НЕЛІНІЙНІ КОЛИВАННЯ т. 2 • № 4 • 1999

UDC 517 . 9

ON INVARIANT SETS OF DIFFERENTIAL EQUATIONS WITH IMPULSES*

ПРО ІНВАРІАНТНІ МНОЖИНИ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ІМПУЛЬСАМИ

V.I. Tkachenko

Inst. Math. Nat. Acad. Sci. Ukraine,
Ukraine, 252601, Kyiv 4, Tereshchenkivs'ka str.,3
e-mail: vitk@imath.kiev.ua

For a system of ordinary differential equations depending on a small parameter, defined on the direct product of a torus and aEuclidean space, and subjected to impulsive action on a submanifold of codimension 1 of the torus, we study the problem of existence of a piecewise smooth invariant set.

Вивчається задача існування кусково-гладкої інваріантної множини системи диференціальних рівнянь, залежних від малого параметра та заданих на прямому добутку тора та евклідового простору з імпульсною дією на підмноговиді тора корозмірності 1.

1. Introduction. We consider an impulsive system of the form

$$\frac{d\varphi}{dt} = a(\varphi, x, \varepsilon), \tag{1}$$

$$\frac{dx}{dt} = A(\varphi, \varepsilon)x + f(\varphi, x, \varepsilon), \quad \varphi \in \mathbb{T}_m \setminus \Gamma, \tag{2}$$

$$\Delta x \Big|_{\varphi \in \Gamma} = B(\varphi, \varepsilon)x + g(\varphi, x, \varepsilon), \tag{3}$$

where $x \in \mathbb{R}^n,\ \varphi \in \mathbb{T}_m,\ \mathbb{T}_m$ is an m-dimensional torus, Γ is a smooth compact submanifold of $\mathbb{T}_m$ of codimension 1, and $\varepsilon \in \mathbb{R}$ is a small parameter. Δx stands for the jump of the function x at the point φ obtained during the motion along the trajectory of equation (1).

We suppose that $f = O(\|x\|^2),\ g = O(\|x\|^2)$ as $\varepsilon = 0$. Therefore, system (1) – (3) has the trivial invariant set $S_0 = \{(0, \varphi) \in \mathbb{R}^n \times \mathbb{T}_m\}$ for $\varepsilon = 0$. We are interested in the existence of piecewise continuous (piecewise smooth) invariant set of system (1) – (3) for small $\varepsilon \neq 0$. Partial results of this paper were communicated in [1]. This problem for systems without impulses was studied by many authors [2 – 6]. The invariant sets in particular cases of the impulsive system were considered in [7 – 13].

* Partially supported by INTAS (Grant No. 96-0915) and the Ukrainian Ministry on Science and Technology (Grant No. 1.4/269).

The article is organized as follows: In section 2, we introduce the concept of exponential dichotomy for the linearized system

$$\frac{d\varphi}{dt} = a_0(\varphi), \tag{4}$$

$$\frac{dx}{dt} = A_0(\varphi)x, \quad \varphi \in \mathbb{T}_m \setminus \Gamma, \tag{5}$$

$$\Delta x \Big|_{\varphi \in \Gamma} = B_0(\varphi)x, \tag{6}$$

where $a_0(\varphi) = a(\varphi, 0, 0)$, $A_0(\varphi) = A(\varphi, 0)$, and $B_0(\varphi) = B(\varphi, 0)$. The properties of separatrix subspaces of the linearized system are studied. In section 3, we prove that the exponential dichotomy for system (4) – (6) is not destroyed by small perturbations of the right-hand sides of the system. In section 4, conditions for the existence of an invariant set of system (1) – (3) for small $\varepsilon \neq 0$ are obtained.

2. Linear system. Let us consider system (4) – (6). We assume that $a_0(\varphi)$ is a Lipschitz function in $\varphi \in \mathbb{T}_m$ and the functions $A_0(\varphi)$, $B_0(\varphi)$ are continuous. Equation (4) has solutions $\varphi \cdot t = \sigma(t, \varphi), \sigma(0, \varphi) = \varphi$. Suppose that solutions $\sigma(t, \varphi)$ intersect the manifold Γ transversally. The set $I(\varphi)$ of points t where the solution $\sigma(t, \varphi)$ intersects the compact manifold Γ is at most countable. Note that it can be finite or empty. We denote by $t_j(\varphi), j \in I(\varphi) \subseteq \mathbb{Z}$ the ascending sequence of points t where $\sigma(t, \varphi)$ intersects the manifold Γ, $t_0(\varphi) = \max\{t < 0 : \varphi \cdot t \in \Gamma\}$, $t_1(\varphi) = \min\{t \geq 0 : \varphi \cdot t \in \Gamma\}$. There exists $\theta > 0$ such that

$$t_j(\varphi) - t_{j-1}(\varphi) \geq \theta \tag{7}$$

for all $\varphi \in \mathbb{T}_m, j \in I(\varphi)$.

For fixed φ, system (4) – (6) has the following form:

$$\frac{dx}{dt} = A_0(\sigma(t, \varphi))x, \quad t \neq t_i(\varphi), \tag{8}$$

$$\Delta x \Big|_{t = t_i(\varphi)} = B_0(\sigma_i(\varphi))x, \tag{9}$$

where $\sigma_i(\varphi) = \sigma(t_i(\varphi), \varphi)$. Let $x(t, \varphi, x_0)$ be a solution of the initial-value problem for (8), (9) with the initial value $x(0, \varphi, x_0) = x_0$. Denote by $X(t, \varphi)$, $t \geq 0$ the fundamental solution for system (8), (9), $X(t, \varphi)x_0 = x(0, \varphi, x_0)$, $X(0, \varphi) = I$, I is the identity matrix. The solution $x(t, \varphi, x_0)$ is piecewise continuous and we assume that it is left-side continuous. It has discontinuities in $t = t_i(\varphi)$. It is supposed that $\det(I + B(\varphi)) = 0$ for some or all $\varphi \in \Gamma$. Therefore, the solutions $x(t, \varphi, x_0)$ cannot be continued on the negative semi-axis $t < 0$ or can be ambiguously continued.

Using the uniqueness of solutions for equation (4) and transversality of intersections $\sigma(t, \varphi)$ with Γ, we conclude that the theorem on continuous dependence on initial conditions and parameters [8, 14] is valid for impulsive system (4) – (6): for a solution $x(t, \varphi_0, x_0)$ of system (4) – (6) and for an arbitrary $\varepsilon > 0$ and $T > 0$, there exists $\delta = \delta(\varepsilon, T) > 0$ such that, for any other solution $x(t, \varphi_1, x_1)$ of (4) – (6) with initial conditions (φ_1, x_1), the inequalities $\|x_0 - x_1\| < \delta,\ \rho(\varphi_0, \varphi_1) < \delta$ imply that $\|x(t, \varphi_0, x_0) - x(t, \varphi_1, x_1)\| < \varepsilon$ for $0 \le t \le T$ satisfying $|t - t_i| > \varepsilon$, where t_i are the moments in which $\sigma(t, \varphi_0)$ intersects the manifold Γ, and $\rho(.,.)$ is a metric on the torus $\mathbb{T}_m$.

We distinguish the the left-hand and right-hand sides of manifold Γ. We call a sequence $\varphi_n \to \varphi \in \Gamma$ negative if there exists a sequence of positive numbers $\delta_n \to 0, n \to \infty$ such that $\varphi_n \cdot \delta_n \in \Gamma$. Analogously, a sequence $\varphi_n \to \varphi \in \Gamma$ is said to be positive if there exists a sequence of negative numbers $\delta_n \to 0, n \to \infty$ such that $\varphi_n \cdot \delta_n \in \Gamma$.

Denote by $C^s(\mathbb{T}_m)$ the space of s times continuously differentiable functions or matrices on $\mathbb{T}_m$. By $C^s_\Gamma(\mathbb{T}_m)$ we denote the space of functions or matrices $a(\varphi)$ with the following properties:

i) $a(\varphi)$ has continuous partial derivatives up to the order s inclusively for $\varphi \in \mathbb{T}_m \setminus \Gamma$;

ii) all partial derivatives of $a(\varphi)$ have continuous continuations to the left-hand and right-hand sides of manifold Γ.

For $f(\varphi) \in C^s_\Gamma(\mathbb{T}_m)$, we denote the norm

$$\|f(\varphi)\|_s = \max_{0 \le |j| \le s} \sup_{\varphi \in \mathbb{T}_m \setminus \Gamma} \left\| \frac{\partial^{|j|} f(\varphi)}{\partial \varphi^j} \right\|,$$

where $j = (j_1, ..., j_m),\ \varphi^j = (\varphi_1^{j_1} ... \varphi_m^{j_m}),\ |j| = j_1 + ... + j_m$, and $\|.\|$ is the norm in $\mathbb{R}^n$ or in the space of matrices.

Definition 1. *System (4), (6) is said to be exponentially dichotomous if, for all $\varphi \in \mathbb{T}_m$, the space $\mathbb{R}^n$ can be represented in the form of the direct sum of the subspaces $U(\varphi)$ and $S(\varphi)$ of dimensions r and $n - r$, respectively, so that:*

1) any solution of system (8), (9) with $x_0 \in S(\varphi)$ satisfies the inequality

$$\|x(t, \varphi, x_0)\| \le K \exp(-\alpha(t - \tau)) \|x(\tau, \varphi, x_0)\|, \quad t \ge \tau \ge 0; \tag{10}$$

2) any solution with $x_0 \in U(\varphi)$ satisfies the inequality

$$\|x(t, \varphi, x_0)\| \ge K_1 \exp(\alpha(t - \tau)) \|x(\tau, \varphi, x_0)\|, \quad t \ge \tau \ge 0, \tag{11}$$

where positive constants α, K, K_1 are independent of φ, x_0;

3) $X(t, \varphi) S(\varphi) \subseteq S(\varphi \cdot t), \quad X(t, \varphi) U(\varphi) \subseteq U(\varphi \cdot t), \quad t \ge 0$;

4) the projectors $P(\varphi)$ and $Q(\varphi) = I - P(\varphi)$ corresponding to $S(\varphi)$ and $U(\varphi)$ are uniformly bounded

$$\sup_{\varphi \in \mathbb{T}_m} \|P(\varphi)\| + \sup_{\varphi \in \mathbb{T}_m} \|Q(\varphi)\| < \infty.$$

Analogously to the proof of Theorem 1 [15], we prove the following statement:

Theorem 1. *Assume that system (4) – (6) is exponentially dichotomous. Then the projector $P(\varphi)$ is continuous on the set $\mathbb{T}_m \setminus \Gamma$ and has discontinuities of the first kind on the set Γ.*

It follows from Definition 1 that the subspace $U(\varphi)$ has a unique negative continuation such that

$$\|x(t, \varphi, x_0)\| \le K_2 \exp(\alpha t) \|x_0\|, \quad t \le 0, \quad \varphi \in \mathbb{T}_m,\ x_0 \in U(\varphi).$$

Hence, $X(t,\varphi)Q(\varphi)$ is well defined for all $t \leq 0$, and we can define the Green function for system (4) – (6)

$$G(t,\tau,\varphi) = \begin{cases} X(t-\tau,\varphi\cdot\tau)P(\varphi\cdot\tau), & t \geq \tau; \\ -X(t-\tau,\varphi\cdot\tau)Q(\varphi\cdot\tau), & \tau \geq t. \end{cases} \tag{12}$$

For $t \neq \tau$, the Green function $G(t,\tau,\varphi)$ satisfies equations (8), (9). If system (4) – (6) has exponential dichotomy, then the Green function $G(t,\tau,\varphi)$ is bounded by an exponent:

$$\|G(t,\tau,\varphi)\| \leq K_3 \exp(-\alpha|t-\tau|), \quad t,\tau \in \mathbb{R}, \quad K_3, \alpha > 0. \tag{13}$$

The linear inhomogeneous system

$$\frac{dx}{dt} = A_0(\varphi\cdot t)x + f(t), \quad t \neq t_i(\varphi),$$

$$\Delta x\Big|_{t=t_i(\varphi)} = B_0(\sigma_i(\varphi))x + g_i$$

has the following unique bounded solution:

$$u(t,\varphi) = \int\limits_{-\infty}^{\infty} G(t,\tau,\varphi)f(\tau)d\tau + \sum_{i\in I(\varphi)} G(t,t_i(\varphi),\varphi)g_i. \tag{14}$$

Theorem 2. *Suppose that:*
1) Γ is a smooth manifold of the class $C^s, s \geq 1$;
2) $a_0(\varphi), A_0(\varphi) \in C^s_\Gamma(\mathbb{T}_m),\ B_0(\varphi) \in C^s(\Gamma)$;
3) solutions of equation (4) intersect the manifold Γ transversally;
4) system (4) – (6) is exponentially dichotomous with constants α, K, K_1.
Then the projector $P(\varphi)$ and the Green function $G(t,s,\varphi)$ have continuous partial derivatives of order s with respect to φ on the set $\mathbb{T}_m \setminus \Gamma$ and, moreover,

$$\left\|\frac{\partial^{|j|}G(t,\tau,\varphi)}{\partial\varphi^j}\right\| \leq \tilde{K}_j \exp(-(\alpha_1 - |j|\omega)|t-\tau|), \tag{15}$$

where $j = (j_1,...,j_m)$, $|j| = j_1+...+j_m$, $|j| \leq s$, $\varphi^j = \varphi_1^{j_1},...,\varphi_m^{j_m}$, $\alpha_1 = \alpha-\varepsilon$, ε is an arbitrarily small positive value, $\tilde{K}_j = \tilde{K}_j(\varepsilon)$ is a constant independent of $\varphi \in \mathbb{T}_m$, and $\omega = \|\partial a(\varphi)/\partial\varphi\|_0$.

Proof. Let $\delta\varphi_i$ be an increment of the i-th coordinate of φ and $\varphi + \delta\varphi_i = (\varphi_1,...,\varphi_i + +\delta\varphi_i,...,\varphi_n)$. Let us consider the difference $R = G(t,\tau,\varphi+\delta\varphi_i) - G(t,\tau,\varphi)$, where the points φ and $\varphi+\delta\varphi_i$ are located at the same side of Γ and do not belong to Γ. The difference R satisfies the following system:

$$\frac{dR}{dt} = A_0(\sigma(t,\varphi))R + (A_0(\sigma(t,\varphi+\delta\varphi_i)) - A_0(\sigma(t,\varphi)))G(t,\tau,\varphi+\delta\varphi_i),$$

$$\Delta R\Big|_{t=t_j^1} = B_0(\sigma_j^1)R - B_0(\sigma_j^1)G(t_j^1,\tau,\varphi+\delta\varphi_i),$$

$$\Delta R\Big|_{t=t_j^2} = B_0(\sigma_j^2)G(t_j^2,\tau,\varphi+\delta\varphi_i),$$

where $t_j^1 = t_j(\varphi), t_j^2 = t_j(\varphi + \delta\varphi_i), \sigma_j^1 = \sigma(t_j(\varphi), \varphi), \sigma_j^2 = \sigma(t_j(\varphi + \delta\varphi_i), \varphi + \delta\varphi_i)$. By (14), one has

$$G(0, \tau, \varphi + \delta\varphi_i) - G(0, \tau, \varphi) =$$

$$= \int_{-\infty}^{\infty} G(0, s, \varphi)(A_0(\sigma(s, \varphi + \delta\varphi_i)) - A_0(\sigma(s, \varphi)))G(s, \tau, \varphi + \delta\varphi_i)ds+$$

$$+ \sum_{j \in I(\varphi+\delta\varphi_i)} G(0, t_j^2, \varphi)B(\sigma_j^2)(G(t_j^1, \tau, \varphi + \delta\varphi_i) - G(t_j^2, \tau, \varphi + \delta\varphi_i))+$$

$$+ \sum_{j \in I(\varphi)\cup I(\varphi+\delta\varphi_i)} (G(0, t_j^2, \varphi) - G(0, t_j^1, \varphi))B_0(\sigma_j^2))G(t_j^1, \tau, \varphi + \delta\varphi_i)+$$

$$+ \sum_{j \in I(\varphi)} G(0, t_j^1, \varphi)(B_0(\sigma_j^2) - B_0(\sigma_j^1))G(t_j^1, \tau, \varphi + \delta\varphi_i). \tag{16}$$

Since $a_0(\varphi) \in C^s(\varphi), s \geq 1$, we have $\|a_0(\varphi_1) - a_0(\varphi_2)\| \leq \omega\|\varphi_1 - \varphi_2\|$, where $\omega = \|\partial a_0(\varphi)/\partial\varphi\|_0$. Hence,

$$\|\sigma(t, \varphi_1) - \sigma(t, \varphi_2)\| \leq e^{\omega|t|}\|\varphi_1 - \varphi_2\|, \tag{17}$$

$$\left\|\frac{\partial\sigma(t, \varphi)}{\partial\varphi}\right\| \leq e^{\omega|t|}. \tag{18}$$

Let the manifold Γ be defined by $F(\varphi) = 0$ with some smooth function F. By definition, $\sigma(t_j(\varphi), \varphi) \in \Gamma$ or $F(\sigma(t_j(\varphi), \varphi)) = 0,\ j \in I(\varphi),\ \varphi \in \mathbb{T}_m$. Therefore,

$$\frac{\partial F(\sigma(t_j(\varphi), \varphi))}{\partial\varphi_i} = 0, \quad \text{and} \quad \left(\frac{\partial F}{\partial\sigma}, \frac{\partial\sigma_j(\varphi)}{\partial t}\frac{\partial t_j(\varphi)}{\partial\varphi_i} + \frac{\partial\sigma_j(\varphi)}{\partial\varphi_i}\right) = 0,$$

where $\sigma_j = \sigma_j(\varphi) = \sigma(t_j(\varphi), \varphi),\ j \in I(\varphi), i = 1, ..., m;$ $(.,.)$ is a scalar product in $\mathbb{R}^n$. Let us make the transformation

$$\left(\frac{\partial F(\sigma_j)}{\partial\sigma}, a_0(\sigma_j)\right)\frac{\partial t_j(\varphi)}{\partial\varphi_i} + \left(\frac{\partial F(\sigma_j)}{\partial\sigma}, \frac{\partial\sigma_j(\varphi)}{\partial\varphi_i}\right) = 0. \tag{19}$$

The intersections of the solution $\sigma(t, \varphi)$ with the compact manifold Γ are transversal, and, therefore,

$$\left(\frac{\partial F(\sigma_j)}{\partial\sigma}, a_0(\sigma_j)\right) \geq C_1 > 0, \quad C_1 \neq C_1(\varphi).$$

By (18) and (19), we see that

$$\left|\frac{\partial t_j(\varphi)}{\partial\varphi}\right| \leq \frac{C_2}{C_1}e^{\omega|t_j(\varphi)|} = C_3 e^{\omega|t_j(\varphi)|}, \tag{20}$$

where $C_2 \geq \|\partial F/\partial\sigma\|$.

The second derivative $\partial^2\sigma(t,\varphi)/\partial\varphi_i\partial\varphi_j,\ i,j=1,...,m$, satisfies the following equation:

$$\frac{d}{dt}\frac{\partial^2\sigma(t,\varphi)}{\partial\varphi_i\partial\varphi_j}=\frac{\partial a_0(\sigma(t,\varphi))}{\partial\sigma}\frac{\partial^2\sigma(t,\varphi)}{\partial\varphi_i\partial\varphi_j}+$$

$$+\sum_{j,k=1}^{m}\frac{\partial^2 a_0(\sigma(t,\varphi))}{\partial\sigma_k\partial\sigma_l}\frac{\partial\sigma_k(t,\varphi)}{\partial\varphi_i}\frac{\partial\sigma_l(t,\varphi_0)}{\partial\varphi_j}. \tag{21}$$

Here, $\sigma_k(t,\varphi)$ is the k-component of the vector $\sigma(t,\varphi)$. Taking into account (18) and (21), we obtain

$$\left\|\frac{\partial^2\sigma(t,\varphi)}{\partial\varphi_i\partial\varphi_j}\right\|\leq e^{\omega|t|}(a_1+a_2e^{\omega|t|})\leq a_3e^{(2\omega+\varepsilon)|t|},$$

where ε is an arbitrarily small positive value and a_1,a_2,a_3 are positive constants, $a_3=a_3(\varepsilon)$.

The higher derivatives are estimated similarly:

$$\left\|\frac{\partial^{|l|}\sigma(t,\varphi)}{\partial\varphi^l}\right\|\leq a_le^{(|l|\omega+\varepsilon)|t|}, \tag{22}$$

where l is a multiindex, $l=(l_1,...,l_m),\ \sum_j l_j=|l|,\ \varphi^l=\varphi_1^{l_1}...\varphi_m^{l_m}$, ε is an arbitrarily small positive value, and $a_l=a_l(\varepsilon)>0$.

Differentiating (19) and taking (22) into account, we estimate the higher derivatives of $t_j(\varphi)$:

$$\left\|\frac{\partial^{|l|}t_j(\varphi)}{\partial\varphi^l}\right\|\leq C_le^{(|l|\omega+\varepsilon)|t_j(\varphi)|}, \tag{23}$$

where $C_l=C_l(\varepsilon)>0$ is a constant, and l is multiindex as before.

Analogously to [1], we compute limits

$$\lim_{\delta\varphi_i\to0}\frac{1}{\delta\varphi_i}\Bigg(G(t_j(\varphi+\delta\varphi_i),\tau,\varphi+\delta\varphi_i)-(G(t_j(\varphi),\tau,\varphi+\delta\varphi_i)\Bigg)=$$

$$=A_0(\sigma_j)G(t_j(\varphi),\tau,\varphi))\frac{\partial t_j(\varphi)}{\partial\varphi_i} \tag{24}$$

and

$$\lim_{\delta\varphi_i\to0}\frac{1}{\delta\varphi_i}\Bigg(G(0,t_j(\varphi+\delta\varphi_i),\varphi)-G(0,t_j(\varphi),\varphi)\Bigg)=$$

$$=-G(0,t_j(\varphi),\varphi)A_0(\sigma_j(\varphi))\frac{\partial t_j(\varphi)}{\partial\varphi_i}. \tag{25}$$

Let $t_j(\varphi)\to\infty$ as $\varphi\to\bar{\varphi}$. This means that $j\notin I(\bar{\varphi})$ for sufficiently large j and $G(0,t_j(\varphi),\varphi)\to$ $\to0$ as $\varphi\to\bar{\varphi}$. By (18) and (25), one has

$$\lim_{\varphi\to\bar{\varphi}}\lim_{\delta\varphi_i\to0}\frac{1}{\delta\varphi_i}\Bigg(G(0,t_j(\varphi+\delta\varphi_i),\varphi)-G(0,t_j(\varphi),\varphi)\Bigg)=$$

$$= \lim_{\delta\varphi_i} \frac{1}{\delta\varphi_i} G(0, t_j(\bar\varphi + \delta\bar\varphi), \bar\varphi) =$$

$$- \lim_{\varphi\to\bar\varphi} G(0, t_j(\varphi), \varphi) A_0(\sigma_j(\varphi)) \frac{\partial t_j(\varphi)}{\partial \varphi_i} = 0. \qquad (26)$$

Taking (24), (25) and (26) into account, we get

$$\frac{\partial G(0,\tau,\varphi)}{\partial\varphi_i} = \int_{-\infty}^{\infty} G(0,s,\varphi) \frac{\partial A_0(\sigma(s,\varphi))}{\partial\sigma} \frac{\partial\sigma(s,\varphi)}{\partial\varphi_i} G(s,\tau,\varphi) ds +$$

$$+ \sum_{j\in I(\varphi)} G(0, t_j(\varphi), \varphi) \frac{\partial B_0(\sigma_j(\varphi))}{\partial\sigma} \frac{\partial\sigma_j(\varphi)}{\partial\varphi_i} G(t_j(\varphi), \tau, \varphi) +$$

$$+ \sum_{j\in I(\varphi)} G(0, t_j(\varphi), \varphi) A_0(\sigma_j) B_0(\sigma_j) G(t_j(\varphi), \tau, \varphi) \frac{\partial t_j(\varphi)}{\partial\varphi_i} -$$

$$- \sum_{j\in I(\varphi)} G(0, t_j(\varphi), \varphi) B_0(\sigma_j) A_0(\sigma_j) G(t_j(\varphi), \tau, \varphi) \frac{\partial t_j(\varphi)}{\partial\varphi_i}. \qquad (27)$$

The matrix $\partial A(\sigma(t,\varphi))/\partial\sigma)(\partial\sigma(s,\varphi)/\partial\varphi_i)$ has the elements

$$\sum_{j=1}^{m} \frac{\partial a_{kl}(\sigma(s,\varphi))}{\partial\sigma_j} \frac{\partial\sigma_j(s,\varphi)}{\partial\varphi_i},$$

where $A(\varphi) = \{a_{kl}\}$, and $\sigma = (\sigma_1, ..., \sigma_m)$.

The derivative $\partial G(0,\tau,\varphi)/\partial\varphi_i$ exists if the integral and series in (27) are convergent. Using (13) and (18), we estimate

$$\int_{-\infty}^{\infty} \left\| G(0,s,\varphi) \frac{\partial A_0(\sigma(s,\varphi))}{\partial\sigma} \frac{\partial\sigma(s,\varphi)}{\partial\varphi_i} G(s,\tau,\varphi) \right\| ds \le$$

$$\le \int_{-\infty}^{\infty} K_3^2 M e^{-(\alpha-\omega)|s| - \alpha|\tau - s|} ds \le K_3^2 M \left(\frac{2}{2\alpha - \omega} + |\tau| \right) e^{-(\alpha-\omega)|\tau|},$$

where $\|A_0(\varphi)\|_s \le M$, $\|B_0(\varphi)\|_s \le M$, and $\|a_0(\varphi)\|_s \le M$. The integral converges if $2\alpha - \omega > 0$. By (7), (13), (18) and (20), we get

$$\sum_j \left\| G(0, t_j(\varphi), \varphi) \frac{\partial B_0(\sigma(t_j,\varphi))}{\partial\sigma} \frac{\partial\sigma(t_j(\varphi),\varphi)}{\partial\varphi_i} G(t_j(\varphi), \tau, \varphi) \right\| \le$$

$$\le K_3^2 M (1 + C_3 M) \left(\frac{2}{1 - e^{-(2\alpha-\omega)\theta}} + \frac{|\tau|}{\theta} \right) e^{-(\alpha-\omega)|\tau|}.$$

The last sums in (27) are estimated similarly. Therefore,

$$\left\|\frac{\partial G(0,\tau,\varphi)}{\partial \varphi_i}\right\| \le (K_4 + K_5|\tau|)e^{-(\alpha-\omega)|\tau|} \le K_6 e^{-(\alpha_1-\omega)|\tau|}, \tag{28}$$

where $\alpha_1 = \alpha - \varepsilon$, ε is an arbitrarily small positive value, and $K_4, K_5, K_6 = K_6(\varepsilon)$ are positive constants independent of $\varphi \in \mathbb{T}_m$.

Differentiating (12) with respect to τ and taking (13) and (28) into account, we get

$$\frac{\partial G(t,\tau,\varphi)}{\partial \tau} = -A(\varphi \cdot t)G(t,\tau,\varphi) + \frac{\partial G(t,\tau,\varphi)}{\partial \varphi} a(\varphi \cdot \tau),$$

and

$$\left\|\frac{\partial G(0,\tau,\varphi))}{\partial \tau}\right\| \le \bar{K}_6 e^{-(\alpha_1-\omega)|\tau|}, \tag{29}$$

where $t, \tau \neq t_j(\varphi),\ j \in I(\varphi)$, and $\bar{K}_6 = \bar{K}_6(\varepsilon)$ is a positive constants independent of $\varphi \in \mathbb{T}_m$.

To estimate higher-order derivatives of $G(0,\tau,\varphi)$ and $P(\varphi)$ (up to the s-th order), we continue the above approach. Successively differentiating (27) and estimating the i-th derivative of the integrant by $\exp(-(\alpha_1 - i\omega)|s| - \alpha|\tau - s|)$ and the j-th terms in all series by $\exp(-(\alpha_1 - i\omega)|t_j| - \alpha|\tau - t_j|)$, we conclude that the integral and all series are convergent. Thus, we prove the existence of derivatives (up to the s-th order) of the projector $P(\varphi)$ and the Green function $G(t,\tau,\varphi)$ and estimate (15). The theorem is proved.

Remark. Differentiating (12) with respect to τ and using estimate (15), we get

$$\left\|\frac{\partial^s G(0,\tau,\varphi))}{\partial \tau^s}\right\| \le \bar{K}_7 e^{-(\alpha_1-s\omega)|\tau|}, \tag{30}$$

where $t, \tau \neq t_j(\varphi), j \in I(\varphi)$, and $\bar{K}_7 = \bar{K}_7(\varepsilon)$ is a positive constant independent of $\varphi \in \mathbb{T}_m$.

3. Perturbation theorem. Denote by $\mathcal{L}(\delta)$ the set of Lipschitz vectors or matrices $a(\varphi)$ on $\mathbb{T}_m$ such that $\|a(\varphi)\| \le \delta$ and $\operatorname{Lip} a \le \delta$, where $\operatorname{Lip} a = \inf\{\lambda > 0 :\ \|a(\varphi_1) - a(\varphi_2)\| \le \lambda\rho(\varphi_1,\varphi_2)\}$.

We consider a perturbed system

$$\frac{d\varphi}{dt} = a_0(\varphi) + \tilde{a}(\varphi), \tag{31}$$

$$\frac{dx}{dt} = (A_0(\varphi) + \tilde{A}(\varphi))x, \quad \varphi \in \mathbb{T}_m \setminus \Gamma, \tag{32}$$

$$\Delta x\Big|_{\varphi\in\Gamma} = (B_0(\varphi) + \tilde{B}(\varphi))x, \tag{33}$$

where $\tilde{a}(\varphi), \tilde{A}(\varphi), \tilde{B}(\varphi) \in C_{Lip}(\mathbb{T}_m)$. Using the properties of system (1) – (3), one can show that, for sufficiently small δ such that $\tilde{a}(\varphi) \in \mathcal{L}(\delta)$, solutions $\sigma(t,\varphi,\tilde{a})$ of equation (31) intersect manifold Γ transversally. Let $t_j(\varphi,\tilde{a}), j \in I(\varphi,\tilde{a})$, be the sequence of points where $\sigma(t,\varphi,\tilde{a})$ intersects Γ. Using the compactness of Γ and transversality of intersections of $\sigma(t,\varphi) =$

$= \sigma(t,\varphi,0)$ with Γ, we get the estimate $t_j(\varphi,\tilde a) - t_{j-1}(\varphi,\tilde a) \ge \tilde\theta > 0,\ j \in I(\varphi,\tilde a)$, with some positive $\tilde\theta$. Denote

$$\mathcal{A}(\delta) = \{(b_1(\varphi), b_2(\varphi), b_3(\varphi)) :\ b_i(\varphi) \in \mathcal{L}(\delta), i = 1,2,3\}. \tag{34}$$

Theorem 3. *Let system (4) – (6) be exponentially dichotomous. Then there exists a sufficiently small $\delta > 0$ such that system (31) – (33) with $(\tilde a(\varphi), \tilde A(\varphi), \tilde B(\varphi)) \in \mathcal{A}(\delta)$ has exponential dichotomy.*

To prove the theorem, we use ideas of [16, 17].

Denote $\mathcal{M} = \mathcal{M}(\delta) = \mathbb{T}_m \times \mathcal{A}(\delta)$. We define a flow on the set $\mathcal{M}(\delta)$:

$$p \cdot t = (\sigma(t,p), \hat a),\ \ t \in \mathbb{R},$$

where $p = (\varphi, \hat a) \in \mathcal{M}(\delta),\ \hat a = (\tilde a, \tilde A, \tilde B) \in \mathcal{A}(\delta)$, and $\sigma(t,p)$ is a solution of equation (31). Let $x(t,x_0,p)$ be a solution and let $\Phi(t,p)$ be the fundamental solution of system (31) – (33). The function $\Phi(t,p)$ has discontinuities of the first kind for $t = \bar t$ such that $\bar\varphi = \varphi \cdot \bar t \in \Gamma$, and, moreover,

$$\Phi(\bar t + 0, p) - \Phi(\bar t, p) = B(\bar\varphi)\Phi(\bar t, p).$$

We assume that $\Phi(t,p)$ and $x(t,x_0,p)$ are left-continuous with respect to t.

We define the following piecewise continuous linear skew-product semiflow on $\mathbb{R}^n \times \mathcal{M}(\delta)$:

$$\pi(t,x,p) = (\Phi(t,p)x, p\cdot t), \quad x \in \mathbb{R}^n,\ p \in \mathcal{M}(\delta),\ t \ge 0.$$

A point (x,p) is said to have a negative continuation with respect to π if there exists a piecewise continuous function $\phi : (-\infty, 0] \to \mathbb{R}^n \times \mathcal{M}$ that possesses the following properties:

1) $\phi(t) = (\phi^x(t), p\cdot t)$, where $\phi^x : (-\infty,0] \to \mathbb{R}^n$;

2) $\phi(0) = (x,p)$;

3) $\pi(t,\phi(s)) = \phi(s+t)$ for each $s \le 0$ and $0 \le t \le -s$;

4) $\pi(t,\phi(s)) = \pi(t+s,x,p)$ for each $0 \le -s \le t$.

We define the following sets:

$\Omega(x,p)$ is the set of ω-limit points of the trajectory $\pi(t,x,p)$,

$A(x,p,\phi)$ is the set of α-limit points of the negative continuation ϕ of the point (x,p),

$M = \{(x,p) :\ (x,p)$ has a negative continuation$\}$,

$U = \{(x,p) \in M :$ there is a negative continuation $\phi(t,x,p)$ of (x,p) such that $\|\phi(t,x,p)\| \to$ $\to 0,\ t \to -\infty\}$,

$\mathcal{B}^- = \{(x,\varphi)$: there is a bounded negative continuation $\phi(t,x,p)$ of (x,p), i.e., $\sup\limits_{t\le 0} \|\phi(t,x,p)\| < \infty\}$,

$\mathcal{B}^-_u = \{(x,p) :\ (x,p)$ has a unique bounded negative continuation$\}$,

$\mathcal{B}^+ = \{(x,p) : \ \sup\limits_{t\ge 0} \|\Phi(t,p)x\| \le \infty\}$.

$S = \{(x,p) : \ \ \|\Phi(t,p)x\| \to 0,\ t \to +\infty\}$.

$S(p) = \{x :\ (x,p) \in S\}, \quad U(p) = \{x :\ (x,p) \in U\}$,

$\mathcal{B} = \mathcal{B}^+ \cap \mathcal{B}^-$ is the bounded set of the semiflow π.

Lemma 1. *Let a point (x,p) have a bounded negative continuation $\phi(t,x,p)$ and $(\bar x, \bar p) \in$ $\in A(x,p,\phi),\ \bar p \notin \bar\Gamma$. Then $x(t,\bar x,\bar p) \in \mathcal{B}$.*

Proof. Denote $A_{\bar{p}}(x,p,\phi) = \{\bar{x} : (\bar{x},\bar{p}) \in A(x,p,\phi)\}$. We prove that

$$\pi(t, A_{\bar{p}}(x,p,\phi), \bar{p}) = (A_{\bar{p}\cdot t}(x,p,\phi), \bar{p}\cdot t) \tag{35}$$

for $\bar{p} \notin \bar{\Gamma},\ \bar{p}\cdot t \notin \bar{\Gamma},\ t \geq 0$. $A_{\bar{p}}(x,p,\phi)$ can be characterized as the collection of all points $(\bar{x},\bar{p})$ such that there exist sequences (x_n,p_n) and $t_n \to -\infty$ such that $p_n = p\cdot t_n \to \bar{p},\ x_n = \phi(t_n,x,p) \to \bar{x}$.

Let us fix $t > 0$ and set $\hat{x}_n = \Phi(t,p_n)x_n$ and $\hat{p}_n = p_n \cdot t$. The sequence $(\hat{x}_n,\hat{p}_n)$ is bounded. Choose a convergent subsequence so that $(\hat{x}_n,\hat{p}_n) \to (\hat{x},\hat{p}) = (\Phi(t,\bar{p})\bar{x}, \bar{p}\cdot t)$. On the other hand, $(\hat{x}_n,\hat{p}_n) = \pi(t,x_n,p_n) = \phi(t_n+t,x,p)$. Hence, $(\hat{x},\hat{p}) \in A_{\bar{p}\cdot t}(x,p,\phi)$.

To prove the inverse inclusion in (35), we consider $\hat{x} \in A_{\bar{p}\cdot t}(x,p,\phi)$ and sequences $(\hat{x}_n,\hat{p}_n)$ and $\hat{t}_n \to -\infty$ such that

$$\hat{p}_n = p\cdot \hat{t}_n \to \bar{p}\cdot t,\ \hat{x}_n = \phi(\hat{t}_n,x,p) \to \hat{x},\ n\to\infty.$$

The bounded sequence $\phi(\hat{t}_n - t,x,p)$ has a convergent subsequence such that $\phi(\hat{t}_n - t,x,p) \to \tilde{x},\ p\cdot(\hat{t}_n - t) \to \bar{p}$. Hence, $\tilde{x} \in A_{\bar{p}}(x,p,\phi)$. We have proved (35), i.e., π maps $A_{\bar{p}}(x,p,\phi)$ onto $A_{\bar{p}\cdot t}(x,p,\phi)$ and every $(\bar{x},\bar{p}) \in A_{\bar{p}}(x,p,\phi), \bar{p} \notin \bar{\Gamma}$ has a negative continuation. Clearly, $(\bar{x},\bar{p}) \in \mathcal{B}$.

Lemma 2. *Suppose that* $x(t,x,p) \in \mathcal{B}^+$ *and* $(\bar{x},\bar{p}) \in \Omega(x,p),\ \bar{p} \notin \bar{\Gamma}$; *then* $x(t,\bar{x},\bar{p}) \in \mathcal{B}$.

Proof. Denote $\Omega_{\bar{p}}(x,p) = \{\bar{x} : (\bar{x},\bar{p}) \in \Omega(x,p)\}$. By analogy with the proof of Lemma 1, we prove that

$$\pi(t, \Omega_{\bar{p}}(x,p), \bar{p}) = (\Omega_{\bar{p}\cdot t}(x,p,\phi), \bar{p}\cdot t) \tag{36}$$

for $\bar{p} \notin \bar{\Gamma},\ \bar{p}\cdot t \notin \bar{\Gamma},\ t \geq 0$. Then every point $(\bar{x},\bar{p}) \in \Omega(x,p),\ \bar{p} \notin \bar{\Gamma}$, has a negative continuation and $(\bar{x},\bar{p}) \in \mathcal{B}$.

Assumption. In the next lemmas, we assume that $\mathcal{B} = \{0\} \times \mathcal{M}$.

Lemma 3. *Let* $t_k \to -\infty$ *and let there exist continuations of points* (x_k,p_k) *on* $[t_k,0]$ *such that*

$$\|\phi(t,x_k,p_k)\| \leq M \quad for \quad t \in [t_k,0].$$

Assume that $(\bar{x},\bar{p}) = \lim_{k\to\infty}(x_k,p_k),\ \bar{p} \notin \bar{\Gamma}$; *then* $(\bar{x},\bar{p})$ *has a negative continuation and* $(\bar{x},\bar{p}) \in U$, *i.e.* $\|\phi(t,\bar{x},\bar{p})\| \to 0,\ t \to -\infty$.

Proof. The sequence $(\phi(t_1,x_k,p_k), p_k\cdot t_1),\ k = 1,2,...$, is bounded. Assume that there exist limits (otherwise, we consider subsequences)

$$x_k^1 = \phi(t_1,x_k,p_k) \to \bar{x}_1,\ \ p_k^1 = p_k\cdot t_1 \to \bar{p}_1,\ \ k\to\infty.$$

If $\bar{p}_1 \in \bar{\Gamma}$, we consider the sequence $t_1 + \varepsilon$ with sufficiently small $\varepsilon > 0$.

By the theorem on continuous dependence of solutions of impulsive system on parameters, we get

$$\pi(-t_1,\bar{x}_1,\bar{p}_1) = (\Phi(-t_1,\bar{p}_1)\bar{x}_1, \bar{p}_1\cdot(-t_1)) = \lim_{k\to\infty}\pi(-t_1,\phi(t_1,x_k,p_k)) =$$

$$= \lim_{k\to\infty}(\Phi(-t_1,p_k^1)x_k^1, p_k^1\cdot(-t_1)) = \lim_{k\to\infty}(x_k,p_k) = (\bar{x},\bar{p}).$$

Hence, the point $(\bar{x},\bar{p})$ has a continuation on $[t_1,0]$, which is bounded by a constant M.

Next, we consider the sequence $\phi(t_2 - t_1, x_k^1, p_k^1),\ p_k^1 \cdot (t_2 - t_1)),\ k = 2, 3, ...,$ and show analogously a continuability of the point $(\bar{x}, \bar{p})$ on $[t_2, 0]$, and so on. Hence, a continuation of the point $(\bar{x}, \bar{p})$ exists for $t \leq 0$ and is bounded.

By Lemma 1, the α-limit set of the point $(\bar{x}, \bar{p})$ belongs to $\mathcal{B}$. Since $\bar{B}$ is trivial, we get $x(t, \bar{x}, \bar{p}) \to 0,\ t \to -\infty$, i.e., $(\bar{x}, \bar{p}) \in U$.

Lemma 4. $S = \mathcal{B}^+$. *The set* S *is closed and there exist constants* $K \geq 1$ *and* $\beta > 0$ *such that, for all* $(x, p) \in S$, *one has*

$$\|\Phi(t, p)x\| \leq K e^{-\beta t}\|x\|, \quad t \geq 0. \tag{37}$$

If $p \in \bar{\Gamma}$, *then* $S(p)$ *is closed in* $p - 0$ *and* $p + 0$.

Proof. Let $(x_k, p_k) \in S$ and let $(x_k, p_k) \to (x, p),\ k \to \infty,\ p \notin \bar{\Gamma}$. If $x = 0$, then $(x, p) \in S$. If $x \neq 0$, we consider the solution $x(t, x, p)$. It is bounded. By Lemma 3, $\Omega(x, p) \in \mathcal{B}$. Using the triviality of $\mathcal{B}$, we have $x(t, x, p) \to 0,\ t \to \infty$, i.e., $x(t, x, p) \in S$.

If $p \in \bar{\Gamma}$, we consider positive and negative sequences $p_k \to p$ and prove analogously that $S(p)$ is closed in $p - 0$ and $p + 0$.

There exists $T > 0$ such that, for all $(x, p) \in S$, one has

$$\|\Phi(t, p)x\| \leq \frac{1}{2}\|x\| \quad \text{for} \quad t \geq T. \tag{38}$$

If this were not true, then there would exist $(x_k, p_k) \in S$ and $t_k \to \infty$ such that $\|\Phi(t_k, p_k)x_k\| \geq$ $\geq \frac{1}{2}\|x_k\|$. Let $\|x_k\| = 1$. Then $\|\Phi(t_k, p_k)x_k\| \geq \frac{1}{2}$. Denote $\hat{x}_k = \Phi(t_k, p_k)x_k,\ \hat{p}_k = p_k \cdot t_k$. These sequences are bounded. Therefore, there exists a convergent subsequence $(\hat{x}_k, \hat{p}_k) \to (\bar{x}, \bar{p})$. Let $\bar{p} \notin \bar{\Gamma}$. Then, by Lemma 4, $(\bar{x}, \bar{p}) \in S$. On the other hand, by Lemma 1, $(\bar{x}, \bar{p}) \in U$. Hence, $\|\bar{x}\| = 0$. This contradicts $\|\bar{x}\| \geq \frac{1}{2}$.

Let now $\bar{p} \in \bar{\Gamma}$. Assume that there exists an infinite subsequence t_{k_j} of the sequence t_k such that points $p_{k_j} \cdot t_{k_j}$ are located on the positive side of $\bar{\Gamma}$ obtained during the motion along the trajectories $p \cdot t$. We consider the subsequence $t_{k_j} + \varepsilon$ with sufficiently small $\varepsilon > 0$. Using the piecewise continuity of $\Phi(t, p)$, one has $\|\Phi(\varepsilon, p_{k_j} \cdot t_{k_j})x\| \geq \frac{\|x\|}{2}$, hence

$$\|\Phi(\varepsilon + t_{k_j}, p_{k_j})x_{k_j}\| \geq \|\Phi(\varepsilon, p_{k_j} \cdot t_{k_j})\Phi(t_{k_j}, p_{k_j})x_{k_j}\| \geq \frac{1}{4}\|x_{k_j}\| = \frac{1}{4}.$$

Taking boundedness into account, we conclude that, there exists a convergent subsequence $\Phi(\varepsilon + t_{k_j}, p_{k_j})x_{k_j} \to x^*,\ p_{k_j} \cdot (\varepsilon + t_{k_j}) \to p^*,\ k \to \infty$, and $p^* \notin \bar{\Gamma}$. By construction, $(x^*, p^*) \in S$; on the other hand, Lemma 1 implies that $(x^*, p^*) \in U$. Then $x^* = 0$, which contradicts $\|x^*\| \geq \frac{\nu}{4}$.

If a positive subsequence t_{k_j} does not exist, we choose another subsequence t_{k_l} such that points $p_{k_l} \cdot t_{k_l}$ are located on the negative side of $\bar{\Gamma}$. In this case, we consider the subsequence $t_{k_l} - \varepsilon$ with sufficiently small $\varepsilon > 0$ and arrive at a contradiction as before.

Define β and K as follows:

$$\beta = \frac{\ln 2}{T}, \quad K = 2 \sup\{\|\Phi(t, \varphi)x\| : (\varphi, x) \in S,\ \|x\| = 1,\ 0 \leq t \leq T\},$$

where T is given in (38). (37) is proved by induction analogously to [16, p. 51].

Lemma 5. *$U = \mathcal{B}_u^-$. If $(x,p) \in U$, the function $\Phi(t,p)x$ is well defined for all $t \le 0$. The set U is closed and there exist constants $K \ge 1$ and $\beta > 0$ such that, for all $(x,p) \in U$, one has*

$$\|\Phi(t,p)x\| \le Ke^{\beta t}\|x\|, \quad t \le 0. \tag{39}$$

If $p \in \bar{\Gamma}$, then $U(p)$ is closed in $p-0$ and $p+0$.

Proof. Let $(x_k, p_k) \in U$ and $(x_k, p_k) \to (x,p),\ p \notin \bar{\Gamma}$. If $x = 0$, then $(x,p) \in U$. Let $x \ne 0$. By Lemma 3, (x,p) has a negative continuation $\phi(t,x,p)$ such that $\phi(t,x,p) \to 0,\ t \to -\infty$, i.e., $(x,p) \in U$.

There exists $T < 0$ such that, for all $(x,p) \in U$, one has

$$\|\Phi(t,p)x\| \le \frac{1}{2}\|x\|, \quad t \in (-\infty, T). \tag{40}$$

If this were not true, then there would exist (x_k, p_k) and $t_k \to -\infty$ such that $\|\Phi(t_k,p_k)x_k\| \ge \ge \dfrac{\|x_k\|}{2}$. Choose $\|x_k\| = 1$; then $\|\Phi(t_k,p_k)x_k\| \ge 1/2$. The sequence $(\hat{x}_k, \hat{p}_k) = (\Phi(t_k,p_k)x_k, \varphi_k \cdot t_k)$ is bounded. Therefore, there exists a convergent subsequence $(\hat{x}_k, \hat{p}_k) \to (\hat{x}, \hat{p})$. Let $\hat{p} \notin \bar{\Gamma}$. Since U is closed, we have $(\hat{x}, \hat{p}) \in U$. On the other hand, the solutions $x(-t_k, \hat{x}_k, \hat{p}_k)$ are uniformly bounded and $-t_k \to +\infty$ as $k \to \infty$; therefore, $x(t, \hat{x}, \hat{p}) \in S$. Then $\|\hat{x}\| = 0$, which contradicts $\|\hat{x}\| \ge 1/2$. If $\hat{p} \in \bar{\Gamma}$, we consider the sequence $t_k + \varepsilon$ analogously to the proof of Lemma 4.

Define $\beta = -(\ln 2)/T$ and

$$K = \frac{1}{2}\sup\{\|\Phi(t,p)x\| :\ (x,p) \in U,\ \|x\| \le 1,\ t \in [T,0]\},$$

where T is given in (40). (39) is proved analogously to [16, p. 52].

Lemma 6. *For $p \in \mathcal{M}$, one has*

$$\dim U(\eta) \ge n - \dim S(p),$$

where $\eta \in \omega(p)$ ($\omega(p)$ is an ω-limit set of the trajectory $p \cdot t$).

Proof. Let $\mathcal{K}(p)$ be a subspace of $\mathbb{R}^n$ such that

$$\mathcal{K}(p) \cap S(p) = \{0\}, \quad \mathcal{K}(p) \oplus S(p) = \mathbb{R}^n. \tag{41}$$

Let $\{t_k\}$ be a sequence of positive numbers such that $t_k \to +\infty$. Denote

$$\mu_k = \min\{\|x(t_k,x,p)\| :\ x \in \mathcal{K}(p),\ \|x\| = 1\}.$$

Clearly, $\mu_k \to +\infty$ as $t_k \to +\infty$. Let $p \cdot t_k = p_k \to \eta \in \omega(p)$. Denote $K_k(p) = \Phi(t_k,p)\mathcal{K}(p)$. $\Phi(t_k,p)$ is a one-to-one mapping of $\mathcal{K}(p)$ onto the linear subspace $K_k(p)$. For any $x \in K_k(p)$ with $\|x\| \le 1$, one has $\|\Phi(-t_k,p_k)x\| \le \mu_k^{-1}$.

By definition, one has

$$\dim K_k(p) = \dim \mathcal{K}(p) = n - \dim S(p), \quad k \ge 0.$$

There exists a subsequence of t_k such that $K_k(p) \to K,\ k \to \infty$, and $\dim K = \dim \mathcal{K}(p)$. To prove that

$$K \subset U(\eta), \tag{42}$$

we consider a sequence $(x_k, p_k),\ \|x_k\| \leq 1,\ x_k \in K_k,\ p_k = p \cdot t_k$. Suppose that $x_k \to x',\ k \to \infty$. It suffices to prove that $(x', \eta) \in U(\eta)$.

For $x \in K_k, t \in [-t_k, 0]$, the trajectory $x(t, x, p_k)$ is well defined. There exists $M > 0$ such that

$$\sup_{-t_k \leq t \leq 0} \|x(t, x, p_k)\| \leq M \tag{43}$$

for $x \in K_k,\ \|x\| \leq 1,\ k = 1, 2, \ldots$. If this were not true, then there would exist sequences $x_k,\ \|x_k\| = 1,\ x_k \in K_k$, and $\beta_k \to \infty$ such that $\beta_k = \sup_{-t_k \leq t \leq 0} \|x(t, x_k, p_k)\|$. Denote $\tau_k \in [0, t_k]$ such that $\beta_k/2 \leq x(-\tau_k, x, p_k) \leq \beta_k$. Let us consider the sequence $(\xi_k, \eta_k) = (\beta_k^{-1} x(-\tau_k, x_k, p_k),\ p_k \cdot (-\tau_k))$. Obviously,

$$1/2 \leq \|\xi_k\| \leq 1, \tag{44}$$

and

$$\|x(\tau_k, \xi_k, \eta_k)\| = \beta_k^{-1} \|x_k\| \to 0, \quad k \to \infty,$$

$$\|x(-t_k + \tau_k, \xi_k, \eta_k)\| = \beta_k^{-1} \|\Phi(-t_k, p_k) x_k\| \leq \beta_k^{-1} \mu_k^{-1} \to 0,\ k \to \infty.$$

If $(\xi_k, \eta_k) \to (\bar{\xi}, \bar{\eta})$, then $(\bar{\xi}, \bar{\eta}) \in \mathcal{B}$, hence $\bar{\xi} = 0$. This contradicts (44). Therefore, (43) is valid. Using (43) and Lemma 1, one has $(x', \eta) \in U$. Hence, $\dim U(\eta) \geq \dim \mathcal{K}_k(p) = n - \dim S(p)$, which completes the proof of the lemma.

Lemma 7. *Let $p \in \mathcal{M}$. Then the semiflow π admits exponential dichotomy over the ω-limit set $\omega(p)$. The semiflow π admits exponential dichotomy over minimal sets of the flow $\varphi \cdot t$.*

Proof. Analogously to the proof of Lemma 1 in [18], we prove that, for each $p \in \mathcal{M}$, the function $\dim S(p \cdot t)$ is a nonincreasing function of t :

$$\dim S(p \cdot t) \leq \dim S(p \cdot \tau) \ \text{ for } \ t \geq \tau. \tag{45}$$

Inequality (45) implies that there exist limits

$$\lim_{t \to -\infty} \dim S(p \cdot t) = k_1, \ \ \lim_{t \to \infty} \dim S(p \cdot t) = k_2.$$

Taking into account the last limits and the fact that the space $\mathbb{R}^n$ is finite-dimensional, we get $\dim S(\eta) = k_2$ for all $\eta \in \omega(p)$. By Lemma 6, $\dim U(\eta) = n - k_2$. Therefore, the semiflow π admits exponential dichotomy over $\omega(p)$.

Proof of Theorem 3. The semiflow π admits exponential dichotomy over $\mathcal{M}(0)$; therefore, π has no nontrivial bounded solutions, i.e.,

$$\mathcal{B}_0 = \{0\} \times \mathcal{M}(0).$$

We shall show that there exists $\delta > 0$ such that the semiflow π does not have nontrivial bounded solutions over $\mathcal{M}(\delta)$. If this were not true, then there would exist a sequence $\{\delta_n\},\ \delta_n > 0,\ \delta_n \to 0,\ n \to +\infty$ and a sequence $\hat{a}_n(\varphi) \in \mathcal{A}(\delta_n)$, such that system (31) – (33) with $\hat{a}(\varphi) = \hat{a}_n(\varphi) = (\tilde{a}, \tilde{A}, \tilde{B})$ would have a nontrivial bounded solution $x_n(t, x_n^0, \varphi_n^0, \hat{a}_n),\ x_n(0, x_n^0, \varphi_n^0, \hat{a}_n) = x_n^0$. Denote

$$\beta_n = \sup_{t \in \mathbb{R}} \{\|x_n(t, x_n^0, \varphi_n^0, \hat{a}_n)\|\}.$$

Choose $t_n \in \mathbb{R}$ such that $\|x_n(t_n, x_n^0, \varphi_n^0, \hat{a}_n)\| \geq \dfrac{\beta_n}{2}$. Let

$$(\xi_n, \eta_n) = (\beta_n^{-1} x_n(t_n, x_n^0, \varphi_n^0, \hat{a}_n),\ \sigma(t_n, \varphi_n^0, \hat{a}_n)).$$

Then $\|\xi_n\| \geq \dfrac{1}{2}$ and $\|x_n(t, \xi_n^0, \eta_n^0, \hat{a}_n)\| \leq 1$ for all $t \in \mathbb{R}$. The sequence (ξ_n, η_n) is bounded. We choose a convergent subsequence so that $(\xi_n, \eta_n) \to (\xi, \eta)$. Without loss of generality, we may assume that $\eta \notin \bar{\Gamma}$. The point (ξ, η) has the following properties: $\eta \in \mathcal{M}(0),\ \|\xi\| \geq \dfrac{1}{2}$. By Lemma 3, the solution $x(t, \xi, \eta, 0)$ has a negative continuation and $\|x(t, \xi, \eta, 0)\| \leq 1,\ t \in \mathbb{R}$. This contradicts the triviality of $\mathcal{B}_0$. Hence, there exists $\delta_0 > 0$ such that the semiflow π does not have nontrivial solutions over $\mathcal{M}(\delta_0)$.

Let us consider the set

$$\Theta_k = \{p = (\varphi, \tilde{a}) \in \mathcal{M}(\delta_0) :\ \dim S(p) = k,\ \dim U(p) = n - k\}.$$

The set Θ_k is closed for $p \notin \bar{\Gamma}$ and closed in $p - 0$ and $p + 0$ if $p \in \bar{\Gamma}$. Therefore, for $p \in \mathcal{M}(\delta_0)$, there exists a compact neighborhood of $\Theta_k(p) = \{x :\ (x, p) \in (\Theta_k)$ that is disjoint with the other sets $\Theta_j,\ j \neq k$. Since the compact set $\mathcal{M}(0)$ belongs to Θ_k, one can see that, for some $\delta_1 \leq \delta_0$, the set $\mathcal{M}(\delta_1)$ is disjoint with the other sets $\Theta_j,\ j \neq k$.

We show that $\dim S(p) = k$ for all $p \in \mathcal{M}(\delta_1)$. Let p_0 be a point such that $\dim S(p_0) < k$ (sign "$<$"is chosen for definiteness). The function $\dim S(p \cdot t)$ is nonincreasing; therefore, one has $\dim S(\eta) = k_1 < k$ for all $\eta \in \alpha(p)$ ($\alpha(p)$ is the set of α-limit points of the trajectory $p \cdot t$.) By Lemma 7, the semiflow π is exponentially dichotomous over the minimal set A_0 contained in $\alpha(p)$. Moreover, $\dim S(\xi) = k_1,\ \xi \in A_0$. Hence, $A_0 \subset \Theta_{k_1}$, contrary to the fact that $\mathcal{M}(\delta_1)$ contains only the set Θ_k. We have proved that the semiflow π is exponentially dichotomous over the set $\mathcal{M}(\delta_1)$, and, correspondingly, system (31) – (33) is exponentially dichotomous for $(\tilde{a}, \tilde{A}, \tilde{B}) \in \mathcal{A}(\delta_1)$. The theorem is proved.

Now we consider a linearized system with small parameter

$$\frac{d\varphi}{dt} = a_0(\varphi) + \varepsilon a_1(\varphi, \varepsilon), \tag{46}$$

$$\frac{dx}{dt} = (A_0(\varphi) + \varepsilon A_1(\varphi, \varepsilon))x, \quad \varphi \in \mathbb{T}_m \setminus \Gamma, \tag{47}$$

$$\Delta x\Big|_{\varphi \in \Gamma} = (B_0(\varphi) + \varepsilon B_1(\varphi, \varepsilon))x, \tag{48}$$

where $\varepsilon \in (-\varepsilon_0, \varepsilon_0),\ \varepsilon_0 > 0$.

Theorem 4. *Suppose that the following conditions are satisfied:*

1) for $\varepsilon = 0$, system (46) – (48) satisfies the conditions of Theorem 2;

2) functions a_1, A_1, and B_1 have continuous partial derivatives with respect to φ, ε up to the order s inclusively for $\varphi \in \mathbb{T}_m \setminus \Gamma$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, and all their partial derivatives have continuous continuations to the left-hand and right-hand sides of the manifold Γ and have discontinuities of the first kind for $\varphi \in \Gamma$.

Then there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that, for $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, system (46) – (48) is exponentially dichotomous with the projector $P(\varphi, \varepsilon)$ and the Green function $G(t, \tau, \varphi, \varepsilon)$ which have *continuous partial derivatives with respect to* φ, ε *up to the* s*-th order inclusively on the set* $\varphi \in \mathbb{T}_m \setminus \Gamma,\ \varepsilon \in (-\varepsilon_1, \varepsilon_1)$, *and, moreover,*

$$\left\| \frac{\partial^{|j|} G(t,\tau,\varphi,\varepsilon)}{\partial \varphi^{\bar{j}} \partial \varepsilon^{j_{m+1}}} \right\| \leq \bar{K}_j \exp(-(\alpha_1 - |j|\omega)|t-\tau|), \tag{49}$$

where $j = (\bar{j}, j_{m+1}) = (j_1, ..., j_m, j_{m+1}),\ |j| = j_1 + ... + j_{m+1},\ |j| \leq s,\ \varphi^j = \varphi_1^{j_1}, ..., \varphi_m^{j_m},\ \alpha_1 = \alpha - \nu,\ \nu$ *is an arbitrarily small positive value,* $\bar{K}_j = \bar{K}_j(\nu)$ *is a constant independent of* $\varphi \in \mathbb{T}_m$, *and* $\omega = \|\partial a(\varphi)/\partial\varphi\|_0$.

Proof. By Theorem 3, there exists $\varepsilon_1 > 0$ such that system (46) – (48) has exponential dichotomy for $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$. Instead of equation (46) we consider the equations

$$\frac{d\varphi}{dt} = a_0(\varphi) + \varepsilon a_1(\varphi, \varepsilon), \quad \frac{d\varepsilon}{dt} = 0 \tag{50}$$

defined on the product $\mathbb{T}_m \times (-\varepsilon_1, \varepsilon_1)$. The impulsive system (50), (47), (48) is defined on the direct product of the manifold $\mathbb{T}_m \times (-\varepsilon_1, \varepsilon_1)$ and Euclidean space $\mathbb{R}^n$ and subjected to the impulsive action on the submanifold $\bar{\Gamma} = \Gamma \times (-\varepsilon_1, \varepsilon_1)$ of codimension 1 of the manifold $\mathbb{T}_m \times (-\varepsilon_1, \varepsilon_1)$. System (50), (47), (48) satisfies conditions of Theorem 2. Hence, the projector $P(\varphi, \varepsilon)$ and the Green function $G(t, \tau, \varphi, \varepsilon)$ have continuous partial derivatives with respect to φ, ε up to the s-th order inclusively on the set $\varphi \in \mathbb{T}_m \setminus \Gamma,\ \varepsilon \in (-\varepsilon_1, \varepsilon_1)$. The theorem is proved.

4. Integral set. In this section, by $C^s_\Gamma(\mathbb{T}_m \times (-\varepsilon_0, \varepsilon_0))$ we denote the space of functions or matrices $a(\varphi, \varepsilon)$ with the following properties:

i) $a(\varphi, \varepsilon)$ has continuous partial derivatives with respect to φ, ε up to the order s inclusively for $\varphi \in \mathbb{T}_m \setminus \Gamma,\ \varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

ii) all partial derivatives of $a(\varphi, \varepsilon)$ have continuous continuations to the left-hand and right-hand sides of the manifold $\Gamma \times (-\varepsilon_0, \varepsilon_0)$.

For $f(\varphi, \varepsilon) \in C^s_\Gamma(\mathbb{T}_m \times (-\varepsilon_0, \varepsilon_0)$, we denote

$$\|f(\varphi, \varepsilon)\|_s = \max_{0 \leq |j| \leq s} \sup_{\varphi \in \mathbb{T}_m \setminus \Gamma} \sup_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)} \left\| \frac{\partial^{|j|} f(\varphi, \varepsilon)}{\partial \varphi^{\bar{j}} \partial \varepsilon^{j_{m+1}}} \right\|,$$

where $j = (\bar{j}, j_{m+1}) = (j_1, ..., j_m, j_{m+1}),\ \varphi^j = (\varphi_1^{j_1} ... \varphi_m^{j_m}),\ |j| = j_1 + ... + j_{m+1}$.

Theorem 5. *Assume that system (4) – (6) is exponentially dichotomous and the right-hand sides of system (1) – (3) have continuous partial derivatives with respect to* x, φ, ε *up to the* s*-th* ($s \geq 1$) *order inclusively, where*

$$(x, \varphi, \varepsilon) \in \mathcal{O} = \{\|x\| \leq d,\ \varphi \in \mathbb{T}_m,\ \varepsilon \in (-\varepsilon_0, \varepsilon_0)\}. \tag{51}$$

Then there exists $\varepsilon' \in (0, \varepsilon_0]$ *such that, for each* $\varepsilon \in (-\varepsilon', \varepsilon')$, *system (1) – (3) has a unique integral manifold* $x = u(\varphi, \varepsilon), \varphi \in \mathbb{T}_m$, *where* $u(\varphi, \varepsilon)$ *have continuous partial derivatives with respect to* φ, ε *up to the* $(s-1)$*-th* ($s \geq 1$) *order inclusively for* $\varphi \in \mathbb{T}_m \setminus \Gamma,\ \varepsilon \in (-\varepsilon', \varepsilon)$ *and has the discontinuities of the first kind for* $\varphi \in \Gamma$.

Proof. We rewrite system (1) – (3) in the form

$$\frac{d\varphi}{dt} = a_0(\varphi) + b(\varphi, x, \varepsilon), \tag{52}$$

$$\frac{dx}{dt} = (A_0(\varphi) + A_1(\varphi, x, \varepsilon))x + f(\varphi, \varepsilon), \quad \varphi \in \mathbb{T}_m \setminus \Gamma, \tag{53}$$

$$\Delta x\Big|_{\varphi\in\Gamma} = (B_0(\varphi) + B_1(\varphi, x, \varepsilon))x + g(\varphi, \varepsilon), \tag{54}$$

where $f(\varphi, 0) = g(\varphi, 0) = 0$, $b(\varphi, x, 0) = O(\|x\|)$, $A_1(\varphi, x, 0) = O(\|x\|)$, and $B_1(\varphi, x, 0) = O(\|x\|)$. We construct the sequence of sets

$$\{x = u_k(\varphi, \varepsilon) : \mathbb{T}_m \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n\}, \quad k = 0, 1, ...$$

where $u_0(\varphi, \varepsilon) \equiv 0$ and $u_{k+1}(\varphi, \varepsilon)$ is an invariant set of the system

$$\frac{d\varphi}{dt} = a_0(\varphi) + b(\varphi, u_k(\varphi, \varepsilon), \varepsilon), \tag{55}$$

$$\frac{dx}{dt} = (A_0(\varphi) + A_1(\varphi, u_k(\varphi, \varepsilon), \varepsilon))x + f(\varphi, \varepsilon), \quad \varphi \in \mathbb{T}_m \setminus \Gamma, \tag{56}$$

$$\Delta x\Big|_{\varphi\in\Gamma} = (B_0(\varphi) + B_1(\varphi, u_k(\varphi, \varepsilon), \varepsilon))x + g(\varphi, \varepsilon). \tag{57}$$

There exists $\varepsilon_1 > 0$ such that $\|b(\varphi, 0, \varepsilon)\|_s \le \delta, \|A_1(\varphi, 0, \varepsilon)\|_s \le \delta, \|B_1(\varphi, 0, \varepsilon)\|_s \le \delta$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and the constant $\delta > 0$ defined in Theorem 3. By Theorem 4, the linearized system

$$\frac{d\varphi}{dt} = a_0(\varphi) + b(\varphi, 0, \varepsilon), \tag{58}$$

$$\frac{dx}{dt} = (A_0(\varphi) + A_1(\varphi, 0, \varepsilon))x, \quad \varphi \in \mathbb{T}_m \setminus \Gamma, \tag{59}$$

$$\Delta x\Big|_{\varphi\in\Gamma} = (B_0(\varphi) + B_1(\varphi, 0, \varepsilon))x, \tag{60}$$

is exponentially dichotomous with a piecewise smooth projector $P_1(\varphi, \varepsilon)$ and the Green function $G_1(t, \tau, \varphi, \varepsilon)$ satisfying estimate (49).

The function $u_1(\varphi, \varepsilon)$ is defined by the formula

$$u_1(\varphi, \varepsilon) = \int\limits_{-\infty}^{\infty} G_1(0, \tau, \varphi, \varepsilon) f(\sigma_1(\tau, \varphi, \varepsilon), \varepsilon) d\tau +$$

$$+ \sum_{j \in I_1(\varphi,\varepsilon)} G_1(0, t_j^1(\varphi, \varepsilon), \varphi, \varepsilon) g(\sigma_j^1(\varphi, \varepsilon), \varepsilon), \tag{61}$$

where $\sigma_1(t, \varphi, \varepsilon)$ is the solution of equation (58), $t_j^1(\varphi, \varepsilon)$, $j \in I_1(\varphi, \varepsilon)$ are points where $\sigma_1(t, \varphi, \varepsilon)$ intersects the manifold Γ, and $\varphi_j^1(\varphi, \varepsilon) = \sigma(t_j^1(\varphi, \varepsilon), \varphi, \varepsilon)$.

In view of the smoothness of the functions $t_j^1(\varphi, \varepsilon)$ and the piecewise smoothness of the Green function $G_1(0, \tau, \varphi, \varepsilon)$, the function $u_1(\varphi, \varepsilon)$ has continuous partial derivatives with respect to φ, ε up to the order s inclusively for $\varphi \in \mathbb{T}_m \setminus \Gamma$ and $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ and the functions $\partial^{|j|} u(\varphi, \varepsilon)/\partial\varphi^{\bar{j}} \partial\varepsilon^{j_{m+1}}$, $j = (\bar{j}, j_{m+1}) = (j_1, ..., j_{m+1})$, $1 \le |j| \le s$, have discontinuities of the first kind for $\varphi \in \Gamma$. Differentiating formula (61) s times and taking into account (23) and (49), we get

$$\|u_1(\varphi,\varepsilon)\|_s \leq C_1(\|f(\varphi,\varepsilon)\|_s + \|g(\varphi,\varepsilon)\|_s), \tag{62}$$

where C_1 is some positive constant.

Choose $\varepsilon_2 > 0$ such that, for $\varepsilon \in (-\varepsilon_2, \varepsilon_2)$ and $u(\varphi,\varepsilon) \in C^s_\Gamma(\mathbb{T}_m \times (-\varepsilon_2,\varepsilon_2)$ satisfying (62), the following inequalities are valid:

$$\|b(\varphi,u(\varphi,\varepsilon),\varepsilon)\|_s \leq \delta,\ \|A_1(\varphi,u(\varphi,\varepsilon),\varepsilon)\|_s \leq \delta,\ \|B_1(\varphi,u(\varphi,\varepsilon),\varepsilon)\|_s \leq \delta,$$

where the constant $\delta > 0$ is defined in Theorem 3.

Let a piecewise smooth function $u_p(\varphi,\varepsilon)$, $\varphi \in \mathbb{T}_m, \varepsilon \in (-\varepsilon_2,\varepsilon_2)$, be an invariant set of system (55) – (57) for $k = p$. We assume that $u_p(\varphi,\varepsilon)$ belongs to $C^s_\Gamma(\mathbb{T}_m) \times (-\varepsilon_2,\varepsilon_2)$ and satisfies inequality (62). Then the linearized system (55) – (57) (if $f = g \equiv 0$) is exponentially dichotomous with the projector $P_p(\varphi,\varepsilon)$ and the Green function $G_p(t,\tau,\varphi,\varepsilon)$ satisfying estimate (15). We define the invariant set $u_{p+1}(\varphi,\varepsilon)$ by the formula

$$u_{p+1}(\varphi,\varepsilon) = \int\limits_{-\infty}^{\infty} G_p(0,\tau,\varphi,\varepsilon) f(\sigma_p(\tau,\varphi,\varepsilon),\varepsilon)\,d\tau +$$

$$+ \sum_{j \in I_p(\varphi,\varepsilon)} G_p(0, t^p_j(\varphi,\varepsilon),\varphi,\varepsilon) g(\sigma^p_j(\varphi^p_j(\varphi,\varepsilon),\varepsilon),\varepsilon), \tag{63}$$

where $\sigma_p(\tau,\varphi,\varepsilon)$ is the solution of (55), $t^p_j(\varphi,\varepsilon), j \in I_p(\varphi,\varepsilon)$, are the points of intersections of $\sigma_p(\tau,\varphi,\varepsilon)$ with the manifold Γ, and $\sigma^p_j(\varphi,\varepsilon) = \sigma_p(t^p_j(\varphi,\varepsilon),\varphi,\varepsilon)$.

Differentiating (63) and and taking into account (23) and (49), we get

$$\|u_{p+1}(\varphi,\varepsilon)\|_s \leq C_1(\|f(\varphi,\varepsilon)\|_s + \|g(\varphi,\varepsilon)\|_s). \tag{64}$$

Hence, we obtain the uniform boundedness of the sequence $\{u_n(\varphi,\varepsilon)\}$ for $\varphi \in \mathbb{T}_m, \varepsilon \in \in (-\varepsilon_2,\varepsilon_2)$.

Define $w_{k+1}(\varphi,\varepsilon) = u_{k+1}(\varphi,\varepsilon) - u_k(\varphi,\varepsilon)$. Since the functions $u_k(\varphi,\varepsilon)$ are smooth for $\varphi \in \in \mathbb{T}_m \setminus \Gamma, \varepsilon \in (-\varepsilon_2,\varepsilon_2)$ and have discontinuities of the first kind for $\varphi \in \Gamma$, the function $w_{k+1}(\varphi,\varepsilon)$ satisfies the following equation:

$$\frac{\partial w_{k+1}}{\partial \varphi}(a_0(\varphi) + b(\varphi,u_k,\varepsilon)) + \frac{\partial u_k}{\partial \varphi}(b(\varphi,u_k,\varepsilon) - b(\varphi,u_{k-1},\varepsilon)) =$$

$$= (A_0(\varphi) + A_1(\varphi,u_k,\varepsilon))w_{k+1} + (A_1(\varphi,u_k,\varepsilon) -$$

$$- A_1(\varphi,u_{k-1},\varepsilon))u_k,\ \ \varphi \in \mathbb{T}_m \setminus \Gamma,$$

$$\Delta w_{k+1}\Big|_{\varphi \in \Gamma} = (B_0(\varphi) + B_1(\varphi,u_k,\varepsilon))w_{k+1} +$$

$$+ (B_1(\varphi,u_k,\varepsilon) - B_1(\varphi,u_{k-1},\varepsilon))u_k.$$

Hence, $w_{k+1}(\varphi,\varepsilon)$ determines an invariant set of system (55) – (57) with $f(\varphi,\varepsilon) = (A_1(\varphi,u_k,\varepsilon) - A_1(\varphi,u_{k-1},\varepsilon))u_k - (\partial u_k/\partial\varphi)(b(\varphi,u_k,\varepsilon) - b(\varphi,u_{k-1},\varepsilon)), g(\varphi,\varepsilon) = (B_1(\varphi,u_k,\varepsilon) - B_1(\varphi,u_{k-1},\varepsilon))u_k$. We can express $w_{k+1}(\varphi,\varepsilon)$ in the form

$$w_{k+1}(\varphi,\varepsilon) = \int\limits_{-\infty}^{\infty} G_k(0,\tau,\varphi,\varepsilon)\Bigg[\bigg(A_1(\sigma_k(\tau,\varphi,\varepsilon),u_k(\sigma_k(\tau,\varphi,\varepsilon),\varepsilon),\varepsilon)-$$

$$- A_1(\sigma_k(\tau,\varphi,\varepsilon),u_{k-1}(\sigma_k(\tau,\varphi,\varepsilon),\varepsilon),\varepsilon)\bigg)u_k(\sigma_k(\tau,\varphi,\varepsilon),\varepsilon)-$$

$$-\frac{\partial u_k(\sigma_k(\tau,\varepsilon),\varepsilon)}{\partial\varphi}\bigg(b(\sigma_k(\tau,\varphi,\varepsilon),u_k(\sigma_k(\tau,\varphi,\varepsilon),\varepsilon),\varepsilon)-$$

$$- b(\sigma_k(\tau,\varphi,\varepsilon),u_{k-1}(\sigma_k(\tau,\varphi,\varepsilon),\varepsilon),\varepsilon)\bigg)\Bigg]d\tau+$$

$$+\sum_{j\in I_k(\varphi,\varepsilon)} G_k(0,t_j^k,\varphi,\varepsilon)(B_1(\sigma_j^k,u_k(\sigma_j^k,\varepsilon),\varepsilon) - B_1(\sigma_j^k,u_{k-1}(\sigma_j^k,\varepsilon),\varepsilon)). \tag{65}$$

By (65), we have

$$\|w_{k+1}(\varphi,\varepsilon)\|_0 \le C_0(\varepsilon)\|w_k(\varphi,\varepsilon)\|_0,$$

where $C_0(\varepsilon) \to 0,\ \varepsilon \to 0$. There exists $\varepsilon' \in [0,\varepsilon_2]$ such that $C_0(\varepsilon) \le \rho_0 < 1$ for $\varepsilon \in (-\varepsilon',\varepsilon')$. Then

$$\|w_{k+1}(\varphi,\varepsilon)\|_0 \le \rho_0^{k-1}\|u_1(\varphi,\varepsilon)\|_0 \le \rho_0^{k-1}C_1(\|f(\varphi,\varepsilon)\|_0 + \|g(\varphi,\varepsilon)\|_0). \tag{66}$$

Inequality (66) proves the convergence of the sequence $\{u_p(\varphi,\varepsilon), p = 0,1,...\}$ in the space $C_\Gamma(\mathbb{T}_m \times (-\varepsilon',\varepsilon'))$. To show that $u(\varphi,\varepsilon) \in C_\Gamma^{s-1}(\mathbb{T}_m \times (-\varepsilon',\varepsilon'))$, we use the uniform boundedness and equicontinuity of the sequence $D^r u_k(\varphi,\varepsilon), k = 0,1,...,r \le s$, which follow from estimate (64). By the Arzela lemma, any infinite subsequence of $D^r u_k(\varphi,\varepsilon), k = 0,1,...$, uniformly converges to some function $v^{(r)}(\varphi,\varepsilon)$. This, with the use of the limit $\lim\limits_{k\to\infty} u_k(\varphi,\varepsilon) = u(\varphi,\varepsilon)$, proves that $D^r u(\varphi,\varepsilon) = v^{(r)}(\varphi,\varepsilon)$ and $u(\varphi,\varepsilon) \in C_\Gamma^{s-1}(\mathbb{T}_m \times (-\varepsilon',\varepsilon'))$. The theorem is proved.

1. *Tkachenko V.I.* On exponential dichotomy and invariant sets of impulsive systems // Extended Abstrs Fourth Int. Conf. Different Equat. and Appl. (Poznan, Poland). — 1998. — P. 341 – 344.

2. *Bogolyubov N.N., Mitropolskij Yu.A.* The method of integral manifolds in nonlinear mechanics // Proc. Int. Symp. Anal. Methods in Theory Nonlinear Vibrations (1961). — Kiev: Izdat. Akad. Nauk Ukr. SSR, 1963. — **1.** — P. 93 – 154.

3. *Hale J.K.* Integral manifolds of perturbed differential systems // Ann. Math. — 1961. — **73**. — P. 496 – 531.

4. *Mitropol'skii Yu.A., Lykova O.B.* Integral manifolds in nonlinear mechanics. — Moscow: Nauka, 1973. — 512 p.

5. *Samoilenko A.M.* Elements of mathematical theory of multi-frequency oscillations. — Dordrecht: Kluwer Acad. Publ., 1991. — 313 p.

6. *Yi Y.* A generalized integral manifold theorem // J. Different. Equat. — 1993. — **102.** — P. 153 – 187.

7. *Akhmetov M.U.* Quasiperiodic solutions of impulsive systems // Asymptotic Methods in Equations of Math. Phys. — Kiev: Inst. Math. Acad. Sci. Ukr. SSR, 1989. — P. 12 – 18.

8. *Samoilenko A.M., Perestyuk N.A.* Impulsive differential equations // World Sci. Ser. on Nonlinear Sci. Ser. A. — 1995. — **14**. — 462 p.

9. *Samoilenko A.M., Teplinskii Yu.V., Tsyganovskii N.S.* Behavior of solutions of a quasilinear system of differential equations with impulses in the neighbourhood of an invariant torus // Different. Uravneniya. — 1982. — **18**. — P. 833 – 839.

10. *Tkachenko V.I.* The Green function and conditions for the existence of invariant set of impulsive systems // Ukr. Math. J. — 1989. — **41**, № 10. — P. 1187 – 1190.

11. *Tkachenko V.I.* On the existence of a piecewise-smooth invariant torus of an impulsive system // Method for Investigating Differential and Functional-Differential Equations. — Kiev: Inst. Mat. Acad. Sci. Ukr. SSR, 1990. — P. 91 – 96.

12. *Tkachenko V.I.* On multi-frequency systems with impulses // Nonlinear Oscillations. — 1998. — № 1. — P. 107 – 116.

13. *Trofimchuk S.I.* Impulsive linear extensions // Asymptotic Solutions of Nonlinear Equations with a small Parameter. — Kiev: Inst. Math. Acad. Sci. Ukr. SSR, 1991. — P. 124 – 131.

14. *Lakshmikantham V., Bainov D., Simeonov P.S.* Theory of impulsive differential equatioms. — Singapore: World Sci., 1989. — 273 p.

15. *Sacker R. J., Sell G.R.* Existence of dichotomies and invariant splittings for linear differential systems, I // J. Different. Equat. — 1974. — **15**. — P. 429 – 458.

16. *Sacker R.J., Sell G.R.* Dichotomies for linear evolutionary equations in Banach spaces // Ibid. — 1994. — **113**. — P. 17 – 67.

17. *Sacker R.J., Sell G.R.* A spectral theory for linear differential systems // Ibid. — 1978. — **27**. — P. 320 – 358.

18. *Tkachenko V.I.* On exponential dichotomy of pulse evolution systems // Ukr. Math. J. — 1994. — **46**, № 4. — P. 441 – 448.

Received 15.07.99