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OSCILLATIONS IN DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAYS

КОЛИВАННЯ В ДИФЕРЕНЦІАЛЬНИХ РІВНЯННЯХ ІЗ ЗАГАЮВАННЯМ, ЩО ЗАЛЕЖИТЬ ВІД СТАНУ

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In this paper we study the existence of oscillating solutions of a delay differential equations with the delay depending directly on the state. Necessary and sufficient conditions for oscillations are established.

Вивчається питання існування коливних розв'язків диференціальних рівнянь із загаюванням, що залежить безпосередньо від стану. Отримано необхідні та достатні умови існування коливань.

1. Introduction. We consider the differential equation with state-dependent delays,

$$x'(t) = -f\big(x(t-\sigma(x_t))\big), \quad \text{for} \quad t \geq 0, \qquad x_0 = \phi \in C^0, \tag{1}$$

or more generally,

$$x'(t) = -f\left(\int_{-r}^{0} g\big(x(t-\sigma(x_t)+\theta)\big)d\theta\right), \tag{2}$$

where f is a function defined from $\Re$ into $\Re$, σ is a function defined from C into $[0, M]$, $\sigma(x_t) \geq$ ≥ 0, $x_t : [-\infty, 0] \to \Re$, $x_t \in BC^0([-\infty, 0], \Re)$ is defined by $x_t(\theta) = x(t+\theta)$, for $\theta \in [-M, 0]$.

In many situations the time delay $\sigma(x_t) \geq 0$ and depends on the state x_t via the threshold condition

$$\int_{t-\sigma(x_t)}^{t} k(x(s))ds = k_0,$$

where $k : \Re \to \Re_+$ is continuous and nonnegative, and $k(0) = k_0$. In this case the time delay is an autonomous function $\sigma(x_t) > 0$ with the derivative

$$\frac{\partial}{\partial t}\sigma(x_t) = 1 - \frac{k(x(t))}{k(x(t-\sigma(x_t)))} < 1. \tag{3}$$

Throughout this paper, the following assumptions are imposed on f and σ:

$$uf(u) > 0, \quad \text{for} \quad u \neq 0, \tag{4}$$

$$\lim_{u\to\infty} \inf \frac{f(u)}{u} \geq 1. \tag{5}$$

There exists $\delta > 0$ such that neither

$$\begin{aligned} f(u) \leq u \quad &\text{for} \quad 0 \leq u \leq \delta, \\ f(u) \geq u \quad &\text{for} \quad -\delta \leq u \leq 0; \end{aligned} \tag{6}$$

k is uniformly positive, $k \geq k_1 > 0$.

2. Preliminary. Equations (l) and (2) are of the type

$$x'(t) = F(x_t),$$

where

$$F(\phi) = -f(\phi(-\sigma(\phi))$$

which is a nonlinear functional equation with values of x lying only in a bounded interval completely contained in the past. Notice that the functional F is defined in C, but it is neither differentiable nor locally Lipschitz continuous, whatever the smoothness of f and σ. Thus, together with the differential equation (3) for σ_t, we can conclude the following.

Theorem 1. *Let $f : \Re \to \Re$ be locally Lipschitzian, Lipschitz continuous uniformly on bounded sets, $k : \Re \to \Re_+$ locally Lipschitz continuous. Then for each initial function $\phi \in C$, which is Lipschitz continuous on $[-\sigma(\phi), 0]$, there exists a unique continuously differentiable solution $x : [0, \infty] \to \Re$ equation (1) satisfying $x_0 = \phi$ and depending only on values of ϕ in $[-\sigma(\phi), 0]$.*

A general theorem on existence of solutions was given in [1]. In this paper we present some oscillation results for a class of nonlinear delay differential equations of the type (1). We begin with a definition of the concept of oscillation.

Definition 1. *Let x be a continuous function defined on some infinite interval $a, \infty]$. The function f is said to oscillate or to be oscillatory if x has arbitrarily large zeros. That is, for every $b > a$ there exists a point $c > b$ such that $x(c) = 0$. Otherwise x is called nonoscillatory.*

Definition 2. *Let x be a continuous function defined on some infinite interval $[a, \infty]$. The function f is said to be eventually positive or eventually negative if there exists a $T \in \Re$ such that $x(t)$ is positive for $t \geq T$ or is negative for $t \geq T$.*

Lemma 1. *Every nonoscillatory solution of equation (1) tends to zero as $t \to +\infty$.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). We assume that $x(t)$ is eventually positive. Then eventually,

$$x'(t) = -f(x(t - \sigma(x_t)) < 0$$

and so $l = \lim_{t\to+\infty} x(t)$ exists and is a nonnegative number. We must show that $l = 0$. Otherwise, $l > 0$ and so $\lim_{t\to+\infty} x'(t) < 0$, which implies that $\lim_{t\to+\infty} x(t) = -\infty$. This is a contradiction and the proof of Lemma 1 is complete.

3. Linearization of equation (1) at zero.

Lemma 2. *Assume that $f(0) = 0$ and $f'(0)$ exists. Then the equation (1) is equivalent to a nonlinear neutral functional at ∞.*

Let $p = f'(0)$ and $\sigma_0 = \sigma(\phi)$. Using the Taylor formula for f in the equation (l), we obtain

$$x'(t) = -f(0) - px(t-\sigma(x_t)) - x(t-\sigma(x_t))^2 f''(\theta). \tag{7}$$

We rewrite the term

$$x(t-\sigma(x_t)) = x(t-\sigma_0) + \frac{\partial}{\partial t}\left(\int\limits_{-\sigma_0}^{-\sigma(x_t)} x(t+\theta)d\theta\right) + x(t-\sigma(x_t))\frac{\partial}{\partial t}\sigma(x_t) \tag{8}$$

and we put it in (7),

$$x'(t) = -px(t-\sigma_0) - p\frac{\partial}{\partial t}\left(\int\limits_{-\sigma_0}^{-\sigma(x_t)} x(t+\theta)d\theta\right) - px(t-\sigma(x_t))^2 f''(\theta) - x(t-\sigma(x_t))\frac{\partial}{\partial t}\sigma(x_t).$$

But with (3) we conclude that

$$\frac{\partial}{\partial t}\sigma(x_t) = \varepsilon\big(\|x_t\|_{[-\sigma_0,0]}\big) \tag{9}$$

so equation (1) is equivalent, at ∞, to a nonlinear neutral functional differential equation

$$\frac{d}{dt}[x(t) + G(x_t)] = -px(t-\sigma_0) + H(x_t), \tag{10}$$

where

$$G(\psi) = p\int\limits_{-\sigma_0}^{-\sigma(\psi)} \psi(s)ds$$

and

$$H(\psi) = \varepsilon(\psi).$$

The linearized neutral functional differential equation associaded with equation (10) is of the form

$$\begin{gathered}\frac{d}{dt}\big[x(t) + G(x_t)\big] = -px(t-\sigma_0),\\ x_0 = \phi.\end{gathered} \tag{11}$$

Our first aim in this paper is to prove that equation (1) has the same oscillatory character as the neutral equation (11).

In [2] and [3] the authors discuss existence of slowly oscillating periodic solutions for equation of type (l). Necessary and sufficient conditions for oscillations of the solutions of equation of neutral type in the case of a delay that depends on time are obtained in [4–6].

In this section we deal with the neutral equation in which the delay depends on the state and give necessary and sufficient conditions for every solution of equation (11) to oscillate.

Before we present the oscillations results, we establish the following theorems.

Theorem 2. *Every nonoscillatory solution of equation (11) goes to zero at infinity.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (11), and suppose that $x(t) > 0$. Put $z(t) = x(t) + G(x_t)$. Then

$$z'(t) = -px(t - \sigma_0), \tag{12}$$

$z'(t) < 0$, so $z(t)$ is a decreasing function. We claim that $x(t)$ is bounded. Otherwise there exists a sequence of points t_n such that $\lim\limits_{n\to\infty} t_n = \infty$, $\lim\limits_{n\to\infty} x(t_n) = \infty$, and $x(t_n) = \max\limits_{s\le t_n} x(s)$.

Then

$$z(t_n) = x(t_n) + G(x_{t_n}) = x(t_n) + p \int\limits_{-\sigma_0}^{-\sigma(x_{t_n})} x(t_n + \theta)d\theta.$$

Because $\sigma(x_{t_n}) \le \sigma_0$ and $x(t)$ is decreasing,

$$z(t_n) \ge x(t_n) + px(t_n - \sigma_0)(\sigma_0 - \sigma(x_{t_n})) \ge x_{t_n},$$

which implies that $z(t_n)$ goes to ∞ as $n \to \infty$. This contradicts the fact that $z(t)$ is decreasing. Thus $x(t)$ is bounded and so

$$l = \lim_{t\to\infty} z(t) \in \Re.$$

By integrating both sides of (12) from t_1 to ∞, with t sufficiently large, we find

$$p \int\limits_{t_1}^{\infty} x(s - \sigma_0)ds = z(t_1) - l.$$

This implies that $\lim\limits_{t\to\infty} \inf x(t) = 0$, if we write

$$z(t) = x(t) + P(t)x(t - \sigma_0),$$

where

$$P(t) = \frac{G(x_t)}{x(t - \sigma_0)} = \frac{p \int\limits_{-\sigma_0}^{-\sigma(x_t)} x(t + \theta)d\theta}{x(t - \sigma_0)}.$$

We verify that $P(t) \leq p$ and show that $l = 0$. Consider a sequence of points t_n such that

$$\lim_{n\to\infty} t_n = \infty, \quad \text{and} \quad \lim_{n\to\infty} x(t_n) = 0,$$

$$z(t_n) = x(t_n) + P(t_n)x(t_n - \sigma_0),$$

$$z(t_n + \sigma_0) - z(t_n) = x(t_n + \sigma_0) + P(t_n + \sigma_0)x(t_n) - x(t_n) - P(t_n)x(t_n - \sigma_0) =$$

$$= x(t_n + \sigma_0) + \big[P(t_n + \sigma_0) - 1\big]x(t_n) - P(t_n)x(t_n - \sigma_0).$$

Thus

$$\lim_{t\to\infty} x(t_n + \sigma_0) - P(t_n)x(t_n - \sigma_0) = 0, \quad 0 < x(t_n + \sigma_0) < x(t_n).$$

Hence, $\lim_{t\to\infty} x(t_n + \sigma_0) = 0$, and $\lim_{t\to\infty} P(t_n)x(t_n - \sigma_0) = 0$, so

$$l = \lim_{n\to\infty} z(t_n) = 0,$$

and thus $\lim_{t\to\infty} (x(t) + G(x_t)) = 0$. Let $\varepsilon > 0$ be given. Then, for t sufficiently large, it follows that $0 < x(t) < -G(x_t) + \varepsilon < \varepsilon$. We conclude that

$$\lim_{t\to\infty} x(t) = 0.$$

Theorem 3. *Suppose that $p\sigma_0 e > 1$. Then the solution of equation (11) oscillates.*

Proof. Assume that equation (11) has a nonoscillatory solution $x(t)$. We assume that $x(t)$ is eventually positive. By Theorem 1, $x(t)$ goes to 0 as t goes to ∞ and $G(x_t)$ goes to zero too.

We integrate both sides of (11) from t to ∞, for t sufficiently large. We have

$$x(t) + G(x_t) = -p \int_t^\infty x(s - \sigma_0)ds. \tag{13}$$

Set

$$w(t) = p \int_t^\infty x(s - \sigma_0)ds. \tag{14}$$

Then

$$w'(t) = -px(t - \sigma_0).$$

Eventually $w(t) > 0$ and $w'(t) < 0$. We also have

$$-px(t) = w'(t + \sigma_0). \tag{15}$$

By substituting (14) and (15) into (13) we obtain, for t sufficiently large,

$$w'(t + \sigma_0) - pG(x_t) + pw(t) = 0.$$

Then

$$w'(t) - pG(x_{t-\sigma_0}) - pw(t - \sigma_0) = 0.$$

$$G(x_t + \sigma_0) > 0$$

implies that

$$w'(t) - pw(t - \sigma_0) \geq 0, \quad w(t) > 0. \tag{16}$$

Then the differential inequality (16) has an eventually positive solution. We deduce that the differential equation

$$w'(t) - pw(t - \sigma_0) = 0 \tag{17}$$

also has an eventually positive solution. This contradicts the fact that every solution of equation (17) oscillates, because $p\sigma_0 e > 1$. The proof of Theorem 3 is complete.

4. Necessary and sufficient conditions for the oscillations of (1).

Theorem 4. *Assume that (3) holds. Then every solution of equation (1) oscillates if every solution of the linearized equation (11) oscillates.*

Proof. Assume for the sake of contradiction, that equation (1) has a nonoscillatory solution $x(t)$. We assume that $x(t)$ is eventually positive. By Lemma 1, we know that $\lim\limits_{t\to+\infty} x(t) = 0$. Thus by (5)

$$\lim_{u\to+\infty} \inf \frac{f(x(t - \sigma(x_t))}{x(t - \sigma(x_t))} \geq 1.$$

Let $\varepsilon \in (0, 1)$. Then there exists a $T(\varepsilon)$ such that for every $t > T(\varepsilon)$, $x(t - \sigma(x_t)) > 0$ and $f(x(t - \sigma(x_t)) \geq (1 - \varepsilon)x(t - \sigma(x_t))$.

Hence from equation (1),

$$x'(t) + (1 - \varepsilon)x(t - \sigma(x_t)) \leq 0 \quad \text{for} \quad t \geq T(\varepsilon). \tag{18}$$

Substituting the term $x(t - \sigma(x_t))$, by formula (8), we obtain

$$\left(1 - \frac{\partial}{\partial t}\sigma(x_t)\right) x(t - \sigma(x_t)) = x(t - \sigma_0) + \frac{\partial}{\partial t}\left(\int\limits_{-\sigma_0}^{-\sigma(x_t)} x(t + \theta)d\theta\right)$$

and so, by (8),

$$x(t-\sigma(x_t)) = \left(1+\frac{\partial}{\partial t}\sigma(x_t)+o(t)\right)x(t-\sigma_0)+\left(1+\frac{\partial}{\partial t}\sigma(x_t)+o(t)\right)\frac{1}{p}\frac{\partial}{\partial t}G(x_t).$$

Then the inequality (18) becomes

$$x'(t)+\frac{1-\varepsilon}{p}\left(1+\frac{\partial}{\partial t}\sigma(x_t)+o(t)\right)px(t-\sigma_0)+\frac{1-\varepsilon}{p}\left(1+\frac{\partial}{\partial t}\sigma(x_t)+o(t)\right)\frac{\partial}{\partial t}G(x_t)\leq 0.$$

This inequality is valid for any $\varepsilon \in (0,1)$, and $t > T(\varepsilon)$. Since x_t is continuous, $\frac{\partial}{\partial t}\sigma(x_t) \to 0$ and

$$\big[x(t)+G(x_t)\big]'+px(t-\sigma_0)\leq 0 \quad \text{for} \quad t\geq T(\varepsilon). \tag{19}$$

Inequality (19) has an eventually positive solution. Then equation (9) also has an eventually positive solution. This contradicts the hypothesis that every solution of equation (11) is oscillatory. This ends the proof.

Conversely we have the following.

Theorem 5. *Suppose that (6) holds. If the solution of equation (l) is oscillatory, then* $p\sigma_0 e > 1$.

We assume that $p\sigma_0 e > 1$. Then equation $y'(t)+py(t-\sigma_0) = 0$ has an eventually positive solution $y(t)$.

We know that $\lim\limits_{t\to+\infty} y(t) = 0$ and so there exists a t_0 such that $0 < y(t) < \delta$ for $t_0-\sigma_0 < < t < t_0$.

With the initial function equal to $y(t)$ for $t_0-\sigma_0 < t < t_0$, equation (1) has a solution $x(t)$ which exists at least in some small heighbourhood to the right of t_0. It suffices to show that for as long as $x(t)$ exists,

$$y(t)\leq x(t)<\delta$$

and then $x(t) > 0$, which is a contradiction.

For $t_0 < t < t_0+\sigma_0$, $0 \leq x(t-\sigma(x_t)) < \delta$, and so we have, by (6),

$$x'(t) = -f(x(t-\sigma(x_t))\geq -px(t-\sigma(x_t))\geq -px(t-\sigma_0).$$

So

$$x'(t)+px(t-\sigma_0)\geq 0$$

and

$$y'(t)+py(t-\sigma_0) = 0.$$

By a comparison result for the positive solution of the delay differential inequalities [6],

$$0 < y(t) \leq x(t),$$

$x(t) > 0$, which is a contradiction to the fact that if $x(t)$ oscillates, then $p\sigma_0 e > 1$.

Theorem 6. *Suppose that (5), (6) hold. Then the solution of (1) is oscillating if and only if* $p\sigma_0 e > 1$.

By combining Theorem 3, Theorem 4, and Theorem 5, we have the needed result. This end the proof.

We derive the following lemma.

Lemma 3. *Assume that the conditions (5) and (6) are fulfilled, and σ is bounded by a positive number $\tau > 0$. Then, every solution of equation (1) is oscillating if and only if $p\tau e > 1$.*

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