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EXISTENCE OF POSITIVE SOLUTIONS FOR THE SINGULAR EQUATION $(\varphi_p(y'))' + g(t,y,y') = 0$*

ІСНУВАННЯ ДОДАТНИХ РОЗВ'ЯЗКІВ СИНГУЛЯРНОГО РІВНЯННЯ $(\varphi_p(y'))' + g(t,y,y') = 0$

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Two theorems concerning the existence of positive solutions for the singular equation $(\varphi_p(y'))' + g(t,y,y') = 0$, $y(0) = y(1) = 0$, *are presented. The results are obtained by using the nonlinear alternative of Leray – Schauder and the lower-upper solution method.*

Наведено дві теореми про існування додатних розв'язків сингулярного рівняння $(\varphi_p(y'))' + g(t,y,y') = 0$, $y(0) = y(1) = 0$. *Результати отримано з використанням нелінійної альтернативи Лере – Шаудера і методу верхнього та нижнього розв'язків.*

1. Introduction and results. In this article existence results are presented for the second-order differential equation

$$\begin{aligned} &(\varphi_p(y'))' + g(t,y,y') = 0, \ 0 < t < 1, \\ &y(0) = y(1) = 0, \end{aligned} \tag{1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p \in R$, and $p > 1$. Equations of the above form occur in the study of the n-Laplace equation [1], non-Newtonial fluid theory [2], and the turbulent flow of a gas in a porous medium [3].

Some basic results concerning the boundary-value problem (1) can be found in [4 – 11] (and references therein). In all these papers the argument relies on the fact that $g(t,y,z)$ is continuous or satisfies a Carathèodory condition. In [12], D. O'Regan established existence results

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for the second-order differential equation

$$(\varphi_p(y'))' + q(t)f(t,y,y') = 0,\ 0 < t < 1,\ 1 < p \le 2,$$
$$y(0) = A,\ y(1) = B, \tag{2}$$

where $f : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is continuous, $q \in C(0,1)$, and $\int_0^1 q(s)ds < \infty$.

In this paper, we present existence results for (1) with $p > 1$. The function $g(t,y,z)$ can be singular at both endpoints $t = 0$, and $t = 1$, and also $y = 0$. The present work is a direct extension of some results in [12]. Our technique of proof uses the nonlinear alternative of Leary – Schauder and the method of upper and lower solutions.

The paper is organized as follows. In Part 1, we present our main results: Theorem 1 and Theorem 2. Part 2 is devoted to preparatory work for the proof. Finally, in Part 3 we prove the main theorems.

The main results of this paper are as follows.

Theorem 1. *Let $g : (0,1) \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ be continuous and suppose the following conditions are satisfied:*

(K_1) $\exists L > 0$ *such that for any compact set $l \subset (0,1)$ there exists $\varepsilon_l > 0$ with*

$$g(t,y,z) > L, \text{ for all } (t,y,z) \in l \times (0,\varepsilon_l] \times \mathbb{R};$$

(K_2) *for every $\tau > 0$, there are two positive continuous functions $q_\tau \in C(0,1)$ and $\psi : [0,\infty) \to (0,\infty)$ such that*

$$|g(t,y,z)| \le q_\tau(t)\,\psi(|z|), \text{ for all } (t,y,z) \in (0,1) \times [\tau,\infty) \times \mathbb{R},$$

with

$$\int_0^1 q_\tau(t)dt < \infty, \text{ and } \int_0^\infty \frac{du}{\psi(\varphi_p^{-1}(u))} > \int_0^1 q_\tau(t)dt;$$

here φ_p^{-1} is the inverse function of φ_p.

Then (1) has at least one solution $y \in C[0,1] \cap C^1(0,1)$ with $y(t) > 0$ for $t \in (0,1)$.

Theorem 2. *Let $g : (0,1) \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be continuous and suppose the following conditions are satisfied:*

(G_1) *there exist constants $L > 0$ and $r > 0$ with*

$$g(t,y,z) > L, \text{ for all } (t,y,z) \in (0,1) \times [0,r] \times \mathbb{R};$$

(G_2) *there exist positive continuous functions $\psi : [0,\infty) \to (0,\infty)$ and $q \in C(0,1)$ such that*

$$|g(t,y,z)| \le q(t)\,\psi(|z|), \text{ for all } (t,y,z) \in (0,1) \times [0,\infty) \times \mathbb{R},$$

with

$$\int_0^1 q(t)dt < \infty, \quad \textit{and} \quad \int_0^\infty \frac{du}{\psi(\varphi_p^{-1}(u))} > \int_0^1 q(t)dt;$$

here φ_p^{-1} *is the inverse function of* φ_p.

Then (1) has at least one solution $y \in C^1[0,1]$ *and* $y(t) > 0$ *for* $t \in (0,1)$.

2. Preparatory work. For $\rho \in (0,1]$, define the operator

$$N_\rho : C[0,1] \to C[0,1]$$

by

$$(N_\rho y)(x) := \varphi^{-1}\left(A_y + \rho \int_0^x g(\tau, (J\,y)(\tau), y(\tau))d\,\tau\right), \tag{3}$$

where

$$J\,(y)(\tau) = b - \int_\tau^1 y(s)ds, \quad \text{for all} \;\; 0 \le \tau \le 1, \tag{4}$$

and $A_y \in (-\infty, \infty)$ is such that

$$\int_0^1 \varphi^{-1}\left(A_y + \rho \int_0^x g(\tau, J\,(y)(\tau), y(\tau))\,d\,\tau\right)\,dx = b - a. \tag{5}$$

Lemma 1. *Let* $g : (0,1)\times\mathbb{R}^2 \to \mathbb{R}$ *be continuous and suppose there exist positive continuous functions* $q \in C(0,1)$ *and* $\psi : [0,\infty) \to (0,\infty)$, *with*

$$\int_0^1 q(t) < \infty,$$

and

$$|g(t,y,z)| \le q(t)\,\psi(|z|) \;\; \textit{for all} \;\; (t,y,z) \in (0,1)\times\mathbb{R}^2.$$

Then

(1) $N_\rho : C[0,1] \to C[0,1]$ *is completely continuous;*

(2) if $\Omega \subset \{z \in C[0,1] : \; (N_\rho z)(t) = z(t)\}$ *and* $\sup\{\sup_{t\in[0,1]} |z(t)| : \; z \in \Omega\} < \infty$, *then* Ω *is a relative compact set in* $C[0,1]$.

Proof. From the proof of Theorem 2.3 in [12], we easily obtain that A_y exists and is unique for every $y \in C[0,1]$.

Let $\Omega \in C[0,1]$ be bounded, i. e. $\sup_{t\in[0,1]} |y(t)| \leq N_\Omega$ for all $y \in \Omega$ (here $N_\Omega > 0$ is a constant). Now (5) and the Mean Value Theorem for integrals implies that there exists $\xi \in (0,1)$ with

$$\varphi^{-1}\left(A_y + \rho\int\limits_0^\xi g(\tau,(J\,y)(\tau),y(\tau))\,d\,\tau\right) = b-a.$$

Consequently,

$$A_y = \varphi(b-a) - \rho\int\limits_0^\xi g(\tau,(J\,y)(\tau),y(\tau))\,d\,\tau,$$

which implies

$$\left|A_y + \rho\int\limits_0^x g(\tau,(J\,y)(\tau),y(\tau))\,d\tau\right| \leq |\varphi(b-a)| + 2\,M_\Omega\,Q(1) =: P_\Omega \tag{6}$$

for $0 \leq x \leq 1$; here $M_\Omega = \sup_{z\in[-N_\Omega,N_\Omega]} |\psi(|z|)|$, and $Q(x) = \int_0^x q(s)ds$.

We first show that $N_\rho\Omega$ is bounded. This follows since

$$|(N_\rho y)(t)| \leq \sup_{-P_\Omega\leq u\leq P_\Omega} |\varphi_p^{-1}(u)|,\ 0\leq t\leq 1,\ \forall y\in\Omega.$$

We next show the equicontinuity of $N_\rho\Omega$ on $[0,1]$. Notice $\varphi_p^{-1}(u)$ is uniformly continuous on $[-P_\Omega, P_\Omega]$, so for $\varepsilon > 0$, there exists $\delta_1 > 0$, such that

$$\left|\varphi^{-1}(u_1) - \varphi^{-1}(u_2)\right| < \varepsilon, \text{ if } u_1,\, u_2 \in [-P_\Omega, P_\Omega] \text{ and } |u_1 - u_2| < \delta_1.$$

Let

$$u_k := A_y + \rho\int\limits_0^{x_k} g(\tau,(J\,y)(\tau),y(\tau))\,d\,\tau,\ 0\leq x_k\leq 1,\ k=1,\,2.$$

Since Q is uniformly continuous on $[0,1]$, we know for the above fixed $\delta_1 > 0$, that there exists $\delta > 0$, such that

$$|u_1 - u_2| < M_\Omega\,|Q(x_1) - Q(x_2)| < \delta_1,$$

if $x_1,\ x_2 \in [0,1]$, and $|x_1 - x_2| < \delta$.

Consequently, for $\varepsilon > 0$, there exists $\delta > 0$, such that if $x_1,\ x_2 \in [0,1]$, and $|x_1 - x_2| < \delta$, then

$$|(N_\rho y)(x_1) - (N_\rho y)(x_2)| < \epsilon,$$

for every $y \in \Omega$. This shows that $N_\rho \Omega$ is equicontinuous. The Arzela – Ascoli theorem guarantees that N_ρ is completely continuous.

We now claim that $N_\rho : C[0,1] \to C[0,1]$ is continuous. Let $y_k \in C[0,1]$, $k = 0, 1, \ldots$, and $y_k \to y_0$ uniformly on $[0,1]$. We must show $N_\rho y_k \to N_\rho y_0$ uniformly on $[0,1]$. Associate A_{y_k} with y_k in (3). Then

$$(N_\rho y_k)(t) - (N_\rho y_0)(t) = \varphi_p^{-1}\left(A_{y_k} + \rho\int_0^t g\left(s, (J\, y_k)(s), y_k(s)\right)\, ds\right) -$$

$$- \varphi_p^{-1}\left(A_{y_0} + \rho\int_0^t g\left(s, (J\, y_0)(s), y_0(s)\right)\, ds\right), \quad k = 1, 2, \ldots, \tag{7}$$

where A_{y_k} is such that

$$\int_0^1 \varphi_p^{-1}\left(A_{y_k} + \rho\int_0^x g\left(s, (J\, y_k)(s), y_k(s)\right)\, ds\right)\, dx -$$

$$- \int_0^1 \varphi_p^{-1}\left(A_{y_0} + \rho\int_0^t g\left(s, (J\, y_0)(s), y_0(s)\right)\, ds\right)\, dx = 0.$$

The Mean Value Theorem for integrals implies that there exists $\xi_k \in (0,1)$ with

$$\varphi_p^{-1}\left(A_{y_k} + \rho\int_0^{\xi_k} g(s, (J\, y_k)(s), y_k(s))\, ds\right) -$$

$$- \varphi_p^{-1}\left(A_{y_0} + \rho\int_0^{\xi_k} g(s, (J\, y_0)(s), y_0(s))\, ds\right) = 0, \tag{8}$$

for $k = 1, 2, \ldots.$. Thus for $k = 1, 2, \ldots.$ we have

$$A_{y_k} - A_{y_0} = \rho\int_0^{\xi_k} \left(g\left(s, (J\, y_0)(s), y_0(s)\right) - g\left(s, (J\, y_k)(s), y_k(s)\right)\right)\, ds.$$

Now since $y_k \to y_0$ uniformly on $[0,1]$, we have $\lim_{k\to\infty} A\, y_k = A\, y_0$.

Then (7), and the fact that $\varphi_p^{-1}(u)$ is strictly increasing and continuous, guarantees that $N_\rho : C[0,1] \to C[0,1]$ is continuous.

(2). The proof is easy and omitted.

Lemma 2. *Let*

$$e_n = \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}\right], \quad n \geq 1, \quad e_0 = \emptyset.$$

If $0 < \varepsilon_n < 1$ *and* $\varepsilon_n \downarrow 0$*, then there exists* $\lambda \in C^1[0,1]$*, with*

(1) $\varphi_p(\lambda') \in C^1[0,1]$ *and* $\max_{0\leq t\leq 1} |(\varphi_p(\lambda'(t)))'| > 0$*,*

and

(2) $\lambda(0) = \lambda(1) = 0$ *with* $0 < \lambda(t) \leq \varepsilon_n$*,* $t \in e_n \setminus e_{n-1}$*,* $n \geq 1$.

Proof. Let $r : [0,1] \to [0,\infty)$ be such that $r(0) = r(1) = 0$ and $r(t) = \varepsilon_n^{p-1}$ for all $t \in e_n \setminus e_{n-1}$, $n \geq 1$. Moreover, let

$$u(t) = \int_0^t r(s)\, ds, \quad v(t) = \left[\int_0^t u(s)\, ds\right]^{\frac{1}{p-1}},$$

and $w(t) = \int_0^t v(s)\, ds$. It is obvious that u, v, and $w : \left[0, \frac{1}{2}\right] \to [0,\infty)$ are continuous and strictly increasing, with $w\left(\frac{1}{4}\right) < \varepsilon_1$.

Choose a natural number $k \geq 2$ with

$$\frac{\left[(4\,k+1)\, v\left(\frac{1}{4}\right) + 4\, v'\left(\frac{1}{4}\right)\right]}{[\,16\,k\,(k+1)]} \leq \varepsilon_1 - w\left(\frac{1}{4}\right)$$

and $(2k+1)\,(p-1) > 1$ if $1 < p < 2$. Let

$$c_0 = \frac{4^{2k}\left[(2k-1)\, v\left(\frac{1}{4}\right) + 4\, v'\left(\frac{1}{4}\right)\right]}{(k+1)},$$

$$c_1 = -\frac{4^{2(k-1)}\left[(2k-1)\, v\left(\frac{1}{4}\right) + 4\, v'\left(\frac{1}{4}\right)\right]}{k},$$

$$c_2 = w\left(\frac{1}{4}\right) + \frac{\left[(4\,k+1)\, v\left(\frac{1}{4}\right) + 4\, v'\left(\frac{1}{4}\right)\right]}{[\,16\,k\,(k+1)]},$$

and

$$p(t) = c_0\left(t - \frac{1}{2}\right)^{2(k+1)} + c_1\left(t - \frac{1}{2}\right)^{2k} + c_2.$$

Define $\lambda : [0,1] \to [0,\infty)$ as follows:

$$\lambda(t) = \begin{cases} w(t), & 0 \le t \le \dfrac{1}{4}; \\ p(t), & \dfrac{1}{4} \le t \le \dfrac{3}{4}; \\ w(1-t), & \dfrac{3}{4} \le t \le 1. \end{cases}$$

Then λ satisfies (1) and (2).

The lemma is proved.

Let's consider the two-point boundary-value problem

$$\begin{aligned} &(\varphi_p(y'))' + g(t,y,y') = 0, \quad 0 < t < 1, \\ &y(0) = A, \ \ y(1) = B. \end{aligned} \tag{9}$$

We say α is a lower solution for (9) if $\alpha \in C^1[0,1]$, $\varphi_p(\alpha') \in C^1(0,1)$ and

$$(\varphi_p(\alpha'(t)))' + g(t,\alpha(t),\alpha'(t)) \ge 0, \quad 0 < t < 1. \tag{10}$$

The definition of an upper solution for (9) is given in a similar way (just reverse the above inequality).

Lemma 3. *Let $g : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $q \in C(0,1)$ with*

$$\int\limits_0^1 q(t)dt < \infty,$$

and

$$|g(t,y,z)| \le M\, q(t), \quad \textit{for all} \ \ (t,y,z) \in (0,1) \times \mathbb{R}^2,$$

where $M > 0$ is a constant. Then (9) has at least one solution $y \in C^1[0,1]$ with $\varphi_p(y') \in$ $\in C^1(0,1)$.

The proof is easily obtained using the Leray – Schauder nonlinear alternative and Lemma 1. We leave the details to the reader.

Lemma 4. *Suppose*

(H_1) *$g : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.*

In addition suppose $\alpha \in C^1[0,1]$, $\beta \in C^1[0,1]$ are lower and upper solutions of (9), *$q \in$ $\in C(0,1)$, and $\psi : [0,\infty) \to (0,\infty)$. Assume the following conditions are satisfied:*

(H_2) *$\alpha(t) \le \beta(t)$ for all $0 \le t \le 1$,*

(H_3) *$\alpha(0) \le A \le \beta(0)$ and $\alpha(1) \le B \le \beta(1)$,*

(H_4) *$|g(t,y,z)| \le q(t)\,\psi(|z|)$ for all $(t,y,z) \in D_{\alpha\beta} \times \mathbb{R}$, where $D_{\alpha\beta} = \{\,(t,y) : \ 0 < t <$ $< 1, \ \alpha(t) \le y \le \beta(t)\,\}$,*

(H_5) $q(t) > 0$ *for all* $t \in (0,1)$ *and* $\int\limits_0^1 q(t)\,dt < \infty$,

and

(H_6) $\psi : [0,\infty) \to (0,\infty)$ *is continuous with*

$$\int\limits_{\varphi_p(|B-A|)}^{\infty} \frac{du}{\psi(\varphi_p^{-1}(u))} > \int\limits_0^1 q(t)\,dt.$$

Then (9) has at least one solution $y \in C^1[0,1]$, $\varphi_p(y') \in C^1(0,1)$, *with* $\alpha(t) \le y(t) \le \beta(t)$ *for all* $0 \le t \le 1$.

Proof. Choose

$$N > \max\left\{ \sup_{t\in[0,1]} |\alpha'(t)|, \sup_{t\in[0,1]} |\beta'(t)|, |B-A| \right\}$$

with

$$\int\limits_{\varphi_p(|B-A|)}^{\varphi_p(N)} \frac{du}{\psi(\varphi_p^{-1}(u))} > \int\limits_0^1 q(t)\,dt. \tag{11}$$

Consider the boundary-value problem

$$\begin{aligned} &(\varphi_p(y'))' + g^*(t,y,y') = 0, \quad 0 < t < 1, \\ &y(0) = A, \quad y(1) = B, \end{aligned} \tag{12}$$

where

$$g^*(t,y,z) = \begin{cases} g(t,\beta(t),u^*) + q(t)\,\dfrac{\beta(t)-y}{1+y-\beta(t)}, & y > \beta(t); \\ g(t,y,u^*), & \alpha(t) \le y \le \beta(t); \\ g(t,\alpha(t),u^*) + q(t)\,\dfrac{\alpha(t)-y}{1-y+\alpha(t)}, & y < \alpha(t), \end{cases}$$

and

$$u^* = \begin{cases} N & \text{if} \quad u > N; \\ u & \text{if} \quad -N \le u \le N; \\ -N & \text{if} \quad u < -N. \end{cases}$$

Notice $g^* : (0,1)\times\mathbb{R}^2 \to \mathbb{R}$ is continuous with

$$|g^*(t,y,z)| \le C\,q(t) \quad \text{for} \quad (t,y,z) \in (0,1)\times\mathbb{R}^2; \tag{13}$$

here $C = \sup_{z\in[N,-N]} \psi(|z|) + 1$. Now Lemma 3 guarantees that (12) has at least one solution $y \in C^1[0,1]$, with $\varphi_p(y') \in C^1(0,1)$.

If we show

$$\alpha(t) \le y(t) \le \beta(t) \ \text{ for } \ t \in [0,1] \ \text{ and } \ \sup_{t\in[0,1]} |y'(t)| \le N, \tag{14}$$

then y is a solution of (9).

First we show $\alpha(t) \le y(t)$. If this is not true, then from (H_3) there exists $t_0 \in (0,1)$ with

$$\min_{t\in[0,1]} (y(t) - \alpha(t)) = y(t_0) - \alpha(t_0) < 0, \quad \text{and} \ \ y'(t_0) = \alpha'(t_0).$$

Notice also that

$$\big(\varphi_p(y')\big)'(t_0) - g\big(t_0, \alpha(t_0), \alpha'(t_0)\big) = q(t_0) \ \frac{y(t_0) - \alpha(t_0)}{1 + y(t_0) - \alpha(t_0)} < 0,$$

since $N > \sup_{t\in[0,1]} |\alpha'(t)| \ge \alpha'(t_0)$. Consequently

$$\big(\varphi_p(y')\big)'(t_0) < \big(\varphi_p(\alpha')\big)'(t_0)\,,$$

so there exists $\varepsilon > 0$ with

$$\big(\varphi_p(y')\big)'(t) < \big(\varphi_p(\alpha')\big)'(t) \quad \text{for} \ \ t \in (t_0, t_0 + \varepsilon).$$

Now since $y'(t_0) = \alpha'(t_0)$ we have

$$\varphi_p(y'(t)) < \varphi_p(\alpha'(t)) \ \ \text{for} \ \ t \in (t_0, t_0 + \varepsilon),$$

and so

$$y'(t) < \alpha'(t) \ \ \text{for} \ \ t \in (t_0, t_0 + \varepsilon),$$

a contradiction. Then $\alpha(t) \le y(t)$ for $t \in [0,1]$. A similar argument shows $y(t) \le \beta(t)$ for $t \in [0,1]$.

It remains to show $\sup_{t\in[0,1]} |y'(t)| \le N$. The Mean Value Theorem guarantees that there exists $\xi \in (0,1)$ with $y'(\xi) = B - A$. Without loss of generality assume $y'(t) \not\le N$. Then there exist $t_1, t_2 \in [0,1]$, with $y'(t_1) = |B - A|$, $y'(t_2) = N$, and

$$|B - A| \le y'(t) \le N \ \ \text{for} \ \ t \ \ \text{between} \ \ t_1 \ \ \text{and} \ \ t_2.$$

Without loss of generality assume $t_1 < t_2$. Now $\alpha(t) \le y(t) \le \beta(t)$, $t \in [0,1]$ and (H_4) guarantees that

$$(\varphi_p(y'))'(t) \le q(t)\psi(y'(t)) \ \ \text{for} \ \ t \in (t_1, t_2)\,.$$

Divide by $\psi(y'(t))$ and integrate from t_1 to t_2 to obtain

$$\int\limits_{\varphi_p(|B-A|)}^{\varphi_p(N)} \frac{du}{\psi(\varphi_p^{-1}(u))} \leq \int\limits_{t_1}^{t_2} q(s)\,ds \leq \int\limits_0^1 q(s)\,ds.$$

This contradicts (11). A similar argument yields a contradiction in the other cases.

3. Proof of Theorems 1 and 2. ***Proof of Theorem 1.*** Let $I = (0, \infty)$, $e_0 = \emptyset$ and

$$e_n = \left[\frac{1}{2^{n+1}}\,,\, 1 - \frac{1}{2^{n+1}}\right] \quad \text{for} \quad n = 1, 2, \ldots .$$

For $n = 1, 2, ...$ let

$$\theta_n(t) = \max\left\{\frac{1}{2^{n+1}}\,,\, \min\left\{t\,,\, 1 - \frac{1}{2^{n+1}}\right\}\right\}, \quad 0 \leq t \leq 1,$$

and

$$f_n(t, y, z) = \max\{g(\theta_n(t), y, z)\,,\, g(t, y, z)\}.$$

Note for each n that $f_n : (0, 1) \times I \times \mathbb{R} \to \mathbb{R}$ is continuous. Next we define, inductively

$$g_1(t, y, z) = f_1(t, y, z),$$

and

$$g_{n+1}(t, y, z) = \min\{g_n(t, y, z),\, f_{n+1}(t, y, z)\}.$$

Each $g_n : (0, 1) \times I \times \mathbb{R} \to \mathbb{R}$ is continuous. In addition

$$g(t, y, z) \leq \ldots \leq g_{n+1}(t, y, z) \leq g_n(t, y, z) \leq \ldots \leq g_1(t, y, z)$$

for $(t, y, z) \in (0, 1) \times I \times \mathbb{R}$, and

$$g_n(t, y, z) = g(t, y, z) \quad \text{for} \quad (t, y, z) \in e_n \times I \times \mathbb{R}.$$

It follows easily that for all $n \geq 1$ and $(t, y, z) \in (0, 1) \times I \times \mathbb{R}$ we have

$$|g_n(t, y, z)| \leq \sum_{i=1}^{n} |g(\theta_i(t), y, z)| + |g(t, y, z)|.$$

Now condition (K_1) guarantees that there is a constant $L > 0$ such that for any e_n, there exists a $\varepsilon_n > 0$ with

$$g(t, y, z) > L \quad \text{for all} \quad (t, y, z) \in e_n \times (0, \varepsilon_n] \times \mathbb{R}. \tag{15}$$

Without loss of generality, we may assume that $\{\varepsilon_n\}$ is a decreasing sequence and $\lim_{n\to\infty}\varepsilon_n = 0$.

Let's consider the two-point boundary-value problem

$$(\varphi_p(y'))' + g_n(t, y, y') = 0, \ \ 0 < t < 1, \qquad y(0) = y(1) = \varepsilon_n. \tag{16_n}$$

Next we establish four propositions.

Proposition 1. *If* $c_n \in (0, \varepsilon_n]$ *and* $\alpha_n(t) = c_n$ *for all* $0 \le t \le 1$, *then*

$$(\varphi_p(\alpha_n'(t)))' + g_n(t, \alpha_n(t), \alpha_n'(t)) \ge 0 \ \ for \ \ 0 < t < 1$$

(i.e. $\alpha_n(t)$ *is a lower solution for* (16_n)*).*

Proof. We must show

$$g_n(t, c_n, 0) \ge 0 \ \ \text{for all} \ \ c_n \in (0, \varepsilon_n]. \tag{17}$$

We prove this by induction. Let $c_1 \in (0, \varepsilon_1]$. Then (15) implies

$$g_1(t, c_1, 0) = f_1(t, c_1, 0) = \max\{g(\theta_1(t), c_1, 0), \, g(t, c_1, 0)\} \ge$$

$$\ge g(\theta_1(t), c_1, 0) \ge \min_{t\in e_1} g(t, c_1, 0) \ > \ L \ > \ 0.$$

Suppose that (17) holds for a given index $n \ge 1$. Let's check its validity for $n+1$. If $c_{n+1} \in (0, \varepsilon_{n+1}] \subset (0, \varepsilon_n]$, then

$$g_{n+1}(t, c_{n+1}, 0) = \min\{g_n(t, c_{n+1}, 0), \, f_{n+1}(t, c_{n+1}, 0)\} \ge$$

$$\ge \min\{0, \, \max\{g(\theta_{n+1}(t), c_{n+1}, 0), \, g(t, c_{n+1}, 0)\}\} \ge$$

$$\ge \min\{0, L\} = 0.$$

Proposition 2. *If* $y_n \in C^1[0,1]$, $\varphi_p(y_n') \in C^1(0,1)$*, is a solution of* (16_n), *then*

$$(\varphi_p(y_n'(t)))' + g_{n+1}(t, y_n(t), y_n'(t)) \le 0 \ \ for \ \ 0 < t < 1$$

(i.e. y_n *is an upper solution of* (16_{n+1})*).*

Proof. Notice

$$(\varphi_p(y_n'(t)))' + g_{n+1}(t, y_n(t), y_n'(t)) \le (\varphi_p(y_n'(t)))' + g_n(t, y_n(t), y_n'(t)) = 0,$$

and we are finished.

Proposition 3. *For all* $n \geq 1$, (16_n) *has at least one solution* $y_n \in C^1[0,1]$, $\varphi_p(y_n') \in C^1(0,1)$, *with* $\varepsilon_{n+1} \leq y_{n+1}(t) \leq y_n(t)$ *for all* $0 \leq t \leq 1$.

Proof. From (K_2), for all $n \geq 1$, there exists $q_n \in C(0,1)$ with $\int\limits_0^1 q_n(t)dt < \infty$ and

$$|g(t,y,z)| \leq q_n(t)\,\psi(|z|) \quad \text{for all} \quad (t,y,z) \in (0,1) \times [\varepsilon_n, \infty) \times \mathbb{R}.$$

Let

$$\widetilde{q}_n(t) = \sum_{i=1}^{n} q_n(\theta_i(t)) + q_n(t) \quad \text{for} \quad 0 < t < 1.$$

Then $\widetilde{q}_n \in C(0,1)$, $\int\limits_0^1 \widetilde{q}_n(t)dt < \infty$, and

$$|g_n(t,y,z)| \leq \widetilde{q}_n(t)\,\psi(|z|) \quad \text{for} \quad (t,y,z) \in (0,1) \times [\varepsilon_n, \infty) \times \mathbb{R}.$$

On the other hand, we can easily verify, using Lemma 1 and the Leray–Schauder nonlinear alternative, that the two-point boundary-value problem

$$(\varphi_p(y'(t)))' + \widetilde{q}_1(t)\,\psi(|y'|) = 0, \quad 0 < t < 1,$$
$$y(0) = y(1) = \varepsilon_1,$$

has at least one solution $y_0 \in C^1[0,1]$, $\varphi_p(y_0') \in C^1(0,1)$, and $\varepsilon_1 \leq y_0(t)$ for $0 \leq t \leq 1$.

Since $(t, y_0(t), y_0'(t)) \in (0,1) \times [\varepsilon_1, \infty) \times \mathbb{R}$, we have

$$(\varphi_p(y_0'(t)))' + g_1(t, y_0(t), y_0'(t)) \leq -\widetilde{q}_1(t)\,\psi(|y_0'|) + g_1(t, y_0(t), y_0'(t)) \leq 0.$$

Thus y_0 is an upper solution for (16_1).

Proposition 1 guarantees that $\alpha_1(t) = \varepsilon_1$, $0 \leq t \leq 1$, is a lower solution of (16_1), and

$$\alpha_1(t) = \varepsilon_1 \leq y_0(t) \quad \text{for} \quad 0 \leq t \leq 1.$$

From Lemma 4 we deduce that (16_1) has at least one solution $y_1 \in C^1[0,1]$ with $\varphi_p(y_1') \in C^1(0,1)$ and

$$\alpha_1(t) = \varepsilon_1 \leq y_1(t) \leq y_0(t) \quad \text{for} \quad 0 \leq t \leq 1;$$

to see this apply Lemma 4 to g_1^* where

$$g_1^*(t,y,z) = \begin{cases} g_1(t,y,z), & y \geq \varepsilon_1; \\ g_1(t,\varepsilon_1,z), & y < \varepsilon_1. \end{cases}$$

Suppose now that (16_n) has a solution $y_n \in C^1[0,1]$ with $\varphi_p(y_n') \in C^1(0,1)$ and $\varepsilon_n \le y_n(t)$ for $0 \le t \le 1$. Proposition 1 guarantees that

$$\alpha_{n+1}(t) = \varepsilon_{n+1},\ 0 \le t \le 1,$$

is a lower solution for (16_{n+1}). From Proposition 2 we deduce that y_n is a upper solution for (16_{n+1}). Now

$$\alpha_{n+1}(t) = \varepsilon_{n+1} \le \varepsilon_n \le y_n(t) \quad \text{for} \quad 0 \le t \le 1,$$

and Lemma 4 guarantee that (16_{n+1}) has at least one solution $y_{n+1} \in C^1[0,1]$ with $\varphi_p(y_{n+1}') \in$ $\in C^1(0,1)$ and

$$\alpha_{n+1}(t) = \varepsilon_{n+1} \le y_{n+1}(t) \le y_n(t) \quad \text{for} \quad 0 \le t \le 1.$$

Proposition 4. *Suppose $h : (0,1) \times I \times \mathbb{R} \to \mathbb{R}$ is continuous with*

$$h(t,y,z) \ge g(t,y,z) \quad \textit{for all} \quad (t,y,z) \in (0,1) \times I \times \mathbb{R}.$$

Let $\tilde{y} \in C^1[0,1]$ with $\tilde{y}(t) > 0$ for $0 \le t \le 1$ and

$$\left(\varphi_p(\tilde{y}'(t))\right)' + h(t, \tilde{y}(t), \tilde{y}'(t)) = 0 \quad \textit{for} \quad 0 \le t \le 1.$$

Then, there exists a function $\lambda^* \in C^1[0,1]$, $\lambda^*(0) = \lambda^*(1) = 0$ with $\lambda^*(t) > 0$ for $0 < t < 1$, and $\tilde{y}(t) \ge \lambda^*(t)$ for all $0 \le t \le 1$.

Proof. Lemma 2 guarantees that there exists $\lambda \in C^1[0,1]$ with $\varphi_p(\lambda') \in C^1(0,1)$, $\lambda(0) =$ $= \lambda(1) = 0$, $M = \sup_{0 \le t \le 1} |(\varphi_p(\lambda'(t))'| > 0$, and

$$0 < \lambda(t) < \varepsilon_n \quad \text{for} \quad t \in e_n \setminus e_{n-1}, \quad n \ge 1.$$

Let $m = \min\left\{1, \left(\dfrac{L}{M}\right)^{1/(p-1)}\right\}$. We now show

$$\tilde{y}(t) - m\,\lambda(t) \ge 0 \quad \text{for} \quad 0 \le t \le 1.$$

Suppose that there exists $t_0 \in (0,1)$ with

$$\min_{0 \le t \le 1} \{\tilde{y}(t) - m\lambda(t)\} = \tilde{y}(t_0) - m\lambda(t_0) < 0. \tag{18}$$

Then $\tilde{y}'(t_0) - m\,\lambda'(t_0) = 0$. Also there exists a $\varepsilon > 0$ with

$$\tilde{y}'(t_\varepsilon) - m\,\lambda'(t_\varepsilon) \ge 0 \quad \text{for} \quad t_\varepsilon \in (t_0, t_0 + \varepsilon).$$

Since φ_p is an increasing function, we get

$$(\varphi_p(\tilde{y}'(t)))'|_{t=t_0} = \lim_{\varepsilon\to 0^+} \frac{\varphi_p(\tilde{y}'(t_\varepsilon)) - \varphi_p(\tilde{y}'(t_0))}{t_\varepsilon - t_0} \geq$$

$$\geq \lim_{\varepsilon\to 0^+} \frac{\varphi_p(m\,\lambda'(t_\varepsilon)) - \varphi_p(m\,\lambda'(t_0))}{t_\varepsilon - t_0} =$$

$$= (\varphi_p(m\,\lambda'(t)))'|_{t=t_0}.$$

Suppose $t_0 \in e_n \setminus e_{n-1}$. Then $0 < \lambda(t_0) < \varepsilon_n$. From (18) we obtain

$$0 < \tilde{y}(t_0) < m\,\lambda(t_0) < \varepsilon_n,$$

and so

$$L < g(t_0, \tilde{y}(t_0), \tilde{y}'(t_0)) \leq h(t_0, \tilde{y}(t_0), \tilde{y}'(t_0)) = -(\varphi_p(\tilde{y}'(t)))'|_{t=t_0} \leq$$

$$\leq -\,(\varphi_p(m\,\lambda'(t)))'|_{t=t_0} \leq m^{p-1}\,\left[(\varphi_p(m\,\lambda'(t)))'|_{t=t_0}\right] \leq$$

$$\leq m^{p-1}\,M \leq L,$$

a contradiction. Now let $\lambda^*(t) \equiv m\,\lambda(t)$.

Proposition 3 guarantees that (16_n) has at least one solution $y_n \in C^1[0,1]$ with $\varphi_p(y_n') \in \in C^1(0,1)$,

$$0 < \varepsilon_{n+1} \leq y_{n+1} \leq y_n \leq \ldots \leq y_1, \ \ 0 \leq t \leq 1, \tag{19}$$

and

$$y_n(0) = y_n(1) = \varepsilon_n. \tag{20}$$

Proposition 4 (note $g_n \geq g$) implies that there exists $\lambda^* \in C^1[0,1]$, $\lambda^*(0) = \lambda^*(1) = 0$, with $\lambda^*(t) > 0$ for $0 < t < 1$ and $y_n(t) \geq \lambda^*(t)$ for $0 \leq t \leq 1$, $n \geq 1$. Let

$$y(t) = \lim_{n\to\infty} y_n(t), \quad 0 < t < 1.$$

Now (20) and $y_n(t) \geq \lambda^*(t)$ for $t \in (0,1)$ imply $y(0) = y(1) = 0$, and $y(t) > 0$ for $t \in (0,1)$.

Now let $[a,b] \subset (0,1)$. There is an index n^* with $[a,b] \subset e_n$ for all $n > n^*$ and so for $n > n^*$ we have

$$(\varphi_p(y_n'(t)))' + g(t, y_n(t), y_n'(t)) = 0, \ \ a \leq t \leq b.$$

On the other hand $\lambda^* \in C^1[0,1]$, $\lambda^*(t) > 0$ for $0 < t < 1$, and $r = \min_{a\leq t\leq b} \lambda^*(t) > 0$. Moreover (K_2) guarantees that there exists $q_r \in C(0,1)$ with $\int\limits_0^1 q_r(t)dt < \infty$ and

$$|g(t,y,z)| \leq q_r(t)\,\psi(|z|), \ \ (t,y,z) \in (0,1) \times [r,\infty) \times \mathbb{R}.$$

It is easy to see that there exists a continuous function $\tilde{g} : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ with

$$|\tilde{g}(t,y,z)| \leq q_r(t)\,\psi(|z|) \quad \text{for} \quad (t,y,z) \in (0,1) \times \mathbb{R}^2,$$

and

$$\tilde{g}(t,y,z) = g(t,y,z) \quad \text{for} \quad (t,y,z) \in (0,1) \times [r,\infty) \times \mathbb{R}.$$

It is clear that $y_n(t) \geq r$ for $a \leq t \leq b$, for all $n \geq 1$. Moreover,

$$(\varphi_p(y_n'(t)))' + \tilde{g}(t, y_n(t), y_n'(t)) = 0 \quad \text{for} \quad a \leq t \leq b.$$

It is easy to see (look at Lemma 1) that there exists a subsequence S of $\{n^* + 1, n^* + 2, \dots\}$ with

$$\sup_{a \leq t \leq b} |y_n(t) - y(t)| \to 0 \quad \text{and} \quad \sup_{a \leq t \leq b} |y_n'(t) - y'(t)| \to 0 \quad \text{as} \quad n \to \infty \quad \text{in} \quad S.$$

Consequently $\varphi_p(y') \in C^1(a,b)$ with

$$(\varphi_p(y'(t)))' + g(t, y(t), y'(t)) = 0 \quad \text{for} \quad a \leq t \leq b.$$

Since $[a,b] \subset (0,1)$ is arbitrary, we find that

$$y \in C^1(0,1) \quad \text{and} \quad (\varphi_p(y'(t)))' + g(t, y(t), y'(t)) = 0 \quad \text{for} \quad 0 < t < 1.$$

It remains only to check the continuity of y at $t = 0$ and $t = 1$. This follows immediately (see [14]) from the fact that $y_n(t) \downarrow y(t)$ and $y_n(0) = y_n(1) = \varepsilon_n \downarrow 0$. Thus $y \in C[0,1]$. The proof of Theorem 1 is complete.

Proof of Theorem 2. Note (G_1) and (G_2) imply (K_1) and (K_2).

Remark 1. Notice that the regularity of the solution $y \in C[0,1] \cap C^1(0,1)$ of Theorem 1 can't be improved. For example consider

$$g(t,y) = \begin{cases} \dfrac{(1+y^4)}{4y^4}, & 0 < y \leq 2; \\ \dfrac{17}{4y^3}, & 2 \leq y < \infty. \end{cases}$$

Then g satisfies (K_1). It is clear that the problem

$$(\varphi_p(y'(t)))' + g(t, y(t)) = 0, \quad 0 \leq t \leq \pi,$$
$$y(0) = y(\pi) = 0,$$

has a unique positive solution $y = \sqrt{\sin t}$ (note $y \in C[0,\pi] \cap C^1(0,\pi)$). It is easily seen that $y \notin C^1\,[0,1]$.

Remark 2. (K_1) is more general than (G_1). For example let

$$g_1(t, y, z) = \cos\left(\frac{y\pi}{2t}\right) \sin z.$$

Then g_1 satisfies (K_1) but not (G_1).

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