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DETERMINATION OF AN EXACT SOLUTION TO THE INTEGRAL GELFAND – LEVYTAN – MARCHENKO EQUATION FOR THE STURM – LIOUVILLE OPERATORS WITH THE STEP-TYPE POTENTIAL

РОЗВ'ЯЗОК ІНТЕГРАЛЬНОГО РІВНЯННЯ ГЕЛЬФАНДА – ЛЕВІТАНА – МАРЧЕНКА ДЛЯ ОПЕРАТОРА ШТУРМА – ЛІУВІЛЛЯ ІЗ СТУПІНЧАСТИМ ПОТЕНЦІАЛОМ

V. P. Revenko

Inst. Appl. Problems Mech. and Math. Nat. Acad. Sci. Ukraine
Naukova St., 3B, L'viv, Ukraine

A method for solving the integral Gelfand – Levitan – Marchenko (GLM) equation for the Sturm – Liouville operator with a step-type potential is obtained. The scattering function is found explicitly. An associated system of infinite recurrence equations is solved. The integral operator kernel is presented in an exact form using Bessel function. A series of new integral representations for Bessel functions is obtained for the first time.

Запропоновано спосіб розв'язання інтегрального рівняння Гельфанда – Левітана – Марченка для оператора Штурма – Ліувілля у випадку ступінчастого потенціала, заданого на додатній півосі або на всій дійсній осі. Розв'язано (точно) асоційовану з задачею нескінченну систему лінійних рекурентних рівнянь. Отримано ядра інтегральних операторів у явному вигляді через функції Бесселя. Знайдено вперше ряд нових інтегральних співвідношень для функцій Бесселя.

Introduction. Finding a solution to the integral GLM equation for the Sturm – Liouville operator is one of the main tasks when studying many applied problems of mathematical physics [1 – 4]. At present, only solutions to the "notreflected" potentials and N-soliton solutions are known. In what follows we shall use the notations of [2]. The integral GLM equation of spectral scattering problem on the whole axis R^1 for the Sturm – Liouville equation

$$L \cdot e(z,x) = z^2 e(z,x), \tag{1}$$

where the Sturm – Liouville operator $L = -\dfrac{d^2}{dx^2} + g(x)$ acts in the space $L_2(R^1)$, $g(x)$ is an operator-valued potential, $z \in \mathbb{C}$ is a spectral parameter, reads [2] as

$$K^+(x,y) + F^+(x+y) + \int\limits_x^\infty K^+(x,t)F^+(y+t)dt = 0, \tag{2}$$

where the function $F^+(x)$ is found as the Fourier transform of the scattering function [2 – 4]. This kernel $K^+(x,y)$ defines a solution to problem (1) in the form

$$e^+(z,x) = \exp(izx) + \int\limits_x^\infty K^+(x,y)\exp(izy)dy. \tag{3}$$

The absence of the upper index + in relations (2), (3) will correspond to the scattering problem on the positive semi-axis. A series of papers considers the potential in the form of a finite step-type function $g(x) = U, U \geq 0$ if $x \in [0, a]$, $g(x) = 0$ if $x \in (a, \infty)$. In particular, for such a potential, solutions to equation (1) were studied in [1, p. 36–46] as solutions to the steady-state Schrodinger equation, but the solution to the GLM integral equation (2) for this potential remained still unknown. The aim of our presentation is as follows. First, we solve the problem (2), (3) for the Sturm–Liouville operator with a step-type function on the positive semi-axis R^1, and then this solution is extended to the whole axis R^1.

1. Construction of a scattering function and the corresponding function $F(x)$ for R^1. Not specifying here all details of the deduction, we shall present an explicit form of a continuous solution to equation (1) for a given potential with the asymptotics $\exp(izx)$ at infinity:

$$\text{for } x \in [0, a], \;\; e(z, x) = e^{iaz} \frac{((z + w)e^{iw(x-a)} - (z - w)e^{-iw(x-a)})}{2w}, \tag{4}$$

$$\text{for } x \in [0, a], \;\; e(z, x) = e^{izx}. \tag{4'}$$

Here we put $w = \sqrt{z^2 - U}$. Solution (4) is given on the interval $[0, a]$ and then extended continuously as $\exp(izx)$. In [2] it is shown that the function $F(x)$ can be defined as the Fourier transform

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - S(z))e^{izx} dz,$$

where $S(z) = e(-z, 0)/e(z, 0)$ is a scattering function (obtained from the representation $e(z, x)$ (4)) and analytic in the upper half-plane of the parameter $z \in \mathbb{C}^+$ and, besides, it satisfies the condition

$$|1 - S(z)| \leq \frac{C}{|z|} \exp(2a\mathrm{Im}\,(z)). \tag{5}$$

Having used the classical Jordan lemma and condition (5), we obtain

$$\text{for } \; x > 2a, \;\; F(x) = 0, \tag{6}$$

$$\text{for } x \leq 2a, \;\; F(x) = -\lim_{R \to \infty} \frac{1}{2\pi} \oint_{C_R} (1 - S(z)) \exp(izx) dz, \tag{7}$$

where C_R is the semi-circle with radius R in the upper half-plane $z \in \mathbb{C}^+$ with centre at the origin. To obtain the Fourier transform (7), we use the standard procedure of finding the integrals of many-valued functions. The integrand function has the branching point at $z = \pm\sqrt{U} = \pm b$, in order that it be one-valued on the complex plane $z \in \mathbb{C}^+$, we shall make two linear cuts along the real axis from the point $+b$ to $-b$ and along the imaginany axis from the point 0 to $-\infty$.

Choose the positive branch of the function $\sqrt{z^2-U}$, for which the estimate $\left|\dfrac{\sqrt{z^2-U}-z}{\sqrt{z^2-U}+z}\right| <$ $< \dfrac{|A|}{|z|^2}$ holds for sufficiently large $|z| >> 1$. Expand the denominator of the function $1-S(z)$ into the series,

$$(\sqrt{z^2-U}+z)^{-1}\exp\left(-ia(z-\sqrt{z^2-U})\right)\sum_{k=0}^{\infty}\left(\frac{z-\sqrt{z^2-U}}{z+\sqrt{z^2-U}}\right)^k \exp\left(2aik\sqrt{z^2-U}\right), \quad (8)$$

which is absoluty convergent in the upper half-plane $z \in \mathbb{C}^+$. Substituting expansion (8) into integral (7) and taking into account representations (4), from the Jordan lemma we obtain for $0 \le x \le 2a$, that

$$F(x) = \frac{1}{2\pi}\lim_{R\to\infty}\oint\limits_{C_R}\frac{\sqrt{z^2-U}-z}{\sqrt{z^2-U}+z}\exp\left(iz(x-2a)\right)dz. \quad (9)$$

To find the limit in (9), we complete the semi-circle to a closed curve of two quarter-circles of radius R in lower half-plane of the parameter $z \in \mathbb{C}$, a vertical cut connecting the point $z = -Ri$ and the origin, two small circles of radius $\varepsilon \to 0$ around $\pm b$ (the branching point), and a cut connecting those two branching points. In the domain of the complex parameter, the integrand function (9) constructed in this way is analytic and the contour integral along the domain boundary is equal to zero. Applying once again the Jordan lemma and assuming $R\to\infty, \varepsilon\to 0$, we obtain $F(x) = -\dfrac{2}{\pi U}\displaystyle\int_{-b}^{b} z\sqrt{z^2-b^2}\exp(iz(x-2a))dz$, or making use of [5, p. 69], we reduce this integral to the following form:

$$F(x) = -2\frac{J_2\left(b(2a-x)\right)}{2a-x} \quad (10)$$

for $0 \le x \le 2a$, where $J_2(x)$ is the corresponding Bessel function [6].

2. Solution of the integral equation (GLM) on R^1_+. To simplify the subsequent presentation, we assume for convenience that $b = 1$. Taking into account condition (6) and expression (10), the integral equation (2) is reduced to

$$K\left(x,y\right) - 2\frac{J_2\left(2a-x-y\right)}{2a-x-y} - 2\int_x^{2a-y}K\left(x,t\right)\frac{J_2\left(2a-y-t\right)}{2a-y-t}dt = 0. \quad (11)$$

Moreover (cf. [2]), the kernel $K(x,y) = 0$ if $x > y$ for all $x, y > 0$. According to condition (6), from equation (2) it follows that $K(x,y) = 0$ if $x+y > 2a$. After the change of variables, $v = y-x,\ s = 2a-2x$, and introduction of the notation

$$K(x,y) = K(x, v+x) = K_1(s,v), \quad (12)$$

the integral equation (11) is reduced to the form

$$K_1\,(s,v) - 2\frac{J_2(s-v)}{s-v} - 2\int\limits_0^{s-v} K_1(s,t)\frac{J_2(s-v-t)}{s-v-t}dt\,=\,0. \tag{13}$$

Expand now the kernel $K_1(s,v)$ into the Neumann series [3,5] in the Bessel functions

$$K_1(s,v) = \sum_{n=0}^{\infty} A_n(s) J_n(v). \tag{14}$$

Upon substituting expansion (14) into equation (13) and making use of the convolution formula for Bessel functions [6], we obtain

$$\sum_{n=0}^{\infty} A_n(s)\,J_n(v) - 2\frac{J_2(s-v)}{s-v} - \sum_{n=0}^{\infty} A_n(s)\,J_{n+2}(s-v)\,=\,0. \tag{15}$$

Taking into account values of the variables $s, v \in R_+$ in equation (15), the following functional equation can be obtained

$$\sum_{n=0}^{\infty} A_n(s)\cdot J_n(s-v) - 2\frac{J_2(v)}{v} - \sum_{n=0}^{\infty} A_n(s) J_{n+2}(v)\,=\,0. \tag{16}$$

Having used the addition formulas and recursion relations for Bessel functions, we expand the known functions contained in equation (15), (16) into the corresponding Bessel functions series

$$J_k(s-v)\,=\,J_k(s)J_0(v) + \sum_{j=1}^{\infty}\Big(J_{k+j}(s) + (-1)^k\,J_{j-k}(s)\Big)\cdot J_j(v),$$

$$-2\frac{J_2(s-v)}{s-v}\,=\,\sum_{n=0}^{\infty} d_n(s) J_n(v),$$

$$-2\frac{J_2(v)}{v}\,=\,-\frac{J_1(v)+J_3(v)}{2}\,=\,\sum_{n=0}^{\infty} a_n J_n(v),$$

where

$$d_0\,(s)\,=\,-\frac{2}{s}J_2\,(s)\,,\;d_n(s)\,=\,-\frac{(n+2)\,J_{n+2}(s)+(n-2)\,J_{n-2}(s)}{s},\quad n\in\mathbb{N},$$

$a_1\,=\,-1/2$, $a_3\,=\,-1/2$, and all other coefficients a_n being equal to zero. If we introduce the notations

$$B_k(s)\,=\,\sum_{m=0}^{\infty} A_m(s) J_{m-k}(s), \tag{17}$$

and equate to zero the coefficients of the Bessel functions of order one in equations (15), (16), then we obtain two systems of equations,

$$-B_{-2}(s)+A_0(s)+d_0(s)=0,$$

$$-B_{-n-2}(s)-(-1)^n B_{n-2}(s)+d_n(s)+A_n(s)=0,\ n=1,2,\dots, \tag{18}$$

and

$$B_0(s)=0,\ \ B_{-1}(s)-B_1(s)-\frac{1}{2}=0,$$

$$B_{-n-2}(s)+(-1)^n B_{n+2}(s)+a_{n+2}-A_n(s)=0,\ n=0,1,2,\dots. \tag{19}$$

If we add (for the same $n\in\mathbb{Z}_+$), equation (18) and equation (19), respectively, then the recursion relations are obtained to define the functions $B_n(s)$ in the form

$$B_n(s)=B_{n-4}(s)-(-1)^n d_{n-2}(s)-(-1)^n a_n,\ n=3,4,\dots. \tag{20}$$

In order to solve the recursion relations (20), we have to find four first coefficients $B_n(s)$, $n= -\overline{1,2}$. To find $B_{-1}(s)$, let's integrate equation (15) with respect to v from 0 to s and, after easy calculations, we obtain

$$B_{-1}(s)=-\frac{J_1(s)}{s}+\frac{1}{2}.$$

The coefficients $B_0(s),\ B_1(s),\ B_2(s)$ are easily found from equations (18), (19). If the values of these coefficients are substituted into the recursion relations (20), then an explicit expression for functions $B_k(s)$ is obtained in the form

$$B_k(s)=(-1)^k kJ_k(s)/s,\ k=0,1,2,\dots. \tag{21}$$

The system of equations (21) can be simplified, if relations (17) are differentiated and the dependences (21) are used. Then an infinite system is obtained relative to the unknown functions $X_m(s)=\dfrac{d}{ds}A_m(s)$ in the form

$$\sum_{m=0}^{\infty}X_m(s)\,J_{m-k}(s)=\frac{1}{4}\delta_{0,k},\ \ k=0,1,2,\dots, \tag{22}$$

where $\delta_{0,k}$ is the Kroneker symbol. The infinite system of equations (22) can be solved in an explicit form using the inversion algorithm for linear relations in combinatoric analysis [7]. Without going here through cumbersome calculations, we give values of the functions,

$$X_m(s)=\frac{s^m}{2^{m+2}m!},\ m=0,1,2,\dots, \tag{23}$$

for which system (22) is satisfied identically, which is easy to check if we represent the Bessel functions $J_n(s)$ in the form of a series and substitute relations (23) into system (22). Integrating equations (23) with respect to s and using conditions (21) to find the unknown constants of integration, we obtain

$$A_m(s) = \frac{s^{m+1}}{2^{m+2}(m+1)!} - \frac{1}{2}\delta_{m,1}, \; m = 0, 1, 2, \dots . \tag{24}$$

If we substitute expression (24) into the Neumann series and take into account the change of variables made, we find the unknown kernel in the form

$$K(x,y) = \sum_{n=0}^{\infty} \frac{(a-x)^{n+1}}{2(n+1)!} J_n(y-x) - \frac{J_1(y-x)}{2}. \tag{25}$$

The solution (25) was obtained with the assumption that $U = 1$. For any $U \in R_+, \; b = \sqrt{U}$, solving equation (11) with the method presented above, we obtain the following representation of the kernel (25):

$$K(x,y) = b\sum_{n=0}^{\infty} \frac{(b(a-x))^{n+1}}{2(n+1)!} J_n(b(y-x)) - \frac{bJ_1(b(y-x))}{2}. \tag{26}$$

If in formula (26), an explicit representation of the Bessel functions is used,

$$\begin{gathered} J_n(bx) = \sum_{m=0}^{\infty} \frac{(-1)^m (bx)^{n+2m}}{2^{n+2m}(n+m)!(m)!}, \\ I_n(bx) = \sum_{m=0}^{\infty} \frac{(bx)^{n+2m}}{2^{n+2m}(n+m)!(m)!}, \; n = 0, 1, 2, \dots , \end{gathered} \tag{27}$$

and the order of summation is changed (series (26), according to [6], is absolutely convergent), then after easy transformations we set

$$K(x,y) = \frac{b(2a-x-y)}{2z_1} I_1(bz_1), \text{ where } 0 \le x \le y, \; x+y \le 2a, \tag{28}$$

and, outside the domain indicated, the kernel is equal to zero. Here $z_1 = \sqrt{(2a-x-y)(y-x)}$. The obtained kernel gives a solution to problem (1) – (3), and also it is a solution of the equation in partial derivatives [2]

$$\frac{\partial^2}{\partial x^2}K(x,y) - \frac{\partial^2}{\partial y^2}K(x,y) = g(x)K(x,y), \text{ where } 0 \le x \le y, \; x+y \le 2a,$$

where $g(x)$ is described in the Introduction. By the direct substitution we can verify the following interesting property of the kernel $K(x,y)$: it is infinite together with all its partial derivatives at the point $x = a$, where the potential $g(x)$ has a discontinuity. It is easy to verify the formula [2, p. 168] to obtain the kernel

$$g\left(x\right) = -2\frac{d}{dx}K\left(x,x\right) = U, \text{ if } \ 0 \le x < a, \ g\left(x\right) = 0, \text{ if } \ x > a.$$

Knowing the explicit form of the kernel $K(x, y)$, we can obtain several formulas for the Bessel functions. Thus, if we substitute the kernel (28) into representation (3) and separate the real and imaginary parts, after some transformations that are not complicated, we have

$$bv\int\limits_0^v \frac{I_1\left(b\sqrt{v^2-y^2}\right)}{\sqrt{v^2-y^2}}\cos\left(zy\right)dy = \cos\left(v\sqrt{z^2-b^2}\right) - \cos\left(zv\right), \tag{29}$$

$$b\int\limits_0^v \frac{I_1\left(b\sqrt{v^2-y^2}\right)}{\sqrt{v^2-y^2}}y\sin\left(zy\right)dy = \frac{z}{\lambda}\sin\left(v\sqrt{z^2-b^2}\right) - \sin\left(zv\right), \tag{30}$$

where $v = a - x, \lambda = \sqrt{z^2 - b^2}$. It is easy to see that formulas (29), (30) assign, respectively, the cosine- and sine-Fourier transforms (where the integrand functions are extended by zero for $y > v$) and if the inverse Fourier transform is applied to them, we obtain

$$\int\limits_0^\infty (\cos\left(v\sqrt{z^2-b^2}\right) - \cos\left(zv\right))\cos(zy)dz = \frac{\pi bv}{2}\frac{I_1\left(b\sqrt{v^2-y^2}\right)}{\sqrt{v^2-y^2}}, \tag{31}$$

$$\int\limits_0^\infty (\frac{z}{\lambda}\sin\left(v\sqrt{z^2-b^2}\right) - \sin\left(zv\right))\sin(zy)dz = \frac{\pi by}{2}\frac{I_1\left(b\sqrt{v^2-y^2}\right)}{\sqrt{v^2-y^2}}. \tag{32}$$

In relations (31), (32), the right-hand sides are identically equal to zero if $y > v$. Formulas (29) – (32) are obtained for the first time.

3. Finding the kernel $K^+(x, y)$ on the whole axis R^1. From [2] and expression (11), it follows that for $x \ge 0$, $K^+(x,y) = K(x,y)$, $e^+(z,x) = e(z,x)$, and representation (3) reads

$$e^+(z,x) = e^{izx} + \int\limits_x^{2a-x} K(x,y)e^{izy}dy. \tag{33}$$

On the other hand, if $x < 0$, the presentation of the kernel is $K^+(x,y) = Q_1(y-x) + {}$ $+Q_2(y+x)$, where $Q_1(y)$, $Q_2(y)$ are arbitrary functions that can be defined from the continuity condition for the solution and the derivative $e^+(z,x)$ at the point zero. In addition, representation (3), if $0 \geq x$, can be expressed in the form

$$e^{+}(z,x) = e^{izx} + e^{izx}\int_{0}^{2a} Q_1(y)e^{izy}dy + e^{-izx}\int_{0}^{2a} Q_2(y)e^{izy}dy =$$

$$= a(z)e^{izx} + b(z)e^{-izx}, \tag{34}$$

where the coefficients $a(z)$, $b(z)$ are defined as follows if representations (4) and continuity conditions on the function $e^{+}(z,x)$ at the point $x = 0$ are used,

$$a(z) = e^{iaz}[\cos(a\lambda_1) - i\frac{2z^2 - U}{2z\lambda_1}\sin(a\lambda_1)],$$

$$b(z) = -iUe^{iaz}\sin(a\lambda_1)/(2z\lambda_1),\ \lambda_1 = \sqrt{z^2 - U}.$$

Taking into account that the solution $e^{+}(z,x)$ is continuous at the point $x = 0$, together with the first derivative, and using the expression for the kernel $K(x,y)$ (28), after some transforms, the unknown functions $Q_1(y)$, $Q_2(y)$ are found to be

$$Q_1(y) = G_1(y) + G_2(y),\quad Q_2(y) = G_1(y) - G_2(y), \tag{35}$$

where

$$G_1(y) = \frac{b(2a-y)}{2z_2}I_1(bz_2),\ z_2 = \sqrt{y(2a-y)},$$

$$G_2(y) = \frac{iUae^{iaz}}{8\sin(az)} + \frac{bi}{4zz_2}I_1(bz_2) + \frac{aUi}{4zy}I_2(bz_2).$$

Using relations (34) and taking into account the expression for the functions $Q_1(y)$, $Q_2(y)$ (35), an integral representations for the coefficients $a(z)$, $b(z)$ can be obtained in the form

$$a(z) = 1 + \int_{0}^{2a} Q_1(y)e^{izy}dy,\ b(z) = \int_{0}^{2a} Q_2(y)e^{izy}dy. \tag{36}$$

From relations (36), the functions of kernel, $Q_1(y)$, $Q_2(y)$, can be expressed conversely in terms of the coefficients $a(z)$, $b(z)$ (cf. the derivation of formulas (31), (32) from formulas (29), (30)). Taking into account uniqueness of the solution to the problem (2), (3) (cf. [2]), we have obtained that the kernel $K(x,y)$ (28) for $x \geq 0, y \geq x$, and the kernels $Q_1(y-x)$, $Q_2(y+x)$ for $0 \geq x, y \geq x$ give a solution to the equation of GLM on the whole real axis R^1.

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