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EXISTENCE, UNIQUENESS, AND ASYMPTOTIC STABILITY FOR A THERMOELASTIC PLATE*

ІСНУВАННЯ, ЄДИНІСТЬ ТА АСИМПТОТИЧНА СТІЙКІСТЬ РОЗВ'ЯЗКУ ДЛЯ ТЕРМОЕЛАСТИЧНОЇ ПЛАТІВКИ

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In this note we are concerned with the linear theory of a thermoelastic plate when a rate-type equation is assumed for the heat flux. We consider an initial boundary-value problem for this plate and show the existence, uniqueness, and asymptotic stability of the solution. Thermodynamic restrictions on the assumed constitutive equations are also derived. Finally, we give the expression of a pseudo free energy.

Розглядається лінійна теорія для термоеластичної платівки за умови, що тепловий потік задовольняє рівняння швидкісного типу. Доведено існування, єдиність та асимптотичну стійкість розв'язку граничної задачі з початковими умовами. Знайдено термодинамічні обмеження на рівняння задачі. Також наведено вираз для псевдовільної енергії.

1. Introduction. Many authors have recently considered the thermoelastic model of a thin plate and have studied, in particular, the possibility that the solutions of the thermoelastic plate equations with Dirichlet or Neumann boundary conditions decay exponentially to zero as time goes to infinity [1 – 8].

In [9] analogous problems have been investigated for a thermoelastic plate model characterized by the presence of memory effects on the heat flux vector. Results about existence, uniqueness, and asymptotic stability of the solutions for an initial boundary-value problem have been derived as a consequence of the dissipation properties of the material; moreover, the exponential decay rate of the energy is proved with suitable multiplicative techniques.

In this work we consider the constitutive relation for the heat flux vector proposed by Cattaneo [10 – 12] and examine the modified system of equations which describe the linear theory of a thin thermoelastic plate. Thus, we prove the existence, uniqueness, and asymptotic stability of the solution to the initial boundary-value problem corresponding to homogeneous conditions, under suitable hypotheses on the sources we have introduced in the equations. Finally, in the last section, we derive the restrictions placed by the thermodynamic principles on the physical constants and give the expression of a pseudo free energy.

2. Basic equations and position of the problem. We consider a homogeneous, isotropic, thermoelastic plate with a thin uniform thickness. The middle surface between its faces, denoted by Ω, is a bounded and regular domain of the Euclidean two-dimensional space $\mathbf{R}^2$ with smooth boundary Γ. We are concerned with small deformations and small variations of the temperature referred to a fixed reference configuration with a uniform absolute temperature Θ_0.

Let us denote by ϑ the mean variation of the temperature on the cross section of the plate and by u the vertical deflection in the place $\mathbf{x} \in \Omega$ at time $t \in \mathbf{R}^+$; within the linear approxi-

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mation theory, small vibrations of the thin thermoelastic plate are described by the following equations:

$$u_{tt}(\mathbf{x},t) - \gamma\Delta u_{tt}(\mathbf{x},t) + \Delta^2 u(\mathbf{x},t) + \alpha\Delta\vartheta(\mathbf{x},t) = f(\mathbf{x},t), \tag{2.1}$$

$$\rho\vartheta_t(\mathbf{x},t) - \alpha\Delta u_t(\mathbf{x},t) + \nabla\cdot\mathbf{q}(\mathbf{x},t) = g(\mathbf{x},t), \tag{2.2}$$

to which we must add the relation between the heat flux $\mathbf{q}$ and the temperature gradient $\nabla\vartheta$. Our purpose is to consider the effects of a change of Fourier's law; therefore, we assume the Cattaneo – Maxwell equation

$$\tau\mathbf{q}_t(\mathbf{x},t) + \mathbf{q}(\mathbf{x},t) + k\nabla\vartheta(\mathbf{x},t) = \mathbf{l}(\mathbf{x},t). \tag{2.3}$$

In these equations we have introduced the sources f, g and $\mathbf{l}$, which must be considered as known functions of $(\mathbf{x},t) \in \Omega\times\mathbf{R}^+$; $\tau, k, \rho, \alpha, \gamma$ are physical and constant parameters such that

$$\tau > 0, \quad k > 0, \quad \rho > 0, \quad \alpha \neq 0, \quad \gamma \geq 0. \tag{2.4}$$

To investigate an initial boundary-value problem for the thermoelastic plate, we must consider the initial and boundary conditions, which are assumed homogeneous and are expressed by

$$u(\mathbf{x},0) = 0,\ u_t(\mathbf{x},0) = 0,\ \vartheta(\mathbf{x},0) = 0,\ \mathbf{q}(\mathbf{x},0) = \mathbf{0} \qquad \forall\mathbf{x}\in\Omega, \tag{2.5}$$

$$u(\mathbf{x},t) = 0,\ \frac{\partial u(\mathbf{x},t)}{\partial\nu} = \mathbf{0},\ \vartheta(\mathbf{x},t) = 0 \qquad \forall(\mathbf{x},t)\in\Gamma\times\mathbf{R}^+, \tag{2.6}$$

where $\nu = (\nu_1,\nu_2)$ is the external unit normal to Γ.

3. Existence, uniqueness, and asymptotic stability. In order to give a compact definition of solution of the initial boundary-value problem (2.1) – (2.3) with (2.5), (2.6), we introduce the following functional spaces:

$$\mathcal{H}_0^2(\Omega) = \left\{u\in H^2(\Omega) : u = 0,\ \frac{\partial u}{\partial\nu} = \mathbf{0} \quad \forall\mathbf{x}\in\Gamma\right\},$$

$$\mathcal{U}(\Omega,\mathbf{R}^+) = H^1(\mathbf{R}^+; H^1(\Omega)) \cap L^2(\mathbf{R}^+;\mathcal{H}_0^2(\Omega)),$$

$$\mathcal{T}(\Omega,\mathbf{R}^+) = L^2(\mathbf{R}^+; H_0^1(\Omega)), \qquad \mathcal{Q}(\Omega,\mathbf{R}^+) = L^2(\mathbf{R}^+; L^2(\Omega)),$$

$$\mathcal{F}(\Omega,\mathbf{R}^+) = H^1(\mathbf{R}^+; L^2(\Omega)) \cap \mathcal{T}(\Omega,\mathbf{R}^+),$$

$$\mathcal{V}(\Omega,\mathbf{R}^+) = H^1(\mathbf{R}^+; L^2(\Omega)) \cap L^2(\mathbf{R}^+; H^1(\Omega)).$$

Definition 3.1. *A triplet* $(u, \vartheta, \mathbf{q}) \in \mathcal{U}(\Omega, \mathbf{R}^+) \times \mathcal{T}(\Omega, \mathbf{R}^+) \times \mathcal{Q}(\Omega, \mathbf{R}^+)$ *is said to be a weak solution to the problem (2.1) – (2.3) with (2.5) – (2.6) and sources* $(f, g, \mathbf{l}) \in [\mathcal{Q}(\Omega, \mathbf{R}^+)]^3$ *if the following identity*

$$\int_0^{+\infty}\int_\Omega [-u_t(\mathbf{x},t)v_t(\mathbf{x},t) - \gamma\nabla u_t(\mathbf{x},t)\cdot\nabla v_t(\mathbf{x},t) + \Delta u(\mathbf{x},t)\Delta v(\mathbf{x},t) -$$

$$- \alpha\nabla\vartheta(\mathbf{x},t)\cdot\nabla v(\mathbf{x},t) - \rho\vartheta(\mathbf{x},t)\phi_t(\mathbf{x},t) + \alpha\nabla u_t(\mathbf{x},t)\cdot\nabla\phi(\mathbf{x},t) - \mathbf{q}(\mathbf{x},t)\cdot$$

$$\cdot\nabla\phi(\mathbf{x},t) - \tau\mathbf{q}(\mathbf{x},t)\cdot\mathbf{p}_t(\mathbf{x},t) + \mathbf{q}(\mathbf{x},t)\cdot\mathbf{p}(\mathbf{x},t) - k\vartheta(\mathbf{x},t)\nabla\cdot\mathbf{p}(\mathbf{x},t)]d\mathbf{x}dt =$$

$$= \int_0^{+\infty}\int_\Omega [f(\mathbf{x},t)v(\mathbf{x},t) + g(\mathbf{x},t)\phi(\mathbf{x},t) + \mathbf{l}(\mathbf{x},t)\cdot\mathbf{p}(\mathbf{x},t)]d\mathbf{x}dt \tag{3.1}$$

is satisfied for every triplet $(v, \phi, \mathbf{p}) \in \mathcal{U}(\Omega, \mathbf{R}^+) \times \mathcal{F}(\Omega, \mathbf{R}^+) \times \mathcal{V}(\Omega, \mathbf{R}^+)$.

To study the existence and uniqueness of the solution we identify any function $w : \mathbf{R}^+ \to \mathbf{R}^n$ with its causal extension on $\mathbf{R}$ and introduce the time-Fourier transform $\hat{w}$. We remember that if w and $\hat{w}$ belong to $L^2(\mathbf{R})$ then also the Fourier transforms $\hat{w}$ and $\hat{w}'$ are L^2-functions. Thus, we have

$$\hat{w}(\omega) = \int_{-\infty}^{+\infty} w(s)e^{-i\omega s}ds, \quad \hat{w}'(\omega) = i\omega\hat{w}(\omega) - w(0), \quad w(0) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\hat{w}(\omega)d\omega. \tag{3.2}$$

Because of the isomorphisms which exist between each functional space, we have introduced, and the corresponding space of the Fourier transforms of its functions, denoted with a circumflex ^ , our problem can be transformed as follows.

Plancherel's theorem applied to (3.1), taking account of $(3.2)_2$ where the initial data are zero both for the solutions and for the text functions, yields

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_\Omega \{-i\omega\hat{u}(\mathbf{x},\omega)[i\omega\hat{v}(\mathbf{x},\omega)]^* - i\omega\gamma\nabla\hat{u}(\mathbf{x},\omega)\cdot[i\omega\nabla\hat{v}(\mathbf{x},\omega)]^* +$$

$$+ \Delta\hat{u}(\mathbf{x},\omega)[\Delta\hat{v}(\mathbf{x},\omega)]^* - \alpha\nabla\hat{\vartheta}(\mathbf{x},\omega)\cdot[\nabla\hat{v}(\mathbf{x},\omega)]^* - \rho\hat{\vartheta}(\mathbf{x},\omega)[i\omega\hat{\phi}(\mathbf{x},\omega)]^* +$$

$$+ i\omega\alpha\nabla\hat{u}(\mathbf{x},\omega)\cdot[\nabla\hat{\phi}(\mathbf{x},\omega)]^* - \hat{\mathbf{q}}(\mathbf{x},\omega)\cdot[\nabla\hat{\phi}(\mathbf{x},\omega)]^* - \tau\hat{\mathbf{q}}(\mathbf{x},\omega)\cdot[i\omega\hat{\mathbf{p}}(\mathbf{x},\omega)]^* +$$

$$+\hat{\mathbf{q}}(\mathbf{x},\omega)\cdot\hat{\mathbf{p}}^*(\mathbf{x},\omega) - k\hat{\vartheta}(\mathbf{x},\omega)[\nabla\cdot\hat{\mathbf{p}}(\mathbf{x},\omega)]^*\Big\}d\mathbf{x}d\omega =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\Omega} [\hat{f}(\mathbf{x},\omega)\hat{v}^*(\mathbf{x},\omega) + \hat{g}(\mathbf{x},\omega)\hat{\phi}^*(\mathbf{x},\omega) + \hat{\mathbf{l}}(\mathbf{x},\omega)\cdot\hat{\mathbf{p}}^*(\mathbf{x},\omega)]d\mathbf{x}d\omega, \tag{3.3}$$

where * denotes the complex conjugate.

We can now choose $\hat{v}(\mathbf{x},\omega) = v_1(\mathbf{x})v_2(\omega)$, $\hat{\phi}(\mathbf{x},\omega) = \phi_1(\mathbf{x})\phi_2(\omega)$, $\hat{\mathbf{p}}(\mathbf{x},\omega) = \mathbf{p}_1(\mathbf{x})p_2(\omega)$. The arbitrariness of (v_2,ϕ_2,p_2) allows us to derive from (3.3) the following identity:

$$\int_{\Omega} \Big\{ -\omega^2\hat{u}(\mathbf{x},\omega)v_1^*(\mathbf{x}) - \omega^2\gamma\nabla\hat{u}(\mathbf{x},\omega)\cdot\nabla v_1^*(\mathbf{x}) + \Delta\hat{u}(\mathbf{x},\omega)\Delta v_1^*(\mathbf{x}) - \alpha\nabla\hat{\vartheta}(\mathbf{x},\omega)\cdot$$

$$\cdot\nabla v_1^*(\mathbf{x}) + i\omega\rho\hat{\vartheta}(\mathbf{x},\omega)\phi_1^*(\mathbf{x}) + i\omega\alpha\nabla\hat{u}(\mathbf{x},\omega)\cdot\nabla\phi_1^*(\mathbf{x}) - \hat{\mathbf{q}}(\mathbf{x},\omega)\cdot\nabla\phi_1^*(\mathbf{x}) +$$

$$+ i\omega\tau\hat{\mathbf{q}}(\mathbf{x},\omega)\cdot\mathbf{p}_1^*(\mathbf{x}) + \hat{\mathbf{q}}(\mathbf{x},\omega)\cdot\mathbf{p}_1^*(\mathbf{x}) - k\hat{\vartheta}(\mathbf{x},\omega)\nabla\cdot\mathbf{p}_1^*(\mathbf{x})\Big\} d\mathbf{x} =$$

$$= \int_{\Omega} [\hat{f}(\mathbf{x},\omega)v_1^*(\mathbf{x}) + \hat{g}(\mathbf{x},\omega)\phi_1^*(\mathbf{x}) + \hat{\mathbf{l}}(\mathbf{x},\omega)\cdot\mathbf{p}_1^*(\mathbf{x})]d\mathbf{x}. \tag{3.4}$$

We observe that the problem (2.1) – (2.3) with (2.5), (2.6), in terms of Fourier's transforms, is expressed by the system

$$-\omega^2\hat{u}(\mathbf{x},\omega) + \gamma\omega^2\Delta\hat{u}(\mathbf{x},\omega) + \Delta^2\hat{u}(\mathbf{x},\omega) + \alpha\Delta\hat{\vartheta}(\mathbf{x},\omega) = \hat{f}(\mathbf{x},\omega), \tag{3.5}$$

$$i\omega\rho\hat{\vartheta}(\mathbf{x},\omega) - i\omega\alpha\Delta\hat{u}(\mathbf{x},\omega) + \nabla\cdot\hat{\mathbf{q}}(\mathbf{x},\omega) = \hat{g}(\mathbf{x},\omega), \tag{3.6}$$

$$i\omega\tau\hat{\mathbf{q}}(\mathbf{x},\omega) + \hat{\mathbf{q}}(\mathbf{x},\omega) + k\nabla\hat{\vartheta}(\mathbf{x},\omega) = \hat{\mathbf{l}}(\mathbf{x},\omega) \qquad \forall\mathbf{x}\in\Omega, \tag{3.7}$$

$$\hat{u}(\mathbf{x},\omega) = 0, \qquad \frac{\partial\hat{u}(\mathbf{x},\omega)}{\partial\nu} = \mathbf{0}, \qquad \hat{\vartheta}(\mathbf{x},\omega) = 0 \qquad \forall\mathbf{x}\in\Gamma. \tag{3.8}$$

The dependence on $\mathbf{x}$ is sometimes understood and not written.

Thus, we can give the following definition.

Definition 3.2. *A triplet* $(\hat{u}(\omega),\hat{\vartheta}(\omega),\hat{\mathbf{q}}(\omega)) \in \mathcal{H}_0^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)$ *with* $\omega\in\mathbf{R}$ *is called a weak solution to the problem (3.5) – (3.8) with* $(\hat{f}(\omega),\hat{g}(\omega),\hat{\mathbf{l}}(\omega)) \in [L^2(\Omega)]^3$ *if it satisfies (3.4) for every triplet* $(v_1,\phi_1,\mathbf{p}_1) \in \mathcal{H}_0^2(\Omega)\times H_0^1(\Omega)\times H^1(\Omega)$.

Let us put

$$\mathcal{I}(\omega) = \int_{\Omega} \left[\omega^2\left(|\hat{u}|^2 + |\nabla\hat{u}|^2\right) + |\Delta\hat{u}|^2 + \left|\hat{\vartheta}\right|^2 + \left|\nabla\hat{\vartheta}\right|^2 + |\hat{\mathbf{q}}|^2\right] d\mathbf{x}; \tag{3.9}$$

we have the following result.

Theorem 3.1. *If* $(\hat{u}(\omega), \hat{\vartheta}(\omega), \hat{\mathbf{q}}(\omega)) \in \mathcal{H}_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ *is a weak solution to the problem (3.5) – (3.8) with* $(\hat{f}(\omega), \hat{g}(\omega), \hat{\mathbf{l}}(\omega)) \in [L^2(\Omega)]^3$*, then there exists a positive coefficient* $\delta(\omega)$*, depending on the material constants and* Ω*, such that*

$$\mathcal{I}(\omega) \le \delta^2(\omega) \int_\Omega \left(\left|\hat{f}\right|^2 + |\hat{g}|^2 + \left|\hat{\mathbf{l}}\right|^2 \right) d\mathbf{x}, \tag{3.10}$$

where $\omega \in \mathbf{R}$.

Proof. Let us consider the system (3.5) – (3.7), where we first suppose $\omega \neq 0$.

Multiplying (3.5) by $\hat{u}^*$ we get a relation where we can integrate by parts taking account of the boundary conditions (3.8); thus, it assumes the following form

$$-\omega^2 \left(\int_\Omega |\hat{u}|^2 \, d\mathbf{x} + \gamma \int_\Omega |\nabla \hat{u}|^2 \, d\mathbf{x} \right) + \int_\Omega |\Delta \hat{u}|^2 \, d\mathbf{x} - \alpha \int_\Omega \nabla \hat{\vartheta} \cdot \nabla \hat{u}^* d\mathbf{x} = \int_\Omega \hat{f} \hat{u}^* d\mathbf{x}. \tag{3.11}$$

Analogously, from (3.6) multiplied by $\hat{u}^*$ and $\hat{\vartheta}^*$ we get

$$i\omega \left(\rho \int_\Omega \hat{\vartheta} \hat{u}^* d\mathbf{x} + \alpha \int_\Omega |\nabla \hat{u}|^2 \, d\mathbf{x} \right) - \int_\Omega \hat{\mathbf{q}} \cdot \nabla \hat{u}^* d\mathbf{x} = \int_\Omega \hat{g} \hat{u}^* d\mathbf{x}, \tag{3.12}$$

$$i\omega \left(\rho \int_\Omega \left|\hat{\vartheta}\right|^2 d\mathbf{x} + \alpha \int_\Omega \nabla \hat{u} \cdot \nabla \hat{\vartheta}^* d\mathbf{x} \right) - \int_\Omega \hat{\mathbf{q}} \cdot \nabla \hat{\vartheta}^* d\mathbf{x} = \int_\Omega \hat{g} \hat{\vartheta}^* d\mathbf{x}. \tag{3.13}$$

Then, in a similar manner the scalar products of (3.7) by $\hat{\mathbf{q}}^*$, $\nabla \hat{\vartheta}^*$ and $\nabla \hat{u}^*$ yield

$$(1 + i\omega\tau) \int_\Omega |\hat{\mathbf{q}}|^2 \, d\mathbf{x} + k \int_\Omega \nabla \hat{\vartheta} \cdot \hat{\mathbf{q}}^* d\mathbf{x} = \int_\Omega \hat{\mathbf{l}} \cdot \hat{\mathbf{q}}^* d\mathbf{x}, \tag{3.14}$$

$$(1 + i\omega\tau) \int_\Omega \hat{\mathbf{q}} \cdot \nabla \hat{\vartheta}^* d\mathbf{x} + k \int_\Omega \left|\nabla \hat{\vartheta}\right|^2 d\mathbf{x} = \int_\Omega \hat{\mathbf{l}} \cdot \nabla \hat{\vartheta}^* d\mathbf{x}, \tag{3.15}$$

$$(1 + i\omega\tau) \int_\Omega \hat{\mathbf{q}} \cdot \nabla \hat{u}^* d\mathbf{x} + k \int_\Omega \nabla \hat{\vartheta} \cdot \nabla \hat{u}^* d\mathbf{x} = \int_\Omega \hat{\mathbf{l}} \cdot \nabla \hat{u}^* d\mathbf{x}. \tag{3.16}$$

First, we consider the imaginary part of (3.11)

$$\operatorname{Im} \int_\Omega \nabla \hat{\vartheta} \cdot \nabla \hat{u}^* d\mathbf{x} = -\frac{1}{\alpha} \operatorname{Im} \int_\Omega \hat{f} \hat{u}^* d\mathbf{x}, \tag{3.17}$$

which allows us to write the real part of (3.13) as follows:

$$\mathrm{Re}\int\limits_{\Omega}\hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^{*}d\mathbf{x} = -\,\mathrm{Re}\int\limits_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x} - \mathrm{Im}\int\limits_{\Omega}\hat{f}\omega\hat{u}^{*}d\mathbf{x}; \tag{3.18}$$

thus, from the real part of (3.14), we have

$$\int\limits_{\Omega}|\hat{\mathbf{q}}|^{2}\,d\mathbf{x} = \mathrm{Re}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\hat{\mathbf{q}}^{*}d\mathbf{x} + k\left(\mathrm{Re}\int\limits_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x} + \mathrm{Im}\int\limits_{\Omega}\hat{f}\omega\hat{u}^{*}d\mathbf{x}\right). \tag{3.19}$$

Then, we take the imaginary part of (3.15), which, on account of (3.18), becomes

$$\mathrm{Im}\int\limits_{\Omega}\hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^{*}d\mathbf{x} = \mathrm{Im}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^{*}d\mathbf{x} + \omega\tau\left(\mathrm{Re}\int\limits_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x} + \mathrm{Im}\int\limits_{\Omega}\hat{f}\omega\hat{u}^{*}d\mathbf{x}\right) \tag{3.20}$$

and use it, together with (3.18), to derive from the real part of (3.15) the following relation:

$$\int\limits_{\Omega}\left|\nabla\hat{\vartheta}\right|^{2}d\mathbf{x} = \frac{1}{k}\left[\mathrm{Re}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^{*}d\mathbf{x} + \omega\tau\,\mathrm{Im}\int_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^{*}d\mathbf{x} + \right.$$

$$\left. + (1+\omega^{2}\tau^{2})\left(\mathrm{Re}\int\limits_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x} + \mathrm{Im}\int\limits_{\Omega}\hat{f}\omega\hat{u}^{*}d\mathbf{x}\right)\right]. \tag{3.21}$$

To estimate the other terms of (3.9), some other relations must be determined.

We begin with the imaginary part of (3.13), which, by virtue of (3.20), yields

$$\omega\alpha\,\mathrm{Re}\int\limits_{\Omega}\nabla\hat{u}\cdot\nabla\hat{\vartheta}^{*}d\mathbf{x} = \mathrm{Im}\int\limits_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x} + \mathrm{Im}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^{*}d\mathbf{x} +$$

$$+\omega\tau\left(\mathrm{Re}\int\limits_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x} + \mathrm{Im}\int\limits_{\Omega}\hat{f}\omega\hat{u}^{*}d\mathbf{x}\right) - \omega\rho\int\limits_{\Omega}\left|\hat{\vartheta}\right|^{2}d\mathbf{x}. \tag{3.22}$$

Then, subtracting the real part of (3.16), multiplied by $\omega\tau$, from the imaginary part of (3.16), on account of (3.17) and (3.22), we get

$$\operatorname{Im}\int\limits_{\Omega}\hat{\mathbf{q}}\cdot\nabla\hat{u}^{*}d\mathbf{x}=\frac{1}{1+\omega^{2}\tau^{2}}\Bigg\{\operatorname{Im}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{u}^{*}d\mathbf{x}-\omega\tau\operatorname{Re}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{u}^{*}d\mathbf{x}+$$

$$+\frac{k}{\alpha}(1+\omega^{2}\tau^{2})\operatorname{Im}\int_{\Omega}\hat{f}\hat{u}^{*}d\mathbf{x}+\frac{\tau k}{\alpha}\Bigg[\operatorname{Im}\int\limits_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x}+\operatorname{Im}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^{*}d\mathbf{x}+$$

$$+\omega\tau\operatorname{Re}\int_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x}-\omega\rho\int\limits_{\Omega}\left|\hat{\vartheta}\right|^{2}d\mathbf{x}\Bigg]\Bigg\}. \tag{3.23}$$

Adding the imaginary part of (3.16), multiplied again by $\omega\tau$, to its real part and using (3.17) and (3.22), we obtain

$$\operatorname{Re}\int\limits_{\Omega}\hat{\mathbf{q}}\cdot\nabla\hat{u}^{*}d\mathbf{x}=\frac{1}{1+\omega^{2}\tau^{2}}\Bigg\{\operatorname{Re}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{u}^{*}d\mathbf{x}+\omega\tau\operatorname{Im}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{u}^{*}d\mathbf{x}-\frac{k}{\omega\alpha}\times$$

$$\times\Bigg[\operatorname{Im}\int\limits_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x}+\operatorname{Im}\int\limits_{\Omega}\hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^{*}d\mathbf{x}+\omega\tau\operatorname{Re}\int\limits_{\Omega}\hat{g}\hat{\vartheta}^{*}d\mathbf{x}-\omega\rho\int\limits_{\Omega}\left|\hat{\vartheta}\right|^{2}d\mathbf{x}\Bigg]\Bigg\}.$$

We now observe that

$$\int\limits_{\Omega}|\hat{u}|^{2}\,d\mathbf{x}\leq\lambda_{u}(\Omega)\int\limits_{\Omega}|\nabla\hat{u}|^{2}\,d\mathbf{x},\qquad\int\limits_{\Omega}\left|\hat{\vartheta}\right|^{2}d\mathbf{x}\leq\lambda_{\vartheta}(\Omega)\int\limits_{\Omega}\left|\nabla\hat{\vartheta}\right|^{2}d\mathbf{x} \tag{3.24}$$

by virtue of Poincaré's inequality, where λ_u and λ_ϑ are positive constants depending on the domain Ω.

Hence we can increase the following term

$$-\frac{\rho}{\alpha}\omega^{2}\operatorname{Re}\int\limits_{\Omega}\hat{\vartheta}\hat{u}^{*}d\mathbf{x}\leq\omega^{2}\left(\int\limits_{\Omega}|\hat{u}|^{2}\,d\mathbf{x}\right)^{1/2}\left(\frac{\rho^{2}}{\alpha^{2}}\int\limits_{\Omega}\left|\hat{\vartheta}\right|^{2}d\mathbf{x}\right)^{1/2}\leq$$

$$\leq\frac{\omega^{2}}{2}\left[\int\limits_{\Omega}|\nabla\hat{u}|^{2}\,d\mathbf{x}+\frac{\rho^{2}}{\alpha^{2}}\lambda_{u}(\Omega)\int_{\Omega}\left|\hat{\vartheta}\right|^{2}d\mathbf{x}\right],$$

which appears in the imaginary part of (3.12) multiplied by ω/α, whence, using (3.23), $(3.24)_2$

and (3.21), we have the following inequality:

$$\int_\Omega |\omega\nabla\hat{u}|^2\,d\mathbf{x} \le \frac{1}{k\alpha^2}\left\{2k^2+\omega^2\rho\lambda_\vartheta(\Omega)\left[\rho\lambda_u(\Omega)\left(1+\omega^2\tau^2\right)-2\tau k\right]\right\}\times$$

$$\times\left(\operatorname{Im}\int_\Omega \hat{f}\omega\hat{u}^*d\mathbf{x}+\frac{\omega\tau}{1+\omega^2\tau^2}\operatorname{Im}\int_\Omega \hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^*d\mathbf{x}\right)+\frac{2}{\alpha}\operatorname{Im}\int_\Omega \hat{g}\omega\hat{u}^*d\mathbf{x}+$$

$$+\;\frac{2}{\alpha}\frac{1}{1+\omega^2\tau^2}\left[\operatorname{Im}\int_\Omega \hat{\mathbf{l}}\cdot\omega\nabla\hat{u}^*d\mathbf{x}-\omega\tau\left(\operatorname{Re}\int_\Omega \hat{\mathbf{l}}\cdot\omega\nabla\hat{u}^*d\mathbf{x}-\right.\right.$$

$$\left.\left.-\frac{k}{\alpha}\operatorname{Im}\int_\Omega \hat{g}\hat{\vartheta}^*d\mathbf{x}\right)\right]+\frac{\rho\lambda_\vartheta(\Omega)}{k\alpha^2}\frac{\omega^2}{1+\omega^2\tau^2}[\rho\lambda_u(\Omega)(1+\omega^2\tau^2)-$$

$$-\;2\tau k]\operatorname{Re}\int_\Omega \hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^*d\mathbf{x}+\frac{\omega^2\tau^2}{k\alpha^2}\left\{\frac{2k^2}{1+\omega^2\tau^2}+\frac{\rho\lambda_\vartheta(\Omega)}{\tau^2}[\rho\lambda_u(\Omega)\times\right.$$

$$\left.\times(1+\omega^2\tau^2)-2\tau k]\right\}\operatorname{Re}\int_\Omega \hat{g}\hat{\vartheta}^*d\mathbf{x}. \tag{3.25}$$

Finally, the real part of (3.11), taking into account (3.22) divided by ω, (3.24), (3.25) and (3.21), gives

$$\int_\Omega |\Delta\hat{u}|^2\,d\mathbf{x} \le \frac{1}{\omega}\operatorname{Re}\int_\Omega \hat{f}\omega\hat{u}^*d\mathbf{x}+\frac{2}{\alpha}[\gamma+\lambda_u(\Omega)]\left\{\operatorname{Im}\int_\Omega \hat{g}\omega\hat{u}^*d\mathbf{x}+\frac{1}{1+\omega^2\tau^2}\times\right.$$

$$\left.\times\left[\operatorname{Im}\int_\Omega \hat{\mathbf{l}}\cdot\omega\nabla\hat{u}^*d\mathbf{x}-\omega\tau\operatorname{Re}\int_\Omega \hat{\mathbf{l}}\cdot\omega\nabla\hat{u}^*d\mathbf{x}\right]\right\}+\left\{\tau+\frac{1}{k}\left(\rho\lambda_\vartheta(\Omega)(1+\omega^2\tau^2)+\right.\right.$$

$$\left.\left.+\;\frac{\gamma+\lambda_u(\Omega)}{\alpha^2}\left\{2k^2+\rho\lambda_\vartheta(\Omega)\omega^2[\rho\lambda_u(\Omega)(1+\omega^2\tau^2)-2\tau k]\right\}\right)\right\}\operatorname{Im}\int_\Omega \hat{f}\omega\hat{u}^*d\mathbf{x}+$$

$$+\left\{\tau+\frac{1}{k}\left(\rho\lambda_\vartheta(\Omega)(1+\omega^2\tau^2)+[\gamma+\lambda_u(\Omega)]\frac{\omega^2\tau^2}{\alpha^2}\left\{\frac{2k^2}{1+\omega^2\tau^2}+\frac{\rho\lambda_\vartheta(\Omega)}{\tau^2}\times\right.\right.\right.$$

$$\times [\rho\lambda_u(\Omega)(1+\omega^2\tau^2) - 2\tau k]\Bigg\}\Bigg)\Bigg\} \operatorname{Re}\int\limits_{\Omega} \hat{g}\hat{\vartheta}^* d\mathbf{x} + \frac{1}{\omega}\Bigg\{1 + [\gamma + \lambda_u(\Omega)]\frac{2k}{\alpha^2} \times$$

$$\times \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\Bigg\} \operatorname{Im}\int\limits_{\Omega} \hat{g}\hat{\vartheta}^* d\mathbf{x} + \frac{\rho\lambda_\vartheta(\Omega)}{k}\Bigg\{1 + \frac{\gamma+\lambda_u(\Omega)}{\alpha^2}\frac{\omega^2}{1+\omega^2\tau^2}[\rho\lambda_u(\Omega)\times$$

$$\times (1+\omega^2\tau^2) - 2\tau k]\Bigg\} \operatorname{Re}\int\limits_{\Omega} \hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} + \frac{1}{\omega}\Bigg\{1 + \frac{\omega^2\tau}{k}\Bigg(\rho\lambda_\vartheta(\Omega) +$$

$$+ \frac{\gamma+\lambda_u(\Omega)}{\alpha^2}\frac{1}{1+\omega^2\tau^2}\left\{2k^2 + \rho\lambda_\vartheta(\Omega)\omega^2[\rho\lambda_u(\Omega)\left(1+\omega^2\tau^2\right) -\right.$$

$$\left. - 2\tau k]\right\}\Bigg)\Bigg\} \operatorname{Im}\int\limits_{\Omega} \hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x}. \tag{3.26}$$

We can now increase (3.9), using (3.19), (3.21), (3.25), (3.26), together with (3.24), to obtain

$$\mathcal{I}(\omega) \le [1+\lambda_u(\Omega)]\int\limits_{\Omega} |\omega\nabla\hat{u}|^2\, d\mathbf{x} + \int\limits_{\Omega} |\Delta\hat{u}|^2\, d\mathbf{x} + [1+\lambda_\vartheta(\Omega)]\int\limits_{\Omega} \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x} +$$

$$+ \int\limits_{\Omega} |\hat{\mathbf{q}}|^2\, d\mathbf{x} \le \frac{1}{\omega}\operatorname{Re}\int\limits_{\Omega} \hat{f}\omega\hat{u}^* d\mathbf{x} + \left(c_1\left\{2k + c_4\omega^2[c_3(1+\tau^2\omega^2) - 2\tau k]\right\} +\right.$$

$$\left. + \tau + k + c_2(1+\tau^2\omega^2)\right) \operatorname{Im}\int\limits_{\Omega} \hat{f}\omega\hat{u}^* d\mathbf{x} + 2c_1\alpha \operatorname{Im}\int\limits_{\Omega} \hat{g}\omega\hat{u}^* d\mathbf{x} + \Bigg(c_1\tau^2\omega^2 \times$$

$$\times \left\{\frac{2k}{1+\tau^2\omega^2} + \frac{c_4}{\tau^2}[c_3(1+\tau^2\omega^2) - 2\tau k]\right\} + \tau + k + c_2(1+\tau^2\omega^2)\Bigg) \times$$

$$\times \operatorname{Re}\int\limits_{\Omega} \hat{g}\hat{\vartheta}^* d\mathbf{x} + \frac{1}{\omega}\left(1 + \frac{2c_1 k\tau\omega^2}{1+\tau^2\omega^2}\right) \operatorname{Im}\int\limits_{\Omega} \hat{g}\hat{\vartheta}^* d\mathbf{x} + \operatorname{Re}\int\limits_{\Omega} \hat{\mathbf{l}}\cdot\mathbf{q}^* d\mathbf{x} +$$

$$+ \left(\frac{c_1 c_4\omega^2}{1+\tau^2\omega^2}[c_3(1+\tau^2\omega^2) - 2\tau k] + c_2\right) \operatorname{Re}\int\limits_{\Omega} \hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} +$$

$$+\frac{1}{\omega}\left(1+\tau\omega^2\left(c_1\left\{2k+c_4\omega^2[c_3(1+\tau^2\omega^2)-2\tau k]\right\}\frac{1}{1+\tau^2\omega^2}+c_2\right)\right)\times$$

$$\times\operatorname{Im}\int_\Omega \hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x}+\frac{2c_1\alpha}{1+\tau^2\omega^2}\left(-\tau\omega\operatorname{Re}\int_\Omega \hat{\mathbf{l}}\cdot\omega\nabla\hat{u}^* d\mathbf{x}+\right.$$

$$\left.+\operatorname{Im}\int_\Omega \hat{\mathbf{l}}\cdot\omega\nabla\hat{u}^* d\mathbf{x}\right), \tag{3.27}$$

where we have introduced the following positive constants:

$$c_1=\frac{1+\gamma+2\lambda_u(\Omega)}{\alpha^2},\quad c_2=\frac{1+(1+\rho)\lambda_\vartheta(\Omega)}{k},\quad c_3=\rho\lambda_u(\Omega),\quad c_4=\frac{\rho\lambda_\vartheta(\Omega)}{k}.$$

Putting

$$\gamma_1(\omega)=\frac{1}{\mid\omega\mid}+c_1[2k+c_4\omega^2\mid c_3(1+\tau^2\omega^2)-2\tau k\mid]+\tau+k+c_2(1+\tau^2\omega^2),$$

$$\gamma_2(\omega)=2c_1\mid\alpha\mid,$$

$$\gamma_3(\omega)=c_1\tau^2\omega^2\left[\frac{2k}{1+\tau^2\omega^2}+\frac{c_4}{\tau^2}\mid c_3(1+\tau^2\omega^2)-2\tau k\mid\right]+\tau+k+$$

$$+c_2(1+\tau^2\omega^2)+\frac{1}{\mid\omega\mid}\left(1+\frac{2c_1k\tau\omega^2}{1+\tau^2\omega^2}\right),$$

$$\gamma_4(\omega)=1,$$

$$\gamma_5(\omega)=\frac{c_1c_4\omega^2}{1+\tau^2\omega^2}\mid c_3(1+\tau^2\omega^2)-2\tau k\mid+c_2+\frac{1}{\mid\omega\mid}\left(1+\tau\omega^2\left\{c_1[2k+\right.\right.$$

$$\left.\left.+c_4\omega^2\mid c_3(1+\tau^2\omega^2)-2\tau k\mid]\frac{1}{1+\tau^2\omega^2}+c_2\right\}\right),$$

$$\gamma_6(\omega)=\frac{2c_1\mid\alpha\mid}{1+\tau^2\omega^2}(1+\tau\mid\omega\mid),$$

the inequality (3.27) can be written as follows:

$$\mathcal{I}(\omega) \leq \gamma(\omega)\Bigg(\Big|\int\limits_{\Omega} \hat{f}\omega\hat{u}^* d\mathbf{x}\Big| + \Big|\int\limits_{\Omega} \hat{g}\omega\hat{u}^* d\mathbf{x}\Big| + \int\limits_{\Omega} \hat{g}\hat{\vartheta}^* d\mathbf{x} + \Big|\int\limits_{\Omega} \hat{\mathbf{l}}\cdot\hat{\mathbf{q}}^* d\mathbf{x}\Big| +$$

$$+ \Big|\int\limits_{\Omega} \hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x}\Big| + \Big|\int\limits_{\Omega} \hat{\mathbf{l}}\cdot\omega\nabla\hat{u}^* d\mathbf{x}\Big|\Bigg), \tag{3.28}$$

where the positive function $\gamma(\omega)$ is given by

$$\gamma(\omega) = \max\{\gamma_i(\omega), \quad i = 1, 2, ..., 6\}.$$

From (3.28) we have

$$\mathcal{I}(\omega) \leq \gamma(\omega)\Bigg[\left(\int\limits_{\Omega} |\hat{f}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}} \left(\int\limits_{\Omega} |\omega\hat{u}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}} + \left(\int\limits_{\Omega} |\hat{g}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}} \left(\int\limits_{\Omega} |\omega\hat{u}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}} +$$

$$+ \left(\int\limits_{\Omega} |\hat{g}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}} \left(\int\limits_{\Omega} \left|\hat{\vartheta}\right|^2 d\mathbf{x}\right)^{\frac{1}{2}} + \left(\int\limits_{\Omega} |\hat{\mathbf{l}}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}} \left(\int\limits_{\Omega} |\hat{\mathbf{q}}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}} +$$

$$+ \left(\int\limits_{\Omega} |\hat{\mathbf{l}}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}} \left(\int\limits_{\Omega} \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x}\right)^{\frac{1}{2}} + \left(\int\limits_{\Omega} |\hat{\mathbf{l}}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}} \left(\int\limits_{\Omega} |\omega\nabla\hat{u}|^2 \, d\mathbf{x}\right)^{\frac{1}{2}}\Bigg] \leq$$

$$\leq \delta(\omega)\left[\int\limits_{\Omega} \left(\left|\hat{f}\right|^2 + |\hat{g}|^2 + \left|\hat{\mathbf{l}}\right|^2\right) d\mathbf{x}\right]^{\frac{1}{2}} \Bigg[\int\limits_{\Omega} \Big(|\omega\hat{u}|^2 + |\omega\nabla\hat{u}|^2 + |\Delta\hat{u}|^2 +$$

$$+ \left|\hat{\vartheta}\right|^2 + \left|\nabla\hat{\vartheta}\right|^2 + |\hat{\mathbf{q}}|^2\Big) d\mathbf{x}\Bigg]^{\frac{1}{2}}$$

and hence

$$\mathcal{I}^{\frac{1}{2}}(\omega) \leq \delta(\omega)\left[\int\limits_{\Omega} \left(\left|\hat{f}\right|^2 + |\hat{g}|^2 + \left|\hat{\mathbf{l}}\right|^2\right) d\mathbf{x}\right]^{\frac{1}{2}},$$

whence (3.10) follows.

We now consider the case $\omega = 0$, where the problem (3.5) – (3.8) becomes

$$\Delta^2 \hat{u}(\mathbf{x},0) + \alpha \Delta \hat{\vartheta}(\mathbf{x},0) = \hat{f}(\mathbf{x},0), \tag{3.29}$$

$$\nabla \cdot \hat{\mathbf{q}}(\mathbf{x},0) = \hat{g}(\mathbf{x},0), \tag{3.30}$$

$$\hat{\mathbf{q}}(\mathbf{x},0) + k\nabla\hat{\vartheta}(\mathbf{x},0) = \hat{\mathbf{l}}(\mathbf{x},0) \qquad \forall \mathbf{x} \in \Omega, \tag{3.31}$$

$$\hat{u}(\mathbf{x},0) = 0, \ \frac{\partial \hat{u}(\mathbf{x},0)}{\partial \nu} = \mathbf{0}, \ \hat{\vartheta}(\mathbf{x},0) = 0 \quad \forall \mathbf{x} \in \Gamma. \tag{3.32}$$

It is enough to multiply (3.29) by $\hat{u}^*(\mathbf{x},0)$ and (3.30) by $\hat{\vartheta}^*(\mathbf{x},0)$, to take the inner products of (3.31) with $\hat{\mathbf{q}}^*(\mathbf{x},0)$ and $\nabla\hat{\vartheta}^*(\mathbf{x},0)$ and consider their real parts to obtain

$$\int_\Omega |\Delta\hat{u}|^2\, d\mathbf{x} = \mathrm{Re}\int_\Omega \hat{f}\hat{u}^* d\mathbf{x} + \alpha\,\mathrm{Re}\int_\Omega \nabla\hat{\vartheta}\cdot\nabla\hat{u}^* d\mathbf{x}, \tag{3.33}$$

$$\mathrm{Re}\int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} = -\,\mathrm{Re}\int_\Omega \hat{g}\hat{\vartheta}^* d\mathbf{x}, \tag{3.34}$$

$$\int_\Omega |\hat{\mathbf{q}}|^2\, d\mathbf{x} = \mathrm{Re}\int_\Omega \hat{\mathbf{l}}\cdot\hat{\mathbf{q}}^* d\mathbf{x} - k\,\mathrm{Re}\int_\Omega \nabla\hat{\vartheta}\cdot\hat{\mathbf{q}}^* d\mathbf{x}, \tag{3.35}$$

$$\int_\Omega \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x} = \frac{1}{k}\left(\mathrm{Re}\int_\Omega \hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} - \mathrm{Re}\int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x}\right). \tag{3.36}$$

We now recall that if $\hat{u} \in \mathcal{H}_0^2(\Omega)$ then $\| \hat{u} \| + \| \nabla\hat{u} \| \leq C \| \Delta\hat{u} \|$ [9], where C is a constant, whence we can consider

$$\int_\Omega |\hat{u}|^2\, d\mathbf{x} \leq C^2 \int_\Omega |\Delta\hat{u}|^2\, d\mathbf{x}, \qquad \int_\Omega |\nabla\hat{u}|^2\, d\mathbf{x} \leq C^2 \int_\Omega |\Delta\hat{u}|^2\, d\mathbf{x}$$

together with (3.24).

Thus, it follows that

$$\mathcal{I}_0 = \int_\Omega \left(|\hat{u}|^2 + |\nabla\hat{u}|^2 + |\Delta\hat{u}|^2 + \left|\hat{\vartheta}\right|^2 + \left|\nabla\hat{\vartheta}\right|^2 + |\hat{\mathbf{q}}|^2\right) d\mathbf{x} \leq$$

$$\leq (1+2C^2)\int_\Omega |\Delta\hat{u}|^2\, d\mathbf{x} + [1+\lambda_\vartheta(\Omega)]\int_\Omega \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x} + \int_\Omega |\hat{\mathbf{q}}|^2\, d\mathbf{x}, \tag{3.37}$$

where all the functions must be considered in $(\mathbf{x}, 0)$.

We observe that the last term in (3.33), which appears in (3.37) multiplied by $(1+2C^2)$, can be increased as follows:

$$(1+2C^2)\alpha \operatorname{Re}\int_\Omega \nabla\hat{\vartheta}\cdot\nabla\hat{u}^* d\mathbf{x} \le (1+2C^2) \mid \alpha \operatorname{Re}\int_\Omega \nabla\hat{\vartheta}\cdot\nabla\hat{u}^* d\mathbf{x} \mid \le$$

$$\le \frac{1}{2}\left[\alpha^2(1+2C^2)^2\int_\Omega \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x} + \int_\Omega |\nabla\hat{u}|^2\, d\mathbf{x}\right], \qquad (3.38)$$

where the last integral is present in the expression of $\mathcal{I}_0$ too.

Then, substituting into (3.37) the inequality derived from (3.33) by using (3.38) and the two relations (3.35) and (3.36), after eliminating their last integrals by using (3.34), the inequality (3.37) can be put in the following form:

$$\mathcal{I}_0 \le 2(1+2C^2)\operatorname{Re}\int_\Omega \hat{f}\hat{u}^* d\mathbf{x} + \frac{1}{k}\Big\{2[1+\lambda_\vartheta(\Omega)] + \alpha^2(1+2C^2)^2 +$$

$$+ 2k^2\Big\}\operatorname{Re}\int_\Omega \hat{g}\hat{\vartheta}^* d\mathbf{x} + \frac{1}{k}\left\{2[1+\lambda_\vartheta(\Omega)] + \alpha^2(1+2C^2)^2\right\}\operatorname{Re}\int_\Omega \hat{\mathbf{l}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} +$$

$$+ 2\operatorname{Re}\int_\Omega \hat{\mathbf{l}}\cdot\hat{\mathbf{q}}^* d\mathbf{x}. \qquad (3.39)$$

Therefore, denoting by m the maximum of the coefficients of the four integrals at the right-hand side of (3.39) and proceeding as we have already done to derive (3.10), we see that if $\omega = 0$, (3.10) must be replaced by

$$\mathcal{I}_0 \le M\int_\Omega \left(\left|\hat{f}(\mathbf{x},0)\right|^2 + |\hat{g}(\mathbf{x},0)|^2 + \left|\hat{\mathbf{l}}(\mathbf{x},0)\right|^2\right) d\mathbf{x}, \qquad (3.40)$$

where $M = 16m^2$. This proves the theorem.

We observe that $\delta(\omega)$ tends to infinity as ω^4 if ω approaches infinity. Therefore, we introduce the following space:

$$\mathcal{W}(\Omega,\mathbf{R}^+) = \Big\{(f,g,\mathbf{l}) \in [\mathcal{Q}(\Omega,\mathbf{R}^+)]^3 : \frac{\partial^{n+1}}{\partial t^{n+1}}(f,g,\mathbf{l}) \in [\mathcal{Q}(\Omega,\mathbf{R}^+)]^3,$$

$$\left[\frac{\partial^n}{\partial t^n}(f,g,\mathbf{l})\right]_{t=0} = 0 \quad (n = 0,1,2,3)\Big\}.$$

Theorem 3.2. *If the sources* $(f, g, \mathbf{l}) \in \mathcal{W}(\Omega, \mathbf{R}^+)$*, then the inverse Fourier transforms of* $(\hat{u}, \hat{\vartheta}, \hat{\mathbf{q}})$ *exist and are* L^2*-functions.*

Proof. As we have already observed, in the inequality (3.10) of Theorem 3.1, δ depends on ω in such a way as to assure that the integral of the right-hand side over $\mathbf{R}$ exists if $(\hat{f}, \hat{g}, \hat{\mathbf{l}}) \in \widehat{\mathcal{W}}(\Omega, \mathbf{R})$. That is, we have

$$\int_{-\infty}^{+\infty} \int_{\Omega} \left(\left| \delta(\omega) \hat{f}(\mathbf{x}, \omega) \right|^2 + |\delta(\omega) \hat{g}(\mathbf{x}, \omega)|^2 + \left| \delta(\omega) \hat{\mathbf{l}}(\mathbf{x}, \omega) \right|^2 \right) d\mathbf{x} d\omega < +\infty;$$

therefore, it follows that $\mathcal{I}(\omega)$ is also integrable over $\mathbf{R}$ and hence Plancherel's theorem assures the existence of the inverse transforms of $(\hat{u}, \hat{\vartheta}, \hat{\mathbf{q}})$. This completes the proof of the theorem.

Corollary 3.1. *Under the hypotheses of Theorem 3.2, if we consider two solutions of our problem,* $(\hat{u}^{(i)}, \hat{\vartheta}^{(i)}, \hat{\mathbf{q}}^{(i)})$*, each of which corresponds to two given source fields* $(\hat{f}^{(i)}, \hat{g}^{(i)}, \hat{\mathbf{l}}^{(i)})$*,* $i = 1, 2$*, we have*

$$\left\| (\hat{u}^{(1)} - \hat{u}^{(2)}, \hat{\vartheta}^{(1)} - \hat{\vartheta}^{(2)}, \hat{\mathbf{q}}^{(1)} - \hat{\mathbf{q}}^{(2)}) \right\|^2 \leq$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\Omega} \delta^2(\omega) \left(\left| \hat{f}^{(1)} - \hat{f}^{(2)} \right|^2 + \left| \hat{g}^{(1)} - \hat{g}^{(2)} \right|^2 + \left| \hat{\mathbf{l}}^{(1)} - \hat{\mathbf{l}}^{(2)} \right|^2 \right) d\mathbf{x} d\omega. \quad (3.41)$$

This result follows at once from the linearity of (3.5) – (3.7) and from Theorem 3.1.

Theorem 3.3. *For any fixed* $\omega \in \mathbf{R}$ *and every* $(\hat{f}, \hat{g}, \hat{\mathbf{l}}) \in [L^2(\Omega)]^3$*, the system (3.5) – (3.8) admits at most only one solution* $(\hat{u}, \hat{\vartheta}, \hat{\mathbf{q}}) \in \mathcal{H}_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$*.*

Proof. This uniqueness theorem is a consequence of Theorem 3.1, since it is equivalent to establishing that the homogeneous system given by (3.5) – (3.7), with the homogeneous boundary conditions (3.8), has only the zero solution in $\mathcal{H}_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$. Inequalities (3.10) and (3.40) assure the uniqueness for every $\omega \in \mathbf{R}$, which proves the theorem.

Theorem 3.4. *For any triplet* $(f, g, \mathbf{l}) \in \mathcal{W}(\Omega, \mathbf{R}^+)$ *there exists a solution*

$$(u, \vartheta, \mathbf{q}) \in \mathcal{U}(\Omega, \mathbf{R}^+) \times \mathcal{T}(\Omega, \mathbf{R}^+) \times \mathcal{Q}(\Omega, \mathbf{R}^+)$$

of the problem (2.1) – (2.3) with (2.5), (2.6) in the sense of Definition 3.1.

Proof. In order to prove the existence of a solution to our problem, we show that the set

$$\mathcal{A} = \Big\{ (\hat{f}, \hat{g}, \hat{\mathbf{l}}) \in \widehat{\mathcal{W}}(\Omega, \mathbf{R}) : there\ exists\ (\hat{u}, \hat{\vartheta}, \hat{\mathbf{q}}) \in \widehat{\mathcal{U}}(\Omega, \mathbf{R}) \times \widehat{\mathcal{T}}(\Omega, \mathbf{R}) \times$$

$$\times \widehat{\mathcal{Q}}(\Omega, \mathbf{R})\ which\ satisfies\ (3.3)\ \forall (\hat{v}, \hat{\phi}, \hat{\mathbf{p}}) \in \widehat{\mathcal{U}}(\Omega, \mathbf{R}) \times \widehat{\mathcal{F}}(\Omega, \mathbf{R}) \times$$

$$\times \widehat{\mathcal{V}}(\Omega, \mathbf{R}) \Big\}$$

is dense and closed in $\widehat{\mathcal{W}}(\Omega, \mathbf{R})$.

Denoting by $\xi[(\hat{u},\hat{\vartheta},\hat{\mathbf{q}}),(\hat{v},\hat{\phi},\hat{\mathbf{p}})]$ the expression in the left-hand side of (3.3), we can write this identity as follows:

$$\xi[(\hat{u},\hat{\vartheta},\hat{\mathbf{q}}),(\hat{v},\hat{\phi},\hat{\mathbf{p}})] = \frac{1}{2\pi}\langle(\hat{f},\hat{g},\hat{\mathbf{l}}),(\hat{v},\hat{\phi},\hat{\mathbf{p}})\rangle. \tag{3.42}$$

To prove that $\mathcal{A}$ is dense, we denote by $\overline{\mathcal{A}}$ its closure in $\widehat{\mathcal{W}}(\Omega,\mathbf{R})$ and suppose that there exists $(\hat{f}^{(0)},\hat{g}^{(0)},\hat{\mathbf{l}}^{(0)}) \in \widehat{\mathcal{W}}(\Omega,\mathbf{R})\backslash\overline{\mathcal{A}}$ and $(\hat{f}^{(0)},\hat{g}^{(0)},\hat{\mathbf{l}}^{(0)}) \neq 0$. Thus, the Hahn–Banach theorem states that there exists $(\hat{v}^{(0)},\hat{\phi}^{(0)},\hat{\mathbf{p}}^{(0)}) \in \widehat{\mathcal{U}}(\Omega,\mathbf{R})\times\widehat{\mathcal{F}}(\Omega,\mathbf{R})\times\widehat{\mathcal{V}}(\Omega,\mathbf{R})$ such that

$$\langle(\hat{f}^{(0)},\hat{g}^{(0)},\hat{\mathbf{l}}^{(0)}),(\hat{v}^{(0)},\hat{\phi}^{(0)},\hat{\mathbf{p}}^{(0)})\rangle \neq 0,\ \langle(\hat{f},\hat{g},\hat{\mathbf{l}}),(\hat{v}^{(0)},\hat{\phi}^{(0)},\hat{\mathbf{p}}^{(0)})\rangle = 0 \quad \forall(\hat{f},\hat{g},\hat{\mathbf{l}}) \in \mathcal{A}. \tag{3.43}$$

Conditions $(3.43)_2$ and (3.42) yield

$$\xi[(\hat{u},\hat{\vartheta},\hat{\mathbf{q}}),(\hat{v}^{(0)},\hat{\phi}^{(0)},\hat{\mathbf{p}}^{(0)})] = 0 \quad \forall(\hat{u},\hat{\vartheta},\hat{\mathbf{q}}) \in \widehat{\mathcal{U}}(\Omega,\mathbf{R})\times\widehat{\mathcal{T}}(\Omega,\mathbf{R})\times\widehat{\mathcal{Q}}(\Omega,\mathbf{R})$$

from which, therefore, with the same technique used to prove the uniqueness theorem, we find that

$$(\hat{v}^{(0)},\hat{\phi}^{(0)},\hat{\mathbf{p}}^{(0)}) = 0,$$

against $(3.43)_1$. Hence, the set $\mathcal{A}$ is dense.

To prove that $\mathcal{A}$ is closed, let us consider, for every $(\hat{f},\hat{g},\hat{\mathbf{l}}) \in \widehat{\mathcal{W}}(\Omega,\mathbf{R})$, a sequence

$$\left\{(\hat{f}^{(n)},\hat{g}^{(n)},\hat{\mathbf{l}}^{(n)}) \in \mathcal{A},\ n = 1,2,...\right\}$$

convergent to $(\hat{f},\hat{g},\hat{\mathbf{l}})$ and the sequence of the corresponding solutions $(\hat{u}^{(n)},\hat{\vartheta}^{(n)},\hat{\mathbf{q}}^{(n)}) \in \widehat{\mathcal{U}}(\Omega,\mathbf{R})\times\widehat{\mathcal{T}}(\Omega,\mathbf{R})\times\widehat{\mathcal{Q}}(\Omega,\mathbf{R})$. Using (3.41) of Corollary 3.1, we have

$$\left\|(\hat{u}^{(n)}-\hat{u}^{(m)},\hat{\vartheta}^{(n)}-\hat{\vartheta}^{(m)},\hat{\mathbf{q}}^{(n)}-\hat{\mathbf{q}}^{(m)})\right\|^2 \leq$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{\Omega}\delta^2(\omega)\left(\left|\hat{f}^{(n)}-\hat{f}^{(m)}\right|^2+\left|\hat{g}^{(n)}-\hat{g}^{(m)}\right|^2+\left|\hat{\mathbf{l}}^{(n)}-\hat{\mathbf{l}}^{(m)}\right|^2\right)d\mathbf{x}d\omega,$$

and hence it follows that the sequence $\left\{(\hat{u}^{(n)},\hat{\vartheta}^{(n)},\hat{\mathbf{q}}^{(n)}),\ n = 1,2,...\right\}$ is a Cauchy sequence and

$$\lim_{n\to+\infty}(\hat{u}^{(n)},\hat{\vartheta}^{(n)},\hat{\mathbf{q}}^{(n)}) = (\hat{u},\hat{\vartheta},\hat{\mathbf{q}},) \in \widehat{\mathcal{U}}(\Omega,\mathbf{R})\times\widehat{\mathcal{T}}(\Omega,\mathbf{R})\times\widehat{\mathcal{Q}}(\Omega,\mathbf{R})$$

for the completeness of the space.

Then, we consider the sequence of identities, obtained by substituting into (3.3) each solution with the corresponding triplet of sources of the two sequences now introduced; the limit of these identities as $n \to +\infty$ yields an analogous identity for the limits $(\hat{u}, \hat{\vartheta}, \hat{\mathbf{q}})$ and $(\hat{f}, \hat{g}, \hat{\mathbf{l}})$ and hence $(\hat{f}, \hat{g}, \hat{\mathbf{l}}) \in \mathcal{A}$.

The application of the Plancherel theorem allows us to complete the proof of the existence of the solution to our problem. This ends the proof of the theorem.

4. Thermodynamic restrictions and free energy. In this last section we examine the restrictions placed by the thermodynamic principles on the material constants which characterize the behaviour of the thin homogeneous, isotropic, thermoelastic plate, we have considered in the previous sections; moreover we give an explicit representation of a pseudo free energy.

With the notation already introduced in Section 2, under the hypotheses of small deformations and small variations of the temperature with respect to the given reference configuration and to the absolute temperature Θ_0, we assume the following constitutive equations for the mean stress tensor $\mathbf{T}$ and the rate at which heat is absorbed for a unit volum h:

$$\mathbf{T}(\mathbf{x}, t) = -a\nabla[\Delta u(\mathbf{x}, t)] + b\nabla u_{tt}(\mathbf{x}, t) - \varepsilon \mathbf{g}(\mathbf{x}, t), \tag{4.1}$$

$$\rho_0 h(\mathbf{x}, t) = \Theta_0 \beta \Delta u_t(\mathbf{x}, t) + \rho_0 c \vartheta_t(\mathbf{x}, t), \tag{4.2}$$

where $\mathbf{g}=\nabla\vartheta$, ρ_0 is the mass density, c is the heat capacity and a, b, ε, β are constitutive constants.

The fundamental system of the linear theory of thermoelasticity, when Cattaneo – Maxwell's equation is assumed as the relation between the heat flux and the temperature gradient, is

$$\rho_0 u_{tt}(\mathbf{x}, t) = \nabla \cdot \mathbf{T}(\mathbf{x}, t) + \rho_0 f(\mathbf{x}, t), \tag{4.3}$$

$$\rho_0 h(\mathbf{x}, t) = -\nabla \cdot \mathbf{q}(\mathbf{x}, t) + \rho_0 g(\mathbf{x}, t), \tag{4.4}$$

$$\tau \mathbf{q}_t(\mathbf{x}, t) + \mathbf{q}(\mathbf{x}, t) = -k\nabla\vartheta(\mathbf{x}, t), \tag{4.5}$$

where we have introduced the body force f and the heat source g, k and $\tau(> 0)$ being two constants.

We observe that the constitutive equations (4.1), (4.2) characterize a thermodynamic system when the state is $\sigma(\mathbf{x}) = (\nabla u_t(\mathbf{x}), \Delta u(\mathbf{x}), \vartheta(\mathbf{x}), \mathbf{q}(\mathbf{x}))$ at $\mathbf{x} \in \Omega$ and the thermokinetic process of duration $d_P \in \mathbf{R}^+$ is a piecewise continuous map defined on $[0, d_P)$ by $P(\mathbf{x}, t) = (\nabla[\Delta u(\mathbf{x}, t)],$ $\nabla u_{tt}(\mathbf{x}, t), \Delta u_t(\mathbf{x}, t), \vartheta_t(\mathbf{x}, t), \mathbf{g}(\mathbf{x}, t))$. If we introduce a state-transition function $\tilde{\rho} : \Sigma^{\mathbf{x}} \times \Pi^{\mathbf{x}} \to$ $\to \Sigma^{\mathbf{x}}$ which assigns to the initial state σ^i, of the space $\Sigma^{\mathbf{x}}$ at the point $\mathbf{x}$, and the process P, of the thermokinetic process space $\Pi^{\mathbf{x}}$, the final state σ^f, that is $\sigma^f = \tilde{\rho}(\sigma^i, P)$, we can consider $\Sigma^{\mathbf{x}}_{\sigma_0} = \{\sigma \in \Sigma^{\mathbf{x}} : \exists\ P \in \Pi^{\mathbf{x}},\ \sigma = \tilde{\rho}(\sigma_0, P)\}$, the subset of the states which can be obtained from a fixed state σ_0 with a process P. The assumed constitutive equations are functions of (σ, P), that is, we have $\mathbf{T} = \tilde{\mathbf{T}}(\sigma, P)$, $h = \tilde{h}(P)$, $\mathbf{q} = \tilde{\mathbf{q}}(\sigma, P)$.

Thus, we recall the definition of a *cycle constant on* Γ [9], which is a pair $(\sigma(\mathbf{x}), P(\mathbf{x}))$ such that $\tilde{\rho}(\sigma^0(\mathbf{x}), P(\mathbf{x})) = \sigma^0(\mathbf{x})\ \forall \mathbf{x} \in \Omega$ and $\sigma(\mathbf{x}, t) = \tilde{\rho}(\sigma^0(\mathbf{x}), P_{[0,t)}(\mathbf{x}))$ is constant $\forall \mathbf{x} \in \Gamma$ and $t \in [0, d_P)$, $P_{[0,t)}$ being the restriction of P to $[0, t) \subset [0, d_P)$.

Now, we give the expressions of the two law of thermodynamics [10], the first of which yields the existence of the internal energy $\tilde{e} : \Sigma^{\mathbf{x}}_{\sigma_0} \to \mathbf{R}$ such that

$$\rho_0 \tilde{e}_t(\sigma(\mathbf{x}, t)) = \rho_0 \tilde{h}(P(\mathbf{x}, t)) + \tilde{\mathbf{T}}(\sigma(\mathbf{x}, t), P(\mathbf{x}, t)) \cdot \nabla u_t(\mathbf{x}, t) \tag{4.6}$$

for continuous processes, while the second one states that the inequality

$$\oint \int_{\Omega} \left\{ \frac{\rho_0 \tilde{h}(P(\mathbf{x},t))}{\Theta_0 + \vartheta(\mathbf{x},t)} + \frac{\tilde{\mathbf{q}}(\sigma(\mathbf{x},t), P(\mathbf{x},t)) \cdot \mathbf{g}(\mathbf{x},t)}{[\Theta_0 + \vartheta(\mathbf{x},t)]^2} \right\} d\mathbf{x} dt \le 0 \tag{4.7}$$

holds for an isolated material Ω for every pair $(\sigma^0(\mathbf{x}), P(\mathbf{x}))$ which defines cycle constant on Γ, the equality sign referring to reversible processes.

Since we are concerned with a linear theory, we must derive an approximation of the second law; therefore, neglecting the terms of order greater than two, (4.7) assumes the form

$$\frac{1}{\Theta_0^2} \oint \int_{\Omega} \left\{ \rho_0 \tilde{h}(P(\mathbf{x},t))[\Theta_0 - \vartheta(\mathbf{x},t)] + \tilde{\mathbf{q}}(\sigma(\mathbf{x},t), P(\mathbf{x},t)) \cdot \mathbf{g}(\mathbf{x},t) \right\} d\mathbf{x} dt \le$$

$$\le \frac{1}{\Theta_0^2} \oint \int_{\Gamma} \tilde{\mathbf{F}}(\sigma(\mathbf{x},t), P(\mathbf{x},t)) \cdot \nu(\mathbf{x}) d\Gamma dt, \tag{4.8}$$

where we have added to the right-hand side the surface integral in order to consider the global formulation of the second law in agreement with the existence of the flux $\mathbf{F}$.

From the inequality (4.8), under the hypotheses that the material, we are considering, is self-consistent, that is, when the constitutive equations, relative to $\mathbf{x} \in \Omega$ and $t \in \mathbf{R}^+$, do not depend upon fields outside Ω at time t, it follows that

$$\frac{1}{\Theta_0^2} \oint \left\{ \rho_0 \tilde{h}(P(\mathbf{x},t))[\Theta_0 - \vartheta(\mathbf{x},t)] + \tilde{\mathbf{q}}(\sigma(\mathbf{x},t), P(\mathbf{x},t)) \cdot \mathbf{g}(\mathbf{x},t) - \right.$$

$$\left. - \nabla \cdot \tilde{\mathbf{F}}(\sigma(\mathbf{x},t), P(\mathbf{x},t)) \right\} dt \le 0,$$

for any $\mathbf{x} \in \Omega$, since (4.8) must hold for any subbody of Ω.

Furthermore, as a consequence of the second law, it is possible to show [11] the existence of the entropy $\hat{\eta} : \Sigma_{\sigma_0}^{\mathbf{x}} \to \mathbf{R}$ for any $\mathbf{x} \in \Omega$ such that

$$\dot{\tilde{\eta}}_t(\sigma(\mathbf{x},t)) \ge \frac{1}{\rho_0 \Theta_0^2} \left\{ \rho_0 \tilde{h}(P(\mathbf{x},t))[\Theta_0 - \vartheta(\mathbf{x},t)] + \right.$$

$$\left. + \tilde{\mathbf{q}}(\sigma(\mathbf{x},t), P(\mathbf{x},t)) \cdot \mathbf{g}(\mathbf{x},t) - \nabla \cdot \tilde{\mathbf{F}}(\sigma(\mathbf{x},t), P(\mathbf{x},t)) \right\} \tag{4.9}$$

for any smooth process.

In order to obtain consequences of the laws of thermodynamics on the material constants, we observe that (4.8), elimintaing $\rho_0 \Theta_0 h$ by means of (4.6) and integrating on a cycle, reduces to

$$\oint [\rho_0 \tilde{h}(P(t))\vartheta(t) \ + \Theta_0 \tilde{\mathbf{T}}(\sigma(t), P(t)) \cdot \nabla u_t(t) - \tilde{\mathbf{q}}(\sigma(t), P(t)) \cdot \mathbf{g}(t) +$$

$$+ \nabla \cdot \tilde{\mathbf{F}}(\sigma(t), P(t))]dt \geq 0, \tag{4.10}$$

where the dependence on $\mathbf{x}$ is understood.

Substituting the expression (4.2) of h, assuming for the heat flux $\mathbf{F}$ the following form:

$$\mathbf{F}(\mathbf{x}, t) = \Theta_0[a\Delta u(\mathbf{x}, t)\nabla u_t(\mathbf{x}, t) - \beta\vartheta(\mathbf{x}, t)\nabla u_t(\mathbf{x}, t)],$$

and using (4.5), from (4.10) we get

$$\oint \Bigg\{ \rho_0 c \vartheta_t(t)\vartheta(t) + \Theta_0[b\nabla u_{tt}(\mathbf{x}, t) - \varepsilon \mathbf{g}(\mathbf{x}, t)] \cdot \nabla u_t(t) + \frac{1}{k}[\tau \mathbf{q}_t(t) +$$

$$+ \mathbf{q}(t)] \cdot \mathbf{q}(t) + \Theta_0 a \Delta u_t(t)\Delta u(t) - \beta\Theta_0 \mathbf{g}(t) \cdot \nabla u_t(t)] \Bigg\} dt \geq 0.$$

Since the integral is taken on a cycle, this inequality reduces to

$$\oint \left[\frac{1}{k}\mathbf{q}^2(t) - \Theta_0(\varepsilon + \beta)\mathbf{g}(t) \cdot \nabla u_t(t) \right] dt \geq 0,$$

which holds for every $\mathbf{g}$; therefore, we have the restrictions

$$\varepsilon + \beta = 0, \qquad k > 0. \tag{4.11}$$

Finally, we introduce the following approximate pseudo-free energy:

$$\psi(\mathbf{x}, t) = e(\mathbf{x}, t) - \Theta_0 \eta(\mathbf{x}, t),$$

which, using (4.6) to eliminate $\rho_0\Theta_0 h$, allows us to transform (4.9) as follows:

$$\rho_0 \psi_t(t) \leq \frac{\rho_0}{\Theta_0} h(t)\vartheta(t) + \mathbf{T}(t) \cdot \nabla u_t(t) - \frac{1}{\Theta_0}\mathbf{q}(t) \cdot \mathbf{g}(t) + \frac{1}{\Theta_0}\nabla \cdot \mathbf{F}(t). \tag{4.12}$$

Substituting (4.1), (4.2) and eliminating $\mathbf{g}$ by using of (4.5), (4.12) yields

$$\rho_0 \psi_t(t) \leq \frac{d}{dt}\frac{1}{2}\left\{ a[\Delta u(t)]^2 + b[\nabla u_t(t)]^2 + \frac{\rho_0 c}{\Theta_0}\vartheta^2(t) + \frac{\tau}{k\Theta_0}\mathbf{q}^2(t) \right\} + \frac{1}{k\Theta_0}\mathbf{q}^2(t), \tag{4.13}$$

whence we can assume

$$\rho_0 \psi(\nabla u_t, \Delta u, \vartheta, \mathbf{q}) \leq \frac{1}{2}\left\{ a[\Delta u(t)]^2 + b[\nabla u_t(t)]^2 + \frac{\rho_0 c}{\Theta_0}\vartheta^2(t) + \frac{\tau}{k\Theta_0}\mathbf{q}^2(t) \right\},$$

which satisfies (4.13) on account of $(4.11)_2$.

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