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SECOND ORDER NONLINEAR CAUCHY PROBLEMS IN A FOUR DIMENSIONAL SPACE

НЕЛІНІЙНІ ЗАДАЧІ КОШІ ДРУГОГО ПОРЯДКУ В ЧОТИРИВИМІРНОМУ ПРОСТОРІ

A. Di Bartolomeo

Università degli Studi and INFN, Salerno, Italy
e-mail: Dibant@sa.infn.it

J. Quartieri

Università degli Studi and INFN, Salerno, Italy
e-mail: Quartieri@sa.infn.it

S. Steri

Università degli Studi "Federico II", Naples, Italy
e-mail: Steri@unina.it

In this paper a general second order Cauchy problem in four dimensional space is considered and sufficient conditions for integrability with generalized Lie series method are determined.

Розглянуто загальну задачу Коші другого порядку в чотиривимірному просторі і знайдено достатні умови її інтегровності методом узагальнених рядів Лі.

Introduction. Let us consider in a 4-dimensional space a problem in an implicit form such as the following:

$$F\left(t, x, y, z, \frac{\partial^2 P}{\partial x^2}, \frac{\partial^2 P}{\partial y^2}, \frac{\partial^2 P}{\partial z^2}, \frac{\partial^2 P}{\partial t^2}\right) = 0, \tag{1}$$

$$P(t=0, x, y, z) = \sum_{h,k,l=0}^{+\infty} a_{0,h,k,l} x^h y^k z^l,$$

$$\frac{\partial}{\partial t} P(t, x, y, z)\Big|_{t=0} = \sum_{h,k,l=0}^{+\infty} a_{1,h,k,l} x^h y^k z^l$$

with F, in general, a nonlinear function. In this paper, we will find a solution of such a problem by using the general technique of Lie series. Equation (1) can be considered as a generalization of the Klein – Gordon equation of a relativistic particle in a force field.

The regularization. The first step to solve problem (1) is to make it regular, which means to explicate the equation with respect to the time derivative. Let us consider in the multidimensional space C^8, where C is the complex numbers field, the surface

$$F(X_1, X_2, \dots, X_8) = 0 \tag{2}$$

with$(x_1, x_2, \dots, x_8)$ a point such that

$$\left[\frac{\partial F}{\partial X_8}\right]_{\substack{X_1=x_1\\ X_2=x_2\\ \cdots\\ X_8=x_8}} \neq 0.$$

We will prove that this condition ensures that

$$X_8 = G(X_1, X_2, \dots, X_7) \tag{3}$$

with

$$G = \sum_{\mu_1,\dots,\mu_7=0}^{+\infty} \frac{(X_1-x_1)^{\mu_1}\dots(X_7-x_7)^{\mu_7}}{\mu_1!\dots\mu_7!}\left[D_1^{\mu_1}\dots D_7^{\mu_7}\pi_8\right]_{\substack{\pi_1=x_1\\ \pi_2=x_2\\ \cdots\\ \pi_8=x_8}},$$

where

$$\begin{aligned} D_1 &= \frac{\partial}{\partial\pi_1} - \frac{F_{\pi_1}(\pi_1,\dots,\pi_8)}{F_{\pi_8}(\pi_1,\dots,\pi_8)}\frac{\partial}{\partial\pi_8},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ D_7 &= \frac{\partial}{\partial\pi_7} - \frac{F_{\pi_7}(\pi_1,\dots,\pi_8)}{F_{\pi_8}(\pi_1,\dots,\pi_8)}\frac{\partial}{\partial\pi_8}, \end{aligned} \tag{4}$$

and $F_{\pi_i}(\pi_1,\dots,\pi_8) \equiv \dfrac{\partial F(\pi_1,\dots,\pi_8)}{\partial\pi_i}$, $i = 1,\dots,8$. $D_1 \dots D_7$ are the commuting Groebner's operators [1–5], having the property that

$$D_1F(\pi_1,\dots,\pi_8) = \dots = D_7F(\pi_1,\dots,\pi_8) = 0. \tag{5}$$

To prove our assertion it is sufficient to observe that, being

$$\left[e^{(X_1-x_1)D_1}\dots e^{(X_7-x_7)D_7}\pi_j\right]_{\substack{\pi_1=x_1\\ \pi_2=x_2\\ \cdots\\ \pi_8=x_8}} = X_j, \qquad j = 1,\dots,7, \tag{6}$$

if we add

$$\left[e^{(X_1-x_1)D_1}\dots e^{(X_7-x_7)D_7}\pi_8\right]_{\substack{\pi_1=x_1\\ \pi_2=x_2\\ \cdots\\ \pi_8=x_8}} = X_8, \tag{7}$$

we obtain

$$\Big[e^{(X_1-x_1)D_1}\dots e^{(X_7-x_7)D_7}F(\pi_1,\dots,\pi_8)\Big]_{\substack{\pi_1=x_1\\ \pi_2=x_2\\ \dots\\ \pi_8=x_8}} =$$

$$= F\Big(e^{(X_1-x_1)D_1}\dots e^{(X_7-x_7)D_7}\pi_1,\dots,e^{(X_1-x_1)D_1}\dots e^{(X_7-x_7)D_7}\pi_8\Big) =$$

$$= F(X_1,\dots,X_8). \tag{8}$$

The first equality follows from the exchange theorem [6], the second one holds by (6) and (7).

The integration. Problem (l) can now be reformulated in its regular form,

$$\frac{\partial^2 P}{\partial t} = G\left(t,x,y,z,\frac{\partial^2 P}{\partial x^2},\frac{\partial^2 P}{\partial y^2},\frac{\partial^2 P}{\partial z^2}\right),$$

$$P(t=0,x,y,z) = \sum_{h,k,l=0}^{+\infty} a_{0,h,k,l}x^h y^k z^l, \tag{9}$$

$$\frac{\partial}{\partial t}P(t,x,y,z)\Big|_{t=0} = \sum_{h,k,l=0}^{+\infty} a_{1,h,k,l}x^h y^k z^l,$$

where G has been previously introduced.

Problem (9) can be further written as an autonomous (i. e. time independent) evolution problem,

$$\frac{\partial P}{\partial t} = P_1, \tag{10}$$

$$\frac{\partial P_1}{\partial t} = G\left(\tau,x,y,z,\frac{\partial^2 P}{\partial x^2},\frac{\partial^2 P}{\partial y^2},\frac{\partial^2 P}{\partial z^2}\right), \tag{11}$$

$$\frac{\partial \tau}{\partial t} = 1, \quad \tau(0) = 0, \tag{12}$$

$$P(0,x,y,z) = \sum_{h,k,l=0}^{+\infty} a_{0,h,k,l}x^h y^k z^l, \tag{13}$$

$$P_1(0,x,y,z) = \sum_{h,k,l=0}^{+\infty} a_{1,h,k,l} x^h y^k z^l. \tag{14}$$

To apply the Groebner's method, we need to transform problem (10)–(14) in an equivalent initial value problem for a system of first order differential equations.

This can be achieved by the Taylor transform with a nonsingular initial point, e.g. ($x = 0$, $y = 0$, $z = 0$). Equations (10) and (11) will give infinite equations, which can be written as follows (the upper index indicates that all derivatives are calculated at the initial point ($x = 0$, $y = 0$, $z = 0$), while the derivation variable is specified in the lower index by a number in the position corresponding to (τ, x, y, z), this number indicating the order of the derivation):

$$\begin{aligned}
&\frac{\partial P^0_{1100}}{\partial t} = \Theta_{1100}\left(\tau, P^0_{0300}, P^0_{0120}, P^0_{0102}\right),\\
&\frac{\partial P^0_{1000}}{\partial t} = \Theta_{1000}\left(\tau, P^0_{0200}, P^0_{0020}, P^0_{0002}\right),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{\partial P^0_{1hkl}}{\partial t} = \Theta_{1hkl}\left(\tau, P^0_{0(2+h)kl}, P^0_{0h(2+k)l}, P^0_{0hk(2+l)}\right),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial P^0_{0000}}{\partial t} = \Theta_{0000} = P^0_{1000},\\
&\frac{\partial P^0_{0100}}{\partial t} = \Theta_{0100} = P^0_{1100},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{\partial P^0_{0hkl}}{\partial t} = \Theta_{0hkl} = P^0_{1hkl},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{\partial \tau}{\partial t} = 1,
\end{aligned}$$

while the initial conditions are

$$P^0_{1hkl}(0) = a_{1hkl},$$

$$P^0_{01hkl}(0) = a_{0hkl},$$

$$\tau(0) = 0,$$

$$h, k, l \in N_0.$$

Then if we construct the following noncommuting Groebner's operators:

$$D_0 = \sum_{h,k,l=0}^{+\infty} \Theta_{0hkl} \frac{\partial}{\partial \pi_{0,h,k,l}},$$

$$D_1 = \sum_{h,k,l=0}^{+\infty} \Theta_{1hkl} \frac{\partial}{\partial \pi_{0,h,k,l}},$$

where the finite coefficients are now depending on parameters named π, the solution of the above initial value system is

$$P^0_{0hkl} = \left[e^{t(D_0+D_1)}\pi_{0hkl}\right]_{\substack{\pi_{0hkl}=a_{0hkl}\\ \pi_{1hkl}=a_{1hkl}}},$$

$$P^0_{1hkl} = \left[e^{t(D_0+D_1)}\pi_{1hkl}\right]_{\substack{\pi_{0hkl}=a_{0hkl}\\ \pi_{1hkl}=a_{1hkl}}}.$$

Therefore the solution of the Cauchy problem is

$$P = \sum_{hkl=0}^{+\infty} P^0_{0hkl} x^h y^k z^l.$$

Conclusions. Groebner's approach is a very suitable tool to integrate both linear and nonlinear second order Cauchy problems in a four dimensional space. Hence it is also very useful in solving problems which arise in particle physics and quantum field theory.

In this paper we have established a sufficient condition to regularize an assigned problem in implicit form. The equivalence to a system of two evolution equations has allowed to apply the generalized Lie series method.

We plan to further extend this subject.

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