 – 88.
23. *Mielke A.* Steady flows of inviscid fluids under localized perturbations // Ibid. — P. 89 – 116.
24. *Mielke A.* Reduction of quasilinear elliptic equations in cylindrical domains with applications // Math. Meth. Appl. Sci. — 1988. — **10**. — P. 51 – 66.
25. *Taniuti T., Yajima N.* Perturbation method for nonlinear wave modulation. I // J. Math. Phys. — 1969. — **10**. — P. 1369 – 1372.
26. *Taniuti T., Najima N.* Perturbation method for nonlinear wave modulation. III // Ibid. — 1973. — **14**. — P. 1389 – 1397.
27. *Stokes G.G.* On the Stokes wave // Cambridge Phil. Soc. — 1998. — P. 441 – 455.
28. *Lange C.G., Newell A.C.* The post-buckling problem for thin elastic shells // SIAM J. Appl. Math. — 1974. — **27**. — 441 p.

Received 16.09.2002,
after revision — 15.02.2003