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ON INVARIANT TORUS OF WEEKLY CONNECTED SYSTEMS OF DIFFERENTIAL EQUATIONS

ПРО ІНВАРІАНТНИЙ ТОР ДЛЯ СЛАБКОЗВ'ЯЗАНИХ СИСТЕМ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

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We consider a family of systems of differential equations depending on a sufficiently small parameter with zero value of which we obtained a couple of independent systems. We used the method of Green – Samoilenko function to construct an invariant manifold of the pertubed system and presented some examples for application.

Розглянуто сім'ю систем диференціальних рівнянь, що залежать від достатньо малого параметра, яка є парою незалежних систем, якщо значення параметра дорівнює нулю. Використано метод функції Гріна – Самойленка для побудови інваріантного многовиду збуреної системи та наведено приклади.

Introduction. Let us consider a system of differential equations of the form

$$x' = X(x, \varepsilon), \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n) \in R^n$, $\varepsilon > 0$ is a sufficiently small parameter.

Suppose that the unpertubed system

$$x' = X(x, 0), \tag{2}$$

has an invariant torus. The classical problem arising here is to find out what we can say about the invariant manifold of the pertubed system?

The main tool of investigation of the above problem was the method of integral mapifolds of nonlinear mechanics by Krylov – Bogolyubov – Mitropolskiy, the method of Levinson – Diliberto and others. Later, by using of the method of Green function, A. M. Samoilenko [1] obtained new results on the theory of invariant torus. The method of Samoilenko was extended to the other lasses of equations by Yu. V. Teplinskiy, D. I. Martynyuk, M. I. Ilolov [2 – 5] and others. We use this approach in a practically interesting case when the dimension of the given space may be reduced.

We shall consider a system of equations of the form

$$\begin{aligned} y' &= Y(y, z, \varepsilon), \\ z' &= Z(y, z, \varepsilon), \end{aligned} \tag{3}$$

where $y = (y_1, y_2, \ldots, y_m)$, $z = (z_1, z_2, \ldots, z_k)$ are vectors in the Euclidean space $R^n = R^m \oplus \oplus R^k$, the parameter $\varepsilon > 0$ is a sufficiently small.

Rewrite the system (3) to the following form:

$$\begin{aligned} y' &= Y_0(y) + Y_1(y, z, \varepsilon), \\ z' &= Z_0(z) + Z_1(z, y, \varepsilon), \end{aligned} \tag{4}$$

where $Y_0(y) = Y(y, 0, 0)$, $Z_0(z) = Z(z, 0, 0)$, $Y_1(y, z, \varepsilon) = Y(y, z, \varepsilon) - Y_0(y)$, $Z_1(z, y, \varepsilon) = Z(z, y, \varepsilon) - Z_0(z)$.

Definition 1. *A system of differential equations of the form (3) is called weakly connected if for $\varepsilon = 0$ we obtain two independent systems, i.e.,*

$$y' = Y_0(y), \tag{5}$$

$$z' = Z_0(z). \tag{6}$$

Let us give some examples of weakly connected systems of differential equations.

Consider weakly connected oscillators of the form

$$\ddot{x}_i + Q_i(x_i) = \varepsilon q_i(t, x_1, \dot{x}_1, \ldots, x_N, \dot{x}_N; \varepsilon), \quad i = 1, 2, \ldots, N, \tag{7}$$

where ε is a sufficiently small parameter and q_i are continuous periodic functions of t. This system was investigated by L. D. Akulenko [6]. He constructed the stationary resonance rotating-oscillating solutions of this system on infinite long period of time.

In [7] Yu. A. Mitropolsky and A. M. Samoilenko considered the system of the type

$$\frac{d^2x}{dt^2} + \lambda^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right),$$

where $x = (x_1, x_2, \ldots, x_n)$, $\lambda^2 = \operatorname{diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2)$, $\lambda_j \geq 0$ $(j = \overline{1, n})$, $f = (f_1, f_2, \ldots, f_n)$ are polynomial function of its variables less than of order N; ε is a sufficiently small parameter. They obtained results concerning the problems of existence, exponential stability and exponential dichotomy of the invariant torus in the resonance and nonresonance case.

Weakly connected networks of quasiperiodic oscillators of the form

$$\dot{X}_i = F_i(X_i) + \varepsilon G_i(X_1, X_2, \ldots, X_n, \varepsilon), \quad X_i \in R^m, \quad i = 1, 2, \ldots, n, \ \varepsilon \ll 1,$$

were considered by E. M. Izhikevich [8], who proved that this system csn be transformed into a phase (canonical) model,

$$\dot{\theta}_i = \Omega_i + \varepsilon h_i(\theta_1, \theta_2, \ldots, \theta_n, \varepsilon), \quad i = 1, \ldots, n,$$

by a continuous, possibly noninvertible change of variables, where θ_i is a vector of phases (angles), Ω_i is a vector of frequencies of the ith oscillator X_i. It was shown also that whether or not the oscillators interact depends not only on the existence of connections between them, but also on their frequencies. One can find many examples of this system, especially in neural networks.

Other type of weakly connected systems can be found in the works [9, 10]. In the present work we assume that for $\varepsilon = 0$ each of the uncoupled systems of equations (5), (6) has an asymptotically stable invariant torus of the following form:

$$M\{y = f(\varphi),\ \varphi \in T_r\}, \quad N = \{z = g(\theta),\ \theta \in T_l\}, \tag{8}$$

where $f \in C^1(T_r)$, $g \in C^1(T_l)$,

$$T_r = \{\varphi(\varphi_1, \varphi_2, \ldots, \varphi_r)\ :\ 0 \le \varphi_i \le 2\pi,\ 0 \le i \le r\},$$

$$T_l = \{\theta(\theta_1, \theta_2, \ldots, \theta_l)\ :\ 0 \le \theta_j \le 2\pi,\ 0 \le j \le l\}.$$

We shall consider the problem, when pertubed system (3) has an invariant manifold. We used the method described in [2]. To this end we have to transform the system about the invariant manifolds M, N (8). We showed that under certain conditions the invariant torus of the transformed system, which has dimension $r + l$, can be constructed from the invariant torus of the unpertubed systems with less dimension r and l.

This paper is organized as follows. In Section 1 the local coordinate system about the invariant torus of the uncoupled systems (5), (6) is introduced and the definition of an almost independent function is given. The proof of the preliminary Lemma 2 is also included in this section. The method for constructing the invariant torus of the system (3) and a proof of the main result is given in Section 2. Section 3 contains two examples of weakly connected systems. We apply the method described in Section 2 to this systems of differential equations.

1. A local coordinate system about the invariant torus. To introduce local coordinate we suppose that the functions f, g are such that

$$f(\varphi) \in C^1(T_r), \quad \operatorname{rank} \frac{\partial f(\varphi)}{\partial \varphi} = r \quad \forall\, \varphi \in T_r,$$

$$g(\theta) \in C^1(T_l), \quad \operatorname{rank} \frac{\partial g(\theta)}{\partial \theta} = l \quad \forall\, \theta \in T_l$$

and also assume that the matrix $\dfrac{\partial f(\varphi)}{\partial \varphi}$ and $\dfrac{\partial g(\theta)}{\partial \theta}$ can be completed to a periodic basis in R^m, R^k respectively, and the complement matrix $B(\varphi) \in C^1(T_r)$, $W(\theta) \in C^1(T_l)$ is such that

$$\det \left| \frac{\partial f(\varphi)}{\partial \varphi}, B(\varphi) \right| \ne 0,$$
$$\det \left| \frac{\partial f(\theta)}{\partial \theta}, W(\theta) \right| \ne 0. \tag{9}$$

Under the above assumptions we can define local coordinates φ, h (for (5)) and s, θ (for (6)) in a neighborhood of the invariant manifolds M, N by

$$y = f(\varphi) + B(\varphi)h, \tag{10}$$

$$z = g(\theta + W(\theta)s. \tag{11}$$

Following lemma [2] (A. M. Samoilenko) we can apply the transformation (10), (11) to (3).

Lemma 1. *For every sufficiently small $\delta > 0$ there exists $\delta_1(\delta) \to 0$ as $\delta \to 0$ such that every point y (or z) satisfying $\rho(y, M) < \delta$ (or $\rho(z, N) < \delta$), has local coordinates φ, h (or s, θ), satisfying*

$$\|h\| \leq \delta_1, \quad \varphi \in T_r \quad (or\ \|s\| \leq \delta_1, \quad \theta \in T_l). \tag{12}$$

By substituting (10), (11) into (3) we obtain

$$\left[\frac{\partial f(\varphi)}{\partial \varphi} + \frac{\partial B(\varphi)}{\partial \varphi} h\right] \frac{d\varphi}{dt} + B(\varphi)\frac{dh}{dt} = Y(f(\varphi) + B(\varphi)h, g(\theta) + W(\theta)s, \varepsilon),$$

$$\left[\frac{\partial g(\theta)}{\partial \theta} + \frac{\partial W(\theta)}{\partial \theta} s\right] \frac{d\theta}{dt} + W(\theta)\frac{ds}{dt} = Z(g(\theta) + W(\theta)s, f(\varphi) + B(\varphi)h, \varepsilon),$$

and condition (9) allows us to solve this system with respect to $\frac{d\varphi}{dt}, \frac{dh}{dt}, \frac{d\theta}{dt}, \frac{ds}{dt}$ for all h, s, φ, θ from the domain (12), so that the system of equations (3), after transformation, will have the form

$$\begin{aligned} \frac{d\varphi}{dt} &= a_1(\varphi, \theta, h, s, \varepsilon), \\ \frac{dh}{dt} &= P_1(\varphi, \theta, h, s, \varepsilon), \\ \frac{d\theta}{dt} &= a_2(\varphi, \theta, h, s, \varepsilon), \\ \frac{ds}{dt} &= P_2(\varphi, \theta, h, s, \varepsilon), \end{aligned} \tag{13}$$

where a_1, a_2, P_1, P_2 are periodic functions with respect to φ_i, $i = 1, \ldots, r$, θ_j, $j = 1, \ldots, l$, defined and continuous with respect to $\varphi, \theta, h, s, \varepsilon$ in the domain

$$\|h\| \leq d, \quad \varphi \in T_r, \quad \|s\| \leq d, \quad \theta \in T_l, \quad \varepsilon \in (0, \varepsilon_0], \tag{14}$$

where d and ε_0 are sufficiently small numbers.

It should be noted that for $\varepsilon = 0$ we have following systems:

$$\begin{aligned} \frac{d\varphi}{dt} &= a_1^*(\varphi, h), \\ \frac{dh}{dt} &= P_1^*(\varphi, h), \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{d\theta}{dt} &= a_2^*(\theta, s), \\ \frac{ds}{dt} &= P_2^*(\theta, s), \end{aligned} \tag{16}$$

where $a_1^*(\varphi, h) = a_1(\varphi, 0, h, 0, 0)$, $P_1^*(\varphi, h) = P_1(\varphi, 0, h, 0, 0)$, $a_2^*(\theta, s) = a_2(0, \theta, 0, s, 0)$, $P_2^*(\theta, s) = P_2(0, \theta, 0, s, 0)$.

By assuming that the systems of equations (5), (6) have the invariant manifolds (8) is equivalent to the systems (15) and (16) having the invariant torus

$$h = 0, \quad \varphi \in T_r, \quad s = 0, \quad \theta \in T_l.$$

Definition 2. *A function $a(\varphi, \theta, \sigma) \in C^p(T_l \times T_r)$ is called almost independent of θ of the order p if for any $\varepsilon > 0$ sufficiently small there exists $\sigma > 0$ such that*

$$\left| a(\varphi, \theta, \sigma) - a(\varphi, 0, 0) \right|_p < \varepsilon.$$

Remark. For the class of this functions we may put, for example,

$$H(\varphi, \theta) = F(\varphi) + \sigma G(\varphi, \theta),$$

where $F(\varphi)$, $G(\varphi, \theta)$ are periodic functions with respect to each component of φ and θ; σ is a sufficiently small parameter.

Definition 3. *According to [2] (A. M. Samoilenko), the Green function $G_0(\tau, \varphi)$ of the system*

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dh}{dt} = P(\varphi)h$$

is said to be rough if there exists $\delta = \text{const} > 0$ and an integer number $p \geq 0$ such that the system of equations

$$\frac{d\varphi}{dt} = a(\varphi) + a_1(\varphi), \quad \frac{dh}{dt} = P(\varphi)h, \tag{17}$$

for $a_1 \in C^{p_0}(T_m)$ $(p_0 = \max(1, p))$ and

$$|a_1|_{p_0} \leq \delta, \tag{18}$$

has a Green function $\overline{G}_0(\tau, \varphi)$ satisfying the inequality

$$\left| \overline{G}_0(\tau, \varphi) f(\varphi_r(\varphi)) \right|_p \leq K e^{-\gamma|\tau|} |f|_p \tag{19}$$

where f is a function in $C^{p_0}(T_m)$, $\varphi_\tau(\varphi)$ is a solution of the first equation (17); K, γ are positive constants independent of φ, δ, f.

The following lemma takes a central place in the future investigations.

Lemma 2. *Assume that the functions $a_1(\varphi, \theta, \sigma)$, $P_1(\varphi, \theta, \sigma)$ and $a_2(\varphi, \theta, \sigma)$, $P_2(\varphi, \theta, \sigma)$ are almost independent of θ and φ, respectively. Let each of the systems of equations*

$$\begin{aligned} \frac{d\varphi}{dt} &= a_1(\varphi, 0, 0), \\ \frac{d\theta}{dt} &= a_2(0, \theta, 0), \\ \frac{dh}{dt} &= P_1(\varphi, \theta, \sigma)h \end{aligned} \tag{20}$$

and

$$\begin{aligned}\frac{d\varphi}{dt} &= a_1(\varphi, 0, 0),\\ \frac{d\theta}{dt} &= a_2(0, \theta, 0),\\ \frac{ds}{dt} &= P_2(\varphi, \theta, \sigma)s\end{aligned} \tag{21}$$

have a rough Green function. Then there exist $\mu = \mu(\delta)$ $(\mu \to 0, \delta \to 0)$ *and* $\sigma_0 > 0$ *such that for every* $\sigma \in (0, \sigma_0]$, *functions* $a_1^\bullet$, $P_1^\bullet$, $a_2^\bullet$, $P_2^\bullet \in C^p(T_l \times T_r)$ *satisfying the inequalities*

$$\begin{aligned}|a_1^\bullet|_p + |a_2^\bullet|_p + |P_1^\bullet|_p &< \mu,\\ |a_1^\bullet|_p + |a_2^\bullet|_p + |P_2^\bullet|_p &< \mu,\end{aligned} \tag{22}$$

and functions $f_1, f_2 \in C^p(T_r \times T_l)$, *each of the systems of equations*

$$\begin{aligned}\frac{d\varphi}{dt} &= a_1(\varphi, \theta, \sigma) + a_1^\bullet(\varphi, \theta),\\ \frac{d\theta}{dt} &= a_2(\varphi, \theta, \sigma) + a_2^\bullet(\varphi, \theta),\\ \frac{dh}{dt} &= \big[P_1(\varphi, \theta, \sigma) + P_1^\bullet(\varphi, \theta)\big]h + f_1(\varphi, \theta)\end{aligned} \tag{23}$$

and

$$\begin{aligned}\frac{d\varphi}{dt} &= a_1(\varphi, \theta, \sigma) + a_1^\bullet(\varphi, \theta),\\ \frac{d\theta}{dt} &= a_2(\varphi, \theta, \sigma) + a_2^\bullet(\varphi, \theta),\\ \frac{ds}{dt} &= \big[P_2(\varphi, \theta, \sigma) + P_2^\bullet(\varphi, \theta)\big]s + f_2(\varphi, \theta)\end{aligned} \tag{24}$$

has an invariant torus of the type

$$h = u(\varphi, \theta, \sigma), \quad s = v(\varphi, \theta, \sigma), \quad \varphi \in T_r,\ \theta \in T_l,\ \sigma \in (0, \sigma_0],$$

satisfying the inequality

$$\begin{aligned}|u(\varphi, \theta, \sigma)|_p &\le K_1 |f_1|_p,\\ |v(\varphi, \theta, \sigma)|_p &\le K_2 |f_2|_p,\end{aligned} \tag{25}$$

where K_1, K_2 *are positive constants independent of* μ, f_1, f_2.

Proof. Let us rewrite the system of equations

$$\frac{d\varphi}{dt} = a_1(\varphi, \theta, \sigma),$$

$$\frac{d\theta}{dt} = a_2(\varphi, \theta, \sigma),$$

$$\frac{dh}{dt} = P_1(\varphi, \theta, \sigma)h$$

in the following form:

$$\frac{d\varphi}{dt} = a_1(\varphi, 0, 0) + a_{11}(\varphi, \theta, \sigma),$$

$$\frac{d\theta}{dt} = a_2(0, \theta, 0) + a_{21}(\varphi, \theta, \sigma),$$

$$\frac{dh}{dt} = \big[P_1(\varphi, 0, 0) + P_{11}(\varphi, \theta, \sigma)\big]h,$$

where $a_{11}(\varphi, \theta, \sigma) = a_1(\varphi, \theta, \sigma) - a_1(\varphi, 0, 0)$, $a_{21}(\varphi, \theta, \sigma) = a_2(\varphi, \theta, \sigma) - a_2(0, \theta, 0)$, $P_{11}(\varphi, \theta, \sigma) = P_1(\varphi, \theta, \sigma) - P_1(\varphi, 0, 0)$. Since the functions a_1, P_1 and a_2 are almost independent of θ and φ, respectively, for every $\varepsilon > 0$ we can choose $\sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0]$ the following inequalities hold:

$$|a_{11}(\varphi, \theta, \sigma)|_p < \varepsilon, \quad |a_{21}(\varphi, \theta, \sigma)|_p < \varepsilon, \quad |P_{11}(\varphi, \theta, \sigma)|_p < \varepsilon, \tag{26}$$
$$\varphi \in T_r,\ \theta \in T_l.$$

By using the fact that the system (23) has a rough Green function we can choose $\delta > 0$ in (18) and $\varepsilon > 0$ in (26) such that the system of equations

$$\frac{d\varphi}{dt} = a_1(\varphi, \theta, \sigma) + a_1^{\bullet}(\varphi, \theta),$$

$$\frac{d\theta}{dt} = a_2(\varphi, \theta, \sigma) + a_2^{\bullet}(\varphi, \theta),$$

$$\frac{dh}{dt} = \big[P_1(\varphi, \theta, \sigma) + P_1^{\bullet}(\varphi, \theta)\big]h + f_1(\varphi, \theta)$$

with the functions $a_1^{\bullet}(\varphi, \theta)$, $a_2^{\bullet}(\varphi, \theta)$, $P_1^{\bullet}(\varphi, \theta)$ satisfying the inequality

$$|a_1^{\bullet}|_p + |a_{11}|_p + |a_2^{\bullet}|_p + |a_{21}|_p + |P_1^{\bullet}|_p + |P_{11}|_p < \delta, \quad \varphi \in T_r,\ \theta \in T_l, \tag{27}$$

has a Green function $\overline{G}_0(\tau, \varphi, \theta)$ which satisfies

$$\big|\overline{G}_0(\tau, \varphi, \theta) f_1(\varphi_\tau(\varphi), \theta_\tau(\theta))\big|_p \le K_1 e^{-\gamma_1|\tau|}|f_1|_p. \tag{28}$$

Let us denote by

$$Gf(\varphi,\theta) = \int\limits_{-\infty}^{\infty} \overline{G}_0(\tau,\varphi,\theta) f(\varphi_\tau(\varphi),\theta_\tau(\theta)) d\tau$$

the operator defined on $f \in C^p(T_r \times T_l)$ and let consider the following equation:

$$u = GP_1^\bullet u + Gf_1.$$

A solution of this equation exists if

$$|GP_1^\bullet|_p = d < 1, \tag{29}$$

but we can choose a sufficiently small $\delta > 0$ such that (29) holds and we obtain a unique solution in the form

$$u = \sum_{k=0}^{\infty} (GP_1^\bullet)^k Gf_1, \tag{30}$$

which satisfies the inequality

$$|u(\varphi,\theta,\sigma)|_p \le K_1 |f_1|_p$$

for any $\sigma \in (0,\sigma_0]$. Obviously, $u = u(\varphi,\theta,\sigma)$ defined by (30) for any $\sigma \in (0,\sigma_0]$ is an invariant torus of the system

$$\frac{d\varphi}{dt} = a_1(\varphi,\theta,\sigma) + a_1^\bullet(\varphi,\theta),$$

$$\frac{d\theta}{dt} = a_2(\varphi,\theta,\sigma) + a_2^\bullet(\varphi,\theta),$$

$$\frac{dh}{dt} = \left[P_1(\varphi,\theta,\sigma) + P_1^\bullet(\varphi,\theta)\right] h + f_1(\varphi,\theta).$$

Lemma 2 is proved for $h = u(\varphi,\theta,\sigma)$, $\sigma \in (0,\sigma_0]$, $\varphi \in T_r$, $\theta \in T_l$. For the case when $s = v(\varphi,\theta,\sigma)$, $\varphi \in T_r$, $\theta \in T_l$, we can prove it in the same way.

2. Constructing an invariant torus. Let us, by using the method of Samoilenko, construct an invariant torus of system (13). To this end we select the «linear» part of $P_1(\varphi,\theta,h,s,\varepsilon)$, $P_2(\varphi,\theta, h,s,\varepsilon)$ and obtain

$$\frac{d\varphi}{dt} = a_1(\varphi,\theta,h,s,\varepsilon),$$

$$\frac{dh}{dt} = P_1^*(\varphi,\theta,h,s,\varepsilon)h + f_1(\varphi,\theta,\varepsilon),$$

$$\frac{d\theta}{dt} = a_2(\varphi,\theta,h,s,\varepsilon),$$

$$\frac{ds}{dt} = P_2^*(\varphi,\theta,h,s,\varepsilon)s + f_2(\varphi,\theta,\varepsilon),$$

where

$$P_1^*(\varphi,\theta,h,s,\varepsilon) = \int_0^1 \frac{\partial[P_1(\varphi,\theta,th,s,\varepsilon) - P_1(\varphi,\theta,0,0,\varepsilon)]}{\partial(th)}dt,$$

$$P_2^*(\varphi,\theta,h,s,\varepsilon) = \int_0^1 \frac{\partial[P_2(\varphi,\theta,h,ts,\varepsilon) - P_2(\varphi,\theta,0,0,\varepsilon)]}{\partial(ts)}dt,$$

$f_1(\varphi,\theta,\varepsilon) = P_1(\varphi,\theta,0,0,\varepsilon)$, $f_2(\varphi,\theta,\varepsilon) = P_2(\varphi,\theta,0,0,\varepsilon)$. When $\varepsilon = 0$, system (13) has the invariant torus $h = 0$, $s = 0$, therefore $f_1(\varphi,0,0) = 0$, $f_2(\varphi,0,0) = 0$.

We can also rewrite the above system in the form

$$\frac{d\varphi}{dt} = a_{10}(\varphi) + a_{11}(\varphi,\theta,h,s,\varepsilon),$$

$$\frac{d\theta}{dt} = a_{20}(\theta) + a_{22}(\varphi,\theta,h,s,\varepsilon),$$

$$\frac{dh}{dt} = \left[P_{10}^*(\varphi) + P_{11}^*(\varphi,\theta,h,s,\varepsilon)\right]h + f_1(\varphi,\theta,\varepsilon),$$

$$\frac{ds}{dt} = \left[P_{20}^*(\theta) + P_{21}^*(\varphi,\theta,h,s,\varepsilon)\right]s + f_2(\varphi,\theta,\varepsilon),$$

where

$$a_{10}(\varphi) = a_1(\varphi,0,0,0,0), a_{11}(\varphi,\theta,h,s,\varepsilon) = a_1(\varphi,\theta,h,s,\varepsilon) - a_{10}(\varphi),$$

$$a_{20}(\theta) = a_2(0,\theta,0,0,0), a_{22}(\varphi,\theta,h,s,\varepsilon) = a_2(\varphi,\theta,h,s,\varepsilon) - a_{20}(\theta),$$

$$P_{10}^*(\varphi) = P_1^*(\varphi,0,0,0,0), P_{11}^*(\varphi,\theta,h,s,\varepsilon) = P_1^*(\varphi,\theta,h,s,\varepsilon) - P_{10}^*(\varphi),$$

$$P_{20}^*(\theta) = P_2^*(0,\theta,0,0,0), P_{21}^*(\varphi,\theta,h,s,\varepsilon) = P_2^*(0,\theta,h,s,\varepsilon) - P_{20}^*(\theta).$$

Consider the systems of equations of the type

$$\begin{aligned}\frac{d\varphi}{dt} &= a_{10}(\varphi),\\ \frac{d\theta}{dt} &= a_{20}(\theta),\\ \frac{dh}{dt} &= \left[P_{10}^*(\varphi) + P_{101}(\varphi,\theta,\varepsilon)\right]h,\end{aligned} \qquad \text{and} \qquad \begin{aligned}\frac{d\varphi}{dt} &= a_{10}(\varphi),\\ \frac{d\theta}{dt} &= a_{20}(\theta),\\ \frac{ds}{dt} &= \left[P_{20}^*(\theta) + P_{201}(\varphi,\theta,\varepsilon)\right]s.\end{aligned} \tag{31}$$

We can prove the following theorem for the system (13).

Theorem. *Suppose that for every $P_{101}(\varphi,\theta,\varepsilon)$, $P_{201}(\varphi,\theta,\varepsilon)$, almost independent of θ and φ respectively, each of the system of equations (31) has a rough Green function of order p, satisfying* *inequality (19). Assume that the functions* a_1, a_2, P_1^*, P_2^*, *in the domain*

$$\|h\| \le d, \quad \varphi \in T_r, \qquad \|s\| \le d, \quad \theta \in T_l, \quad \varepsilon \in (0, \varepsilon_0],$$

are continuous together with all derivative with respect to φ, θ, h, s *up to the order* p. *If* $p \ge 1$ *then there exists a sufficiently small* $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in (0, \varepsilon_0]$ *the system of equations (13) has an invariant torus,*

$$H = U(\varphi, \theta, \varepsilon) = \begin{pmatrix} u(\varphi, \theta, \varepsilon) \\ v(\varphi, \theta, \varepsilon) \end{pmatrix}, \quad \varphi \in T_r,\ \theta \in T_l,$$

where $u(\varphi, \theta, \varepsilon), v(\varphi, \theta, \varepsilon) \in C^{p-1}(T_r \times T_l)$ *and satisfy the inequality*

$$|u(\varphi, \theta, \varepsilon)|_{p-1} \le K_1 |f_1|_{p-1},$$

$$|v(\varphi, \theta, \varepsilon)|_{p-1} \le K_2 |f_2|_{p-1},$$

where K_1, K_2 *are positive constants independent of* ε.

Proof. We may find the invariant torus by an iteration process. To this end for the zero approximation we put

$$u_0(\varphi, \theta) \equiv 0, \quad v_0(\varphi, \theta) \equiv 0, \quad \varphi \in T_r,\ \theta \in T_l,$$

and the first approximation $u_1(\varphi, \theta, \varepsilon)$ can be obtained from

$$\begin{aligned} \frac{d\varphi}{dt} &= a_{10}(\varphi) + a_{11}(\varphi, \theta, 0, 0, \varepsilon), \\ \frac{d\theta}{dt} &= a_{20}(\theta), \\ \frac{dh}{dt} &= \big[P_{10}^*(\varphi) + P_{11}^*(\varphi, \theta, 0, 0, \varepsilon)\big] h + f_1(\varphi, \theta, \varepsilon) \end{aligned} \tag{32}$$

and $v_1(\varphi, \theta, \varepsilon)$ from

$$\begin{aligned} \frac{d\varphi}{dt} &= a_{10}(\varphi), \\ \frac{d\theta}{dt} &= a_{20}(\theta) + a_{21}(\varphi, \theta, 0, 0, \varepsilon), \\ \frac{ds}{dt} &= \big[P_{20}^*(\theta) + P_{21}^*(\varphi, \theta, 0, 0, \varepsilon)\big] s + f_1(\varphi, \theta, \varepsilon). \end{aligned} \tag{33}$$

We may choose ε_0 sufficiently small such that for $\varepsilon \in (0, \varepsilon_0]$ the following inequalities hold:

$$|a_{11}(\varphi, \theta, u(\varphi, \theta, \varepsilon), v(\varphi, \theta, \varepsilon), \varepsilon)|_p + |a_{21}(\varphi, \theta, u(\varphi, \theta, \varepsilon), v(\varphi, \theta, \varepsilon), \varepsilon)|_p +$$

$$+ |P_{11}^*(\varphi, \theta, u(\varphi, \theta, \varepsilon), v(\varphi, \theta, \varepsilon), \varepsilon)|_p < \delta, \qquad (34)$$

$$|a_{11}(\varphi, \theta, u(\varphi, \theta, \varepsilon), v(\varphi, \theta, \varepsilon), \varepsilon)|_p + |a_{21}(\varphi, \theta, u(\varphi, \theta, \varepsilon), v(\varphi, \theta, \varepsilon), \varepsilon)|_p +$$

$$+ |P_{21}^*(\varphi, \theta, u(\varphi, \theta, \varepsilon), v(\varphi, \theta, \varepsilon), \varepsilon)|_p < \delta \qquad (35)$$

for every $u(\varphi, \theta, \varepsilon), v(\varphi, \theta, \varepsilon) \in C^p(T_r \times T_l)$ satisfying the inequality

$$|u(\varphi, \theta, \varepsilon)|_p \leq K_1 |f_1|_p,$$

$$|v(\varphi, \theta, \varepsilon)|_p \leq K_2 |f_2|_p.$$

By using Lemma 2 we obtain that (32), (33) have an invariant torus satisfying

$$|u_1(\varphi, \theta, \varepsilon)|_p \leq K_1 |f_1|_p,$$

$$|v_1(\varphi, \theta, \varepsilon)|_p \leq K_2 |f_2|_p.$$

Therefore, we can continue the iteration process and at the $(i+1)th$ approximation

$$U_{i+1}(\varphi, \theta, \varepsilon) = \begin{pmatrix} u_{i+1}(\varphi, \theta, \varepsilon) \\ v_{i+1}(\varphi, \theta, \varepsilon) \end{pmatrix},$$

the $u_{i+1}(\varphi, \theta, \varepsilon)$ are obtained from the following system:

$$\begin{aligned} \frac{d\varphi}{dt} &= a_{10}(\varphi) + a_{11}(\varphi, \theta, u_i(\varphi, \theta, \varepsilon), v_i(\varphi, \theta, \varepsilon), \varepsilon), \\ \frac{d\theta}{dt} &= a_{20}(\theta) + a_{21}(\varphi, \theta, u_{i-1}(\varphi, \theta, \varepsilon), v_{i-1}(\varphi, \theta, \varepsilon), \varepsilon), \\ \frac{dh}{dt} &= \left[P_{10}^*(\varphi) + P_{11}^*(\varphi, \theta, u_i(\varphi, \theta, \varepsilon), v_i(\varphi, \theta, \varepsilon), \varepsilon)\right] h + f_1(\varphi, \theta, \varepsilon), \end{aligned} \qquad (36)$$

and $v_{i+1}(\varphi, \theta, \varepsilon)$ from

$$\begin{aligned} \frac{d\theta}{dt} &= a_{20}(\theta) + a_{21}(\theta, \varphi, u_i(\varphi, \theta, \varepsilon), v_i(\varphi, \theta, \varepsilon), \varepsilon), \\ \frac{d\varphi}{dt} &= a_{10}(\varphi) + a_{11}(\varphi, \theta, u_{i-1}(\varphi, \theta, \varepsilon), v_{i-1}(\varphi, \theta, \varepsilon), \varepsilon), \\ \frac{ds}{dt} &= \left[P_{20}^*(\theta) + P_{21}^*(\theta, \varphi, u_i(\varphi, \theta, \varepsilon), v_i(\varphi, \theta, \varepsilon), \varepsilon)\right] s + f_2(\theta, \varphi, \varepsilon). \end{aligned} \qquad (37)$$

Since the functions a_{11}, a_{21}, P_{11}^*, P_{21}^* for all $\varepsilon \in (0, \varepsilon_0]$ satisfy inequalities (34), (35), the invariant torus of (36) and (37) exists and satisfies

$$|u_{i+1}(\varphi,\theta,\varepsilon)|_p \le K_1|f_1|_p,$$
$$|v_{i+1}(\varphi,\theta,\varepsilon)|_p \le K_2|f_2|_p. \tag{38}$$

By using the induction we can prove that each iteration for all $\varepsilon \in (0, \varepsilon_0]$ is defined and satisfies an inequality of type (38).

Now, we need to prove the uniform convergence of the sequence of the invariant tori. To this end we consider the following functions:

$$\omega_i(\varphi,\theta,\varepsilon) = u_{i+1}(\varphi,\theta,\varepsilon) - u_i(\varphi,\theta,\varepsilon), \tag{39}$$

$$\nu_i(\varphi,\theta,\varepsilon) = v_{i+1}(\varphi,\theta,\varepsilon) - v_i(\varphi,\theta,\varepsilon). \tag{40}$$

After the calculation of the differences between

$$\frac{\partial u_{i+1}(\varphi,\theta,\varepsilon)}{\partial\varphi}[a_{10}(\varphi) + a_{11}(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)] +$$
$$+ \frac{\partial u_{i+1}(\varphi,\theta,\varepsilon)}{\partial\theta}[a_{20}(\theta) + a_{21}(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)] =$$
$$= \left[P_{10}^*(fi) + P_{11}^*(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)\right]u_{i+1}(\varphi,\theta,\varepsilon) + f_1(\varphi,\theta,\varepsilon)$$

and

$$\frac{\partial u_i(\varphi,\theta,\varepsilon)}{\partial\varphi}[a_{10}(\varphi) + a_{11}(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)] +$$
$$+ \frac{\partial u_i(\varphi,\theta,\varepsilon)}{\partial\theta}[a_{20}(\theta) + a_{21}(\varphi,\theta,u_{i-2}(\varphi,\theta,\varepsilon),v_{i-2}(\varphi,\theta,\varepsilon),\varepsilon)] =$$
$$= \left[P_{10}^*(\varphi) + P_{11}^*(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)\right]u_i(\varphi,\theta,\varepsilon) + f_1(\varphi,\theta,\varepsilon)$$

we obtain following expression:

$$\frac{\partial \omega_{i+1}(\varphi,\theta,\varepsilon)}{\partial\varphi}[a_{10}(\varphi) + a_{11}(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)] +$$
$$+ \frac{\partial \omega_{i+1}(\varphi,\theta,\varepsilon)}{\partial\theta}[a_{20}(\theta) + a_{21}(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)] =$$
$$= \left[P_{10}^*(\varphi) + P_{11}^*(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)\right]\omega_{i+1}(\varphi,\theta,\varepsilon) + f_i(\varphi,\theta,\varepsilon),$$

where

$$\begin{aligned} f_i(\varphi,\theta,\varepsilon) = {} & [P_{11}^*(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)- \\ & - P_{11}^*(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)]u_i(\varphi,\theta,\varepsilon)+ \\ & + \frac{\partial u_i(\varphi,\theta,\varepsilon)}{\partial\varphi}[a_{11}(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)- \\ & - a_{11}(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)]+ \\ & + \frac{\partial u_i(\varphi,\theta,\varepsilon)}{\partial\theta}[a_{21}(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)- \\ & - a_{11}(\varphi,\theta,u_{i-2}(\varphi,\theta,\varepsilon),v_{i-2}(\varphi,\theta,\varepsilon),\varepsilon)], \end{aligned}$$

which means that the function $\omega_{i+1}(\varphi,\theta,\varepsilon)$ is an invariant torus of the following system:

$$\begin{aligned} \frac{d\varphi}{dt} &= a_{10}(\varphi) + a_{11}(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon), \\ \frac{d\theta}{dt} &= a_{20}(\theta) + a_{21}(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon), \\ \frac{dh}{dt} &= \big[P_{10}^*(\varphi) + P_{11}^*(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)\big]h + f_i(\varphi,\theta,\varepsilon), \end{aligned}$$

and, as mentioned above, the functions a_{11}, a_{21}, P_{11}^* satisfy all conditions of Lemma 2. So we have an invariant torus of the above system which satisfies an inequality of type (38) with $p-1$,

$$|\omega_i(\varphi,\theta,\varepsilon)|_{p-1} \le K_1|f_i|_{p-1}, \tag{41}$$

and for $p = 1$ we have

$$\begin{aligned} |\omega_{i+1}(\varphi,\theta,\varepsilon)|_0 \le K_1\big[& L_1\{|\omega_i(\varphi,\theta,\varepsilon)| + |\nu_i(\varphi,\theta,\varepsilon)|\}|u_i|_0+ \\ & + L_2\{|\omega_i(\varphi,\theta,\varepsilon)| + |\nu_i(\varphi,\theta,\varepsilon)|\}|u_i|_1+ \\ & + L_3\{|\omega_{i-1}(\varphi,\theta,\varepsilon)| + |\nu_{i-1}(\varphi,\theta,\varepsilon)|\}|u_i|_1\big]. \end{aligned} \tag{42}$$

We may choose $L = \max(L_1, L_2, L_3)$ and obtain from (42)

$$\begin{aligned} |\omega_{i+1}(\varphi,\theta,\varepsilon)|_0 \le K_1L\big\{ & \{|\omega_i(\varphi,\theta,\varepsilon)| + |\nu_i(\varphi,\theta,\varepsilon)|\}|u_i|_0+ \\ & + \{|\omega_i(\varphi,\theta,\varepsilon)| + |\nu_i(\varphi,\theta,\varepsilon)|\}|u_i|_1+ \\ & + \{|\omega_{i-1}(\varphi,\theta,\varepsilon)| + |\nu_{i-1}(\varphi,\theta,\varepsilon)|\}|u_i|_1\big\}. \end{aligned} \tag{43}$$

Analogously, we can show that the function $\nu_i(\varphi,\theta,\varepsilon)$ is an invariant torus of the system

$$\begin{aligned}
\frac{d\varphi}{dt} &= a_{10}(\varphi)+a_{11}(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon),\\
\frac{d\theta}{dt} &= a_{20}(\theta)+a_{21}(\varphi,\theta,u_{i}(\varphi,\theta,\varepsilon),v_{i}(\varphi,\theta,\varepsilon),\varepsilon),\\
\frac{ds}{dt} &= \big[P_{20}^*(\theta)+P_{21}^*(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)\big]s+g_i(\varphi,\theta,\varepsilon),
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
g_i(\varphi,\theta,\varepsilon) = {}&\big[P_{21}^*(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)-\\
&- P_{21}^*(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)\big]v_i(\varphi,\theta,\varepsilon)+\\
&+\frac{\partial v_i(\varphi,\theta,\varepsilon)}{\partial\theta}[a_{21}(\varphi,\theta,u_i(\varphi,\theta,\varepsilon),v_i(\varphi,\theta,\varepsilon),\varepsilon)-\\
&- a_{21}(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)]+\\
&+\frac{\partial v_i(\varphi,\theta,\varepsilon)}{\partial\varphi}[a_{11}(\varphi,\theta,u_{i-1}(\varphi,\theta,\varepsilon),v_{i-1}(\varphi,\theta,\varepsilon),\varepsilon)-\\
&- a_{11}(\varphi,\theta,u_{i-2}(\varphi,\theta,\varepsilon),v_{i-2}(\varphi,\theta,\varepsilon),\varepsilon)],
\end{aligned}$$

and, by using the inequality of type (38), we have

$$\begin{aligned}
|\nu_{i+1}(\varphi,\theta,\varepsilon)|_0 \le K_1L\big\{&\{|\omega_i(\varphi,\theta,\varepsilon)|_0+|\nu_i(\varphi,\theta,\varepsilon)|_0\}|v_i|_0+\\
&+\{|\omega_i(\varphi,\theta,\varepsilon)|_0+|\nu_i(\varphi,\theta,\varepsilon)|_0\}|v_i|_1+\\
&+\{|\omega_{i-1}(\varphi,\theta,\varepsilon)|_0+|\nu_{i-1}(\varphi,\theta,\varepsilon)|_0\}|v_i|_1\big\}.
\end{aligned} \tag{45}$$

Adding expression (43) and (45) we obtain

$$\begin{aligned}
|\omega_{i+1}(\varphi,\theta,\varepsilon)|_0+|\nu_{i+1}(\varphi,\theta,\varepsilon)|_0 \le M\big\{&\{|\omega_i(\varphi,\theta,\varepsilon)|_0+|\nu_i(\varphi,\theta,\varepsilon)|_0\}\{|v_i|_0+|u_i|_0\}+\\
&+\{|\omega_i(\varphi,\theta,\varepsilon)|_0+|\nu_i(\varphi,\theta,\varepsilon)|_0\}\{|v_i|_1+|u_i|_1\}+\\
&+\{|\omega_{i-1}(\varphi,\theta,\varepsilon)|_0+|\nu_{i-1}(\varphi,\theta,\varepsilon)|_0\}\{|v_i|_1+|u_i|_1\}\big\},
\end{aligned} \tag{46}$$

where M is a positive constant independent of ε. It is known that

$$|u_i|_1 \le K_1|f_1|_1, \quad |v_1|_1 \le K_1|f_2|_1 \tag{47}$$

and, by denoting

$$\alpha_i = |\omega_i(\varphi, \theta, \varepsilon)|_0 + |\nu_i(\varphi, \theta, \varepsilon)|_0,$$

it is easy to show that (46) is equivalent to

$$\alpha_{i+1} \le K_1 M\{|f_1|_1 + |f_2|_1\}\{\alpha_1 + \alpha_{i-1}\}. \tag{48}$$

The property of the functions f_1, f_2 allows us to choose a sufficiently small ε_0 such that

$$K_1 M\{|f_1|_1 + |f_2|_1\} \le \frac{1}{2},$$

but then

$$\lim_{i\to\infty} \alpha_i = 0,$$

which means that for every $\varepsilon \in (0, \varepsilon_0]$ there exist functions $u(\varphi, \theta, \varepsilon)$, $v(\varphi, \theta, \varepsilon)$ in $C(T_r \times T_l)$ for which

$$\begin{aligned} &\lim_{i\to\infty} u_i(\varphi, \theta, \varepsilon) = u(\varphi, \theta, \varepsilon), \\ &\lim_{i\to\infty} v_i(\varphi, \theta, \varepsilon) = v(\varphi, \theta, \varepsilon) \end{aligned} \tag{49}$$

uniformly with respect to φ, θ.

Since the sequences $u_i(\varphi, \theta, \varepsilon), v_i(\varphi, \theta, \varepsilon)$, $i = 1, 2, \ldots$, are bounded in the space $C^p(T_r \times T_l)$, they are compact in the $C^{p-1}(T_r \times T_l)$ which means that

$$\begin{aligned} &\lim_{i\to\infty} |u_i(\varphi, \theta, \varepsilon) - u(\varphi, \theta, \varepsilon)|_{p-1} = 0, \\ &\lim_{i\to\infty} |v_i(\varphi, \theta, \varepsilon) - v(\varphi, \theta, \varepsilon)|_{p-1} = 0, \end{aligned} \tag{50}$$

and taking limits in (38) we obtain

$$|u(\varphi, \theta, \varepsilon)|_{p-1} \le K_1 |f_1|_{p-1},$$

$$|v(\varphi, \theta, \varepsilon)|_{p-1} \le K_1 |f_2|_{p-1}.$$

The final stage of the proof is to show that the function

$$U(\varphi, \theta, \varepsilon) = \begin{pmatrix} u(\varphi, \theta, \varepsilon) \\ v(\varphi, \theta, \varepsilon) \end{pmatrix}$$

is an invariant torus of the system (13).

Let us consider the system of equations

$$\frac{d\varphi}{dt} = a(\varphi, h), \quad \frac{dh}{dt} = P(\varphi, h)h + f(\varphi) \tag{51}$$

and define a sequence of invariant tori

$$h = u^{i+1}(\varphi), \quad \varphi \in T_r, \quad i = 0, 1, \ldots, \tag{52}$$

each of which is an invariant torus of the system

$$\frac{d\varphi}{dt} = a(\varphi, u^i(\varphi)), \quad \frac{dh}{dt} = P(\varphi, u^i(\varphi))h + f(\varphi). \tag{53}$$

The following lemma is proved in [2].

Lemma 3. *Let functions a, P, f are defined and continuous with respect to φ, h in the domain*

$$\|h\| \leq d, \quad \varphi \in T_r \tag{54}$$

and be periodic with respect to $\varphi_\nu,\ \nu = 0, 1, \ldots, r$, with the period 2π. Suppose that for every $i = 1, 2, \ldots$ the system (46) has an invariant torus of the type (45) belonging to (47). If

$$\lim_{i \to \infty} u^i(\varphi) = u(\varphi) \tag{55}$$

uniformly convergent with respect to $\varphi \in T_r$, then the function $u(\varphi)$ defines an invariant torus of the system (44).

Now, by applying Lemma 3 we finish the prove of the theorem.

3. Application. ***1.*** The interaction of chemical reactors, described by a plane autonomous system of the type

$$x_i = F_i(x_1, x_2),$$
$$y_i = G_i(y_1, y_2),$$
$$i = 1, 2,$$

each admitting an exponentially attractive limit cycle, was considered by J. C. Neu [10], and a further investigation on bifurcation of periodic orbits was made by U. Kirchgraber [11] and A. Freidli, U. Kirchgraber, J. Waldvogel [12]. To get nearly identical reactors they put

$$F_1(x_1, x_2) = F(x_1, x_2) + \Lambda f(x_1, x_2), \quad F_2(x_1, x_2) = G(x_1, x_2) + \Lambda g(x_1, x_2),$$

$$G_1(y_1, y_2) = F(y_1, y_2), \quad G_2(y_1, y_2) = G(y_1, y_2),$$

where Λ is a small parameter. If this reactors are separated from each other by a membrane which allows for diffusion from one reactor to the other, they obtained coupled chemical reactors of the form

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2) + K(y_1 - x_1), \\ \dot{x}_2 &= F_2(x_1, x_2) + K(y_2 - x_2), \\ \dot{y}_1 &= G_1(y_1, y_2) + K(x_1 - y_1), \\ \dot{y}_2 &= G_2(y_1, y_2) + K(x_2 - y_2), \end{aligned} \tag{56}$$

where K is a small coupling parameter.

Let

$$x_1 = X_1(\varphi_1), \qquad x_2 = X_2(\varphi_1),$$

$$y_1 = Y_1(\varphi_2), \qquad y_2 = Y_2(\varphi_2)$$

be limit cycles of the uncoupled systems with periods T_1, T_2, respectively. We can transform them into the unit circle

$$u = u(\phi_i) = \begin{pmatrix} u_1(\varphi_i) \\ u_2(\varphi_i) \end{pmatrix} = \begin{pmatrix} \cos \lambda_i \varphi_i \\ -\sin \lambda_i \varphi_i \end{pmatrix}, \qquad i = 1, 2,$$

$$0 \le \varphi_i \le T_i, \quad \lambda_i = \frac{2\pi}{T_i}, \qquad i = 1, 2,$$

and by using

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = u(\varphi_1)(1 + h),$$
$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = u(\varphi_2)(1 + s),$$

the system (56) can be reduced into the form

$$\frac{dh}{dt} = P_1(\varphi_1, h)h - K(1 + h) + K(1 + s)\cos(\lambda_1\varphi_1 - \lambda_2\varphi_2),$$

$$\frac{d\varphi_1}{dt} = \lambda_1 - K\frac{1 + s}{\lambda_1(1 + h)}\sin(\lambda_1\varphi_1 - \lambda_2\varphi_2),$$

$$\frac{ds}{dt} = P_2(\varphi_2, s)s + K(1 + s) - K(1 + h)\cos(\lambda_2\varphi_2 - \lambda_1\varphi_1),$$

$$\frac{d\varphi_2}{dt} = \lambda_2 + K\frac{1 + h}{\lambda_2(1 + s)}\sin(\lambda_2\varphi_2 - \lambda_1\varphi_1),$$

where

$$P(\varphi_1,h)h = \left[\tilde F(\varphi_1,h)+\lambda \tilde f(\varphi_1,h)\right]\cos\lambda_1\varphi_1 - \left[\tilde G(\varphi_1,h)+\lambda\tilde g(\varphi_1,h)\right]\sin\lambda_1\varphi_1,$$

$$P(\varphi_2,s)s = \tilde F(\varphi_2,s)\cos\lambda_2\varphi_2 - \tilde G(\varphi_2,s)\sin\lambda_2\varphi_2,$$

$$\tilde F(\varphi_1,h) = F(x_1,x_2),\ \tilde F(\varphi_2,s) = F(y_1,y_2),\ \tilde f(\varphi_1,h) = f(x_1,x_2),$$

$$\tilde G(\varphi_1,h) = G(x_1,x_2),\ \tilde G(\varphi_2,s) = G(y_1,y_2),\ \tilde g(\varphi_1,h) = g(x_1,x_2).$$

Let us introduce the notations

$$L(h,s) = \frac{1+h}{1+s},\quad K = \varepsilon k,\quad \Lambda = \varepsilon\lambda,$$

where ε is a sufficiently small parameter; k,λ are fixed numbers. Then

$$\frac{dh}{dt} = P_1(\varphi_1,h)h + \varepsilon k[L(s,0)\cos(\lambda_1\varphi_1-\lambda_2\varphi_2) - L(h,0)],$$

$$\frac{d\varphi_1}{dt} = \lambda_1 - \varepsilon\frac{k}{\lambda_1}L(s,h)\sin(\lambda_1\varphi_1-\lambda_2\varphi_2),$$

$$\frac{ds}{dt} = P_2(\varphi_2,s)s - \varepsilon k[L(h,0)\cos(\lambda_2\varphi_2-\lambda_1\varphi_1) - L(s,0)],$$

$$\frac{d\varphi_2}{dt} = \lambda_2 + \varepsilon\frac{k}{\lambda_2}L(h,s)\sin(\lambda_2\varphi_2-\lambda_1\varphi_1).$$

Let $\varphi_{1t}(\varphi_1) = \lambda_1 t+\varphi_1,\ \varphi_{2t}(\varphi) = \lambda_2 t+\varphi_2,\ P_{01}(\varphi_1) = P_1(\varphi_1,0)$ and suppose that $G_{01}(\tau,\varphi_1)$ is a Green function of the equation

$$\frac{dh}{dt} = P_{01}(\varphi_{1t}(\varphi_1))h.$$

Then we can write

$$G_{01}(\tau,\varphi_1) = \begin{cases} e^{-\int_\tau^0 P_{01}(\varphi_{1t}(\varphi_1))dt}, & \tau\le 0,\\ 0, & \tau>0,\end{cases}$$

and the first approximation is obtained from

$$h^{(1)} = u(\varphi_1,\varphi_2,\varepsilon) = \varepsilon k\int_{-\infty}^{+\infty} G_{01}(\tau,\varphi_1)[\cos(\lambda_1\varphi_1-\lambda_2\varphi_2)-1]d\tau.$$

Since the limit cycle is exponentially attractive, the function $G_{01}(\tau,\varphi_1)$ will satisfy the inequality

$$|G_{01}(\tau,\varphi_1)| \le e^{-\gamma|\tau|},\qquad \gamma>0,$$

and, obviously, this function is a rough Green function.

Analogously we can find $s^{(1)}$ and other iterations of the torus of the system (57).

2. Consider two harmonic weakly connected oscillators

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 + x_1(1 - x_1^2 - x_2^2) + \varepsilon f_1(x_1, x_2, y_1, y_2), \\ \frac{dx_2}{dt} &= -x_1 + x_2(1 - x_1^2 - x_2^2) + \varepsilon f_2(x_1, x_2, y_1, y_2), \\ \frac{dy_1}{dt} &= y_2 + y_1(1 - y_1^2 - y_2^2) + \varepsilon g_1(x_1, x_2, y_1, y_2), \\ \frac{dy_2}{dt} &= -y_1 + y_2(1 - y_1^2 - y_2^2) + \varepsilon g_2(x_1, x_2, y_1, y_2), \end{aligned} \tag{57}$$

where f_1, f_2, g_1, g_2 are continuous with respect to all variables.

Obviously, for $\varepsilon = 0$ we have two independent systems, each of which has a limit cycle since the invariant manifold has the type

$$u = u(\phi) = \begin{pmatrix} u_1(\varphi) \\ u_2(\varphi) \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix}, \quad 0 \le \varphi \le 2\pi.$$

For the transformation we may use

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = u(\varphi)(1 + h),$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = u(\theta)(1 + s),$$

and by substituting it to (57) we obtain

$$\begin{aligned} \frac{d\varphi}{dt} &= 1 - \frac{\varepsilon}{1+h}[\tilde{f}_1 \sin\varphi + \tilde{f}_2 \cos\varphi], \\ \frac{d\theta}{dt} &= 1 - \frac{\varepsilon}{1+s}[\tilde{g}_1 \sin\theta + \tilde{g}_2 \cos\theta], \\ \frac{dh}{dt} &= -(1+h)(2+h)h - \varepsilon[\tilde{f}_2 \sin\varphi - \tilde{f}_1 \cos\varphi], \\ \frac{ds}{dt} &= -(1+s)(2+s)s - \varepsilon[\tilde{g}_2 \sin\theta - \tilde{g}_1 \cos\theta], \end{aligned} \tag{58}$$

where $\tilde{f}_i(\varphi, \theta, h, s) = f_i(x_1, x_2, y_1, y_2)$, $\tilde{g}_i(\varphi, \theta, h, s) = g_i(x_1, x_2, y_1, y_2)$, $i = 1, 2$.

By substituting the zero approximation $h = 0, s = 0$ to (58) we can find the first approximation from

$$\frac{d\theta}{dt} = 1,$$
$$\frac{d\varphi}{dt} = 1 - \varepsilon[\tilde{f}_1^0 \sin\varphi + \tilde{f}_2^0 \cos\varphi], \tag{59}$$
$$\frac{dh}{dt} = -2h - \varepsilon[\tilde{f}_2^0 \sin\varphi - \tilde{f}_1^0 \cos\varphi],$$

and

$$\frac{d\varphi}{dt} = 1,$$
$$\frac{d\theta}{dt} = 1 - \varepsilon[\tilde{g}_1^0 \sin\theta + \tilde{g}_2^0 \cos\theta], \tag{60}$$
$$\frac{ds}{dt} = -2s - \varepsilon[\tilde{g}_2^0 \sin\theta - \tilde{g}_1^0 \cos\theta],$$

where $\tilde{f}_i^0(\varphi,\theta) = \tilde{f}_i(\varphi,\theta,0,0)$, $\tilde{g}_i^0(\varphi,\theta) = \tilde{g}_i(\varphi,\theta,0,0)$, $i = 1,2$.

The Green function has the form

$$G_0(\tau) = \begin{cases} e^{2\tau}, & \tau \le 0, \\ 0 & \tau > 0. \end{cases}$$

We can explicitly find an invariant torus of this equations,

$$h^{(1)}(\varphi,\theta) = \varepsilon \int\limits_{-\infty}^{+\infty} G_0(\tau)\big[\tilde{f}_2^0(\varphi_\tau(\varphi),\theta_\tau(\theta)) \sin\varphi_\tau(\varphi) - \tilde{f}_1^0(\varphi_\tau(\varphi),\theta_\tau(\theta)) \cos\varphi_\tau(\varphi)\big]d\tau,$$

$$s^{(1)}(\varphi,\theta) = \varepsilon \int\limits_{-\infty}^{+\infty} G_0(\tau)\big[\tilde{g}_1^0(\varphi_\tau(\varphi),\theta_\tau(\theta)) \sin\theta_\tau(\theta) - \tilde{g}_1^0(\varphi_\tau(\varphi),\theta_\tau(\theta)) \cos\theta_\tau(\theta)\big]d\tau,$$

where $\varphi_\tau(\varphi)$, $\theta_\tau(\theta)$ are solutions of the first equations of (59) for $h^{(1)}(\varphi,\theta)$, and solutions of the first equations of (60) for $s^{(1)}(\varphi,\theta)$ and satisfying the initial conditions $\varphi_0(\varphi) = \varphi$, $\theta_0(\theta) = \theta$.

The $(i+1)$th approximation $u^{i+1}(\varphi,\theta,\varepsilon) = (h^{i+1}(\varphi,\theta,\varepsilon), s^{i+1}(\varphi,\theta,\varepsilon))$ can be found from following two systems of equations:

$$\frac{d\varphi}{dt} = 1 - \frac{\varepsilon}{1+h^i(\varphi,\theta,\varepsilon)}\big[\tilde{f}_1(\varphi,\theta,h^i(\varphi,\theta,\varepsilon),s^i(\varphi,\theta,\varepsilon))\sin\varphi+$$

$$+\tilde{f}_2(\varphi,\theta,h^i(\varphi,\theta,\varepsilon),s^i(\varphi,\theta,\varepsilon))\cos\varphi\big],$$

$$\frac{d\theta}{dt} = 1 - \frac{\varepsilon}{1+s^{i-1}(\varphi,\theta,\varepsilon)}\big[\tilde{g}_1(\varphi,\theta,h^{i-1}(\varphi,\theta,\varepsilon),s^{i-1}(\varphi,\theta,\varepsilon))\sin\theta+$$

$$+\tilde{g}_2(\varphi,\theta,h^{i-1}(\varphi,\theta,\varepsilon),s^{i-1}(\varphi,\theta,\varepsilon))\cos\theta\big],$$

$$\frac{dh}{dt} = -(1+h^i(\varphi,\theta,\varepsilon))(2+h^i(\varphi,\theta,\varepsilon))h-$$

$$-\varepsilon\big[\tilde{f}_2(\varphi,\theta,h^i(\varphi,\theta,\varepsilon),s^i(\varphi,\theta,\varepsilon))\sin\varphi-\tilde{f}_1(\varphi,\theta,h^i(\varphi,\theta,\varepsilon),s^i(\varphi,\theta,\varepsilon))\cos\varphi\big],$$

and

$$\frac{d\varphi}{dt} = 1 - \frac{\varepsilon}{1+h^{i-1}(\varphi,\theta,\varepsilon)}\big[\tilde{f}_1(\varphi,\theta,h^{i-1}(\varphi,\theta,\varepsilon),s^{i-1}(\varphi,\theta,\varepsilon))\sin\varphi+$$

$$+\tilde{f}_2(\varphi,\theta,h^{i-1}(\varphi,\theta,\varepsilon),s^{i-1}(\varphi,\theta,\varepsilon))\cos\varphi\big],$$

$$\frac{d\theta}{dt} = 1 - \frac{\varepsilon}{1+s^i(\varphi,\theta,\varepsilon)}\big[\tilde{g}_1(\varphi,\theta,h^i(\varphi,\theta,\varepsilon),s^i(\varphi,\theta,\varepsilon))\sin\theta+$$

$$+\tilde{g}_2(\varphi,\theta,h^i(\varphi,\theta,\varepsilon),s^i(\varphi,\theta,\varepsilon))\cos\theta\big],$$

$$\frac{ds}{dt} = -(1+s^i(\varphi,\theta,\varepsilon))(2+s^i(\varphi,\theta,\varepsilon))s-$$

$$-\varepsilon\big[\tilde{g}_2(\varphi,\theta,h^i(\varphi,\theta,\varepsilon),s^i(\varphi,\theta,\varepsilon))\sin\theta-\tilde{g}_1(\varphi,\theta,h^i(\varphi,\theta,\varepsilon),s^i(\varphi,\theta,\varepsilon))\cos\theta\big].$$

It should be noted that the Green function in this case is rough with for order p, where $0 \le \le p \le \infty$. Now, by using the main result from Section 2 we may choose $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0]$ there exists an invariant torus $u(\varphi,\theta,\varepsilon)$ of (57), which we can find by the above process as a limit function, i.e.,

$$u(\varphi,\theta,\varepsilon) = \lim_{k\to\infty} u^{(k)}(\varphi,\theta,\varepsilon) = \lim_{k\to\infty}\left(h^{(k)}(\varphi,\theta,\varepsilon), s^{(k)}(\varphi,\theta,\varepsilon)\right).$$

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