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THE GENERALIZED DE RHAM – HODGE THEORY OF DELSARTE TRANSMUTATION OPERATORS IN MULTIDIMENSION AND ITS APPLICATIONS

УЗАГАЛЬНЕНА ТЕОРІЯ ДЕ РАМА – ХОДЖА ДЛЯ ТРАНСМУТАЦІЙНИХ ОПЕРАТОРІВ ДЕЛЬСАРТА У БАГАТОВИМІРНОМУ ВИПАДКУ ТА ЇЇ ЗАСТОСУВАННЯ

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A study of spectral and differential-geometric properties of Delsarte transmutation operators is given. Their differential geometrical and topological structure in multidimension is analyzed, the relationships with the generalized De Rham – Hodge theory of generalized differential complexes are stated. Some applications to integrable dynamical systems in multidimension are presented.

Вивчено спектральні та диференціально-геометричні властивості трансмутаційних операторів Дельсарта. Проведено аналіз їх диференціально-геометричної та топологічної структур у багатовимірному випадку. Встановлено зв'язок з узагальненою теорією Де Рама – Ходжа узагальнених диференціальних комплексів. Наведено деякі застосування до теорії інтегровних динамічних систем у багатовимірному випадку.

1. Introduction. Consider the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m; \mathbb{C}^N)$, $m, N \in \mathbb{Z}_+$, with the scalar semilinear form on $\mathcal{H}^* \times \mathcal{H}$,

$$(\varphi, \psi) := \int\limits_{\mathbb{R}^m} \overline{\varphi}(x)^{\top} \psi(x) dx \tag{1.1}$$

for any pair $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, where $\mathcal{H}^* \simeq \mathcal{H}$, the sign "$\top$"is the usual matrix transposition. Take also $\mathcal{H}_0$ and $\widetilde{\mathcal{H}}_0$ to be some two closed subspaces of $\mathcal{H}$ and, correspondingly, two linear operators L and $\widetilde{L}$ acting from $\mathcal{H}$ into $\mathcal{H}$.

Definition 1.1 (J. Delsarte and J. Lions [1]). *A linear invertible operator $\mathbf{\Omega}$ defined on the whole $\mathcal{H}$ and acting from $\mathcal{H}_0$ onto $\widetilde{\mathcal{H}}_0$ is called a Delsarte transmutation operator for a pair of linear operators $\widetilde{\mathrm{L}}$ and $\mathrm{L} : \mathcal{H} \to \mathcal{H}$ if the following two conditions hold:*

the operator $\mathbf{\Omega}$ and its inverse $\mathbf{\Omega}^{-1}$ are continuos in $\mathcal{H}$;

the operator identity

$$\widetilde{\mathrm{L}}\mathbf{\Omega} = \mathbf{\Omega}\widetilde{\mathrm{L}} \tag{1.2}$$

is satisfied.

Such transmutation operators were introduced for the first time in [1, 2] for the case of one-dimensional second order differential operators. In particular, for the Sturm–Liouville and Dirac operators, the complete structure of the corresponding Delsarte transmutation operators was described in [3, 4], where extensive applications to spectral theory were also given.

It become apparent just recently, that some special cases of the Delsarte transmutation operators were constructed long before by Darboux and Crum (see [5]). A special generalization of the Delsarte-operators for the two-dimensional Dirac operators was done for the first time in [6], where its applications to the inverse spectral theory and a solution of some nonlinear two-dimensional evolution equations were also presented.

Recently some progress in this direction was made in [7–9] due to analyzing a special operator structure of Darboux type transformations which appeared in [5].

In this work we give, in some sense, a complete description of multidimensional Delsarte transmutation operators based on a natural generalization of the differential-geometric approach originated in [8, 9], and discuss how one can apply these operators to a study of spectral properties of linear multidimensional differential operators.

2. The differential-geometric structure of the generalized Lagrangian identity. Let a multidimensional linear differential operator $\mathrm{L}:\mathcal{H}\to\mathcal{H}$ of order $n(\mathrm{L})\in\mathbb{Z}_+$ be of the form

$$\mathrm{L}(x|\partial) := \sum_{|\alpha|=0}^{n(\mathrm{L})} a_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{2.1}$$

and defined on a dense domain $D(\mathrm{L})\subset\mathcal{H}$, where, as usual, $\alpha\in\mathbb{Z}_+^m$ is a multiindex, $x\in\mathbb{R}^m$, and for brevity one assumes that the coefficients $a_\alpha\in\mathcal{S}(\mathbb{R}^m;\mathrm{End}\mathbb{C}^N)$, $\alpha\in\mathbb{Z}_+^m$. Consider the following easily derivable generalized Lagrangian identity for the differential expression (2.1):

$$< \mathrm{L}^*\varphi,\psi > - < \varphi,\mathrm{L}\psi >= \sum_{i=1}^{m}(-1)^{i+1}\frac{\partial}{\partial x_i}Z_i[\varphi,\psi], \tag{2.2}$$

where $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$, the mappings $Z_i:\mathcal{H}^*\times\mathcal{H}\to\mathbb{C}$, $i=\overline{1,m}$, are semilinear due to the construction and $\mathrm{L}^*:\mathcal{H}^*\to\mathcal{H}^*$ is the corresponding differential expression formally conjugate to (2.1), that is,

$$\mathrm{L}^*(x|\partial) := \sum_{|\alpha|=0}^{n(\mathrm{L})}(-1)^{|\alpha|}\frac{\partial^{|\alpha|}}{\partial x^\alpha}\cdot\bar{a}_\alpha^{\intercal}(x).$$

Multiplying identity (2.2) by the usual oriented Lebesgue measure $dx=\wedge_{j=\overrightarrow{1,m}}dx_j$, we get that

$$< \mathrm{L}^*\varphi,\psi > dx- < \varphi,\mathrm{L}\psi > dx = dZ^{(m-1)}[\varphi,\psi] \tag{2.3}$$

for all $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, where

$$Z^{(m-1)}[\varphi, \psi] := \sum_{i=1}^{m} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{i-1} \wedge Z_i[\varphi, \psi] dx_{i+1} \wedge \ldots \wedge dx_m \tag{2.4}$$

is an $(m-1)$-differential form on $\mathbb{R}^m$.

Consider now all pairs $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0 \subset \mathcal{H}_- \times \mathcal{H}_-$, $\lambda, \mu \in \Sigma$, where

$$D(\mathrm{L}) \subset \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \tag{2.5}$$

is the usual Gelfand triple of Hilbert spaces [10, 11] related with our Hilbert–Schmidt rigged Hilbert space $\mathcal{H}$, $\Sigma \in \mathbb{C}^p$, $p \in \mathbb{Z}_+$, is some fixed measurable space of parameters endowed with a finite Borel measure ρ, such that the differential form (2.4) is exact, that is, there exists a set of $(m-2)$-differential forms $\Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] \in \Lambda^{m-2}(\mathbb{R}^m; \mathbb{C})$, $\lambda, \mu \in \Sigma$, on $\mathbb{R}^m$ satisfying the condition

$$Z^{(m-1)}[\varphi(\lambda), \psi(\mu)] = d\Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)]. \tag{2.6}$$

A way to realize this condition is to take some closed subspaces $\mathcal{H}_0^*$ and $\mathcal{H}_0 \subset \mathcal{H}_-$ as solutions to the corresponding linear differential equations under some boundary conditions,

$$\mathcal{H}_0 := \{\psi(\lambda) \in \mathcal{H}_- : \mathrm{L}\psi(\lambda) = 0,\ \psi(\lambda)|_{x\in\Gamma} = 0,\ \lambda \in \Sigma\},$$

$$\mathcal{H}_0^* := \{\varphi(\lambda) \in \mathcal{H}_-^* : \mathrm{L}^*\varphi(\lambda) = 0,\ \varphi(\lambda)|_{x\in\Gamma} = 0,\ \lambda \in \Sigma\}.$$

The triple (2.5) allows, in a natural way, to properly determine a set of generalized eigenfunctions for the extended operators $\mathrm{L}, \mathrm{L}^* : \mathcal{H}_- \to \mathcal{H}_-$, if $\Gamma \subset \mathbb{R}^m$ is taken as some $(n-1)$-dimensional piece-wise smooth hypersurface imbedded into the configuration space $\mathbb{R}^m$. There can exist, evidently, situations [4] when boundary conditions are not necessary.

Let now $S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \in H_{m-1}(M; \mathbb{C})$ denote some two nonintersecting $(m-1)$-dimensional piece-wise smooth hypersurfaces from the homology group $H_{m-1}(M; \mathbb{C})$ of some topological compactification $M := \bar{\mathbb{R}}^m$, such that their boundaries are the same, that is, $\partial S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) = \sigma_x^{(m-2)} - \sigma_{x_0}^{(m-2)}$ and, additionally, $\partial\big(S_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \cup S_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})\big) = \varnothing$, where $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \in C_{m-2}(\mathbb{R}^m; \mathbb{C})$ are some $(m-2)$-dimensional homological cycles from a suitable chain complex $\mathcal{K}(M)$ parametrized formally by means of two points $x, x_0 \in M$ and related in some way with the chosen above hypersurface $\Gamma \subset M$. Then from (2.6) based on the general Stokes theorem [12, 13] one correspondingly gets easily that

$$\begin{aligned} \int\limits_{S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\lambda), \psi(\mu)] &= \int\limits_{\partial S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] = \\ &= \int\limits_{\sigma_x^{(m-2)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] - \int\limits_{\sigma_{x_0}^{(m-2)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] := \\ &:= \Omega_x(\lambda, \mu) - \Omega_{x_0}(\lambda, \mu), \end{aligned} \tag{2.7}$$

$$\int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \overline{Z}^{(m-1),\intercal}[\varphi(\lambda),\psi(\mu)] = \int\limits_{\partial S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \overline{\Omega}^{(m-2),\intercal}[\varphi(\lambda),\psi(\mu)] =$$

$$= \int\limits_{\sigma_x^{(m-2)}} \overline{\Omega}^{(m-2),\intercal}[\varphi(\lambda),\psi(\mu)] - \int\limits_{\sigma_{x_0}^{(m-2)}} \overline{\Omega}^{(m-2),\intercal}[\varphi(\lambda),\psi(\mu)] :=$$

$$:= \Omega_x^{\circledast}(\lambda,\mu) - \Omega_{x_0}^{\circledast}(\lambda,\mu)$$

for the set of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\lambda,\mu \in \Sigma$, with the operator kernels $\Omega_x(\lambda,\mu)$, $\Omega_x^{\circledast}(\lambda,\mu)$ and $\Omega_{x_0}(\lambda,\mu)$, $\Omega_x^{\circledast}(\lambda,\mu)$, $\lambda,\mu \in \Sigma$, acting naturally in the Hilbert space $L_2^{(\rho)}(\Sigma;\mathbb{C})$. These kernels are assumed further to be nondegenerate in $L_2^{(\rho)}(\Sigma;\mathbb{C})$ and satisfying the homotopy conditions

$$\lim_{x\to x_0}\Omega_x(\lambda,\mu) = \Omega_{x_0}(\lambda,\mu),\ \lim_{x\to x_0}\Omega_x^{\circledast}(\lambda,\mu) = \Omega_{x_0}^{\circledast}(\lambda,\mu).$$

Define now actions of the following two linear Delsarte permutations operators $\mathbf{\Omega}_\pm : \mathcal{H} \to \mathcal{H}$ and $\mathbf{\Omega}_\pm^{\circledast} : \mathcal{H}^* \to \mathcal{H}^*$ still upon a fixed set of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\lambda,\mu \in \Sigma$,

$$\begin{aligned}\widetilde{\psi}(\lambda) = \mathbf{\Omega}_\pm(\psi(\lambda)) := \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\psi(\eta)\Omega_x^{-1}(\eta,\mu)\Omega_{x_0}(\mu,\lambda),\\ \widetilde{\varphi}(\lambda) = \mathbf{\Omega}_\pm^{\circledast}(\varphi(\lambda)) := \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\mu,\eta)\Omega_{x_0}^{\circledast}(\lambda,\mu).\end{aligned} \tag{2.8}$$

Making use of the expressions (2.8), based on arbitrariness of the chosen set of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\lambda,\mu \in \Sigma$, we can easily retrieve the corresponding operator expressions for the operators $\mathbf{\Omega}_\pm$ and $\mathbf{\Omega}_\pm^{\circledast} : \mathcal{H}_- \to \mathcal{H}_-$, forcing the kernels $\Omega_{x_0}(\lambda,\mu)$ and $\Omega_{x_0}^{\circledast}(\lambda,\mu)$, $\lambda,\mu \in \Sigma$, to variate:

$$\widetilde{\psi}(\lambda) = \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\psi(\eta)\Omega_x(\eta,\mu)\Omega_x^{-1}(\mu,\lambda) -$$

$$-\int\limits_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\psi(\eta)\Omega_x^{-1}(\eta,\mu)] \int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),\psi(\lambda)]) =$$

$$= \psi(\lambda) - \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\int\limits_\Sigma d\rho(\nu)\int\limits_\Sigma d\rho(\xi)\psi(\eta)\Omega_x^{-1}(\eta,\nu)\times$$

$$\times\,\Omega_{x_0}(\nu,\xi)]\Omega_{x_0}^{-1}(\xi,\mu)\int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),\psi(\lambda)] =$$

$$= \psi(\lambda) - \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\tilde{\psi}(\eta)\Omega_{x_0}^{-1}(\eta,\mu)]\int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),\psi(\lambda)] =$$

$$= \Bigg(\mathbf{1} - \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\tilde{\psi}(\eta)\Omega_{x_0}^{-1}(\eta,\mu)\times$$

$$\times\int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),(\cdot)]\Bigg)\,\psi(\lambda) := \mathbf{\Omega}_\pm\cdot\psi(\lambda),$$

$$\widetilde{\varphi}(\lambda) = \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\mu,\eta)\Omega_x^{\circledast}(\lambda,\mu) -$$

$$- \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\mu,\eta)\int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \overline{Z}^{(m-1),\intercal}[\varphi(\lambda),\psi(\mu)] =$$

$$= \varphi(\lambda) - \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\nu)\int\limits_\Sigma d\rho(\xi)\int\limits_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\xi,\eta)\times$$

$$\times\,\Omega_{x_0}^{\circledast}(\nu,\xi)\Omega_{x_0}^{\circledast,-1}(\mu,\nu)\int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \overline{Z}^{(m-1),\intercal}[\varphi(\lambda),\psi(\mu)] =$$

$$= \Bigg(\mathbf{1} - \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\tilde{\varphi}(\eta)\Omega_{x_0}^{\circledast,-1}(\mu,\eta)\times$$

$$\times\int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \overline{Z}^{(m-1),\intercal}[(\cdot),\psi(\mu)]\Bigg)\,\varphi(\lambda) := \mathbf{\Omega}_\pm^{\circledast}\cdot\varphi(\lambda),$$

where, by definition,

$$\mathbf{\Omega}_\pm := \mathbf{1} - \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\tilde{\psi}(\eta)\Omega_{x_0}^{-1}(\eta,\mu)\int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),(\cdot)],$$
$$\mathbf{\Omega}_\pm^{\circledast} := \mathbf{1} - \int\limits_\Sigma d\rho(\eta)\int\limits_\Sigma d\rho(\mu)\tilde{\varphi}(\eta)\Omega_{x_0}^{\circledast,-1}(\mu,\eta)\int\limits_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \overline{Z}^{(m-1),\intercal}[(\cdot),\psi(\mu)] \tag{2.9}$$

are multidimensional integral operators of Volterra type. It is to be noted here that the elements $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \mathcal{H}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda, \mu \in \Sigma$, in the expressions (2.9) of the operators are not arbitrary but now fixed. Therefore, the operators (2.9) realize an extension of their actions (2.8) defined on a fixed pair of functions $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\lambda, \mu \in \Sigma$, to the whole functional space $\mathcal{H}^* \times \mathcal{H}$.

Due to the symmetry of expressions (2.8) and (2.9) with respect to two sets of functions $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \mathcal{H}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda, \mu \in \Sigma$, it is very easy to state the following lemma.

Lemma 2.1. *Operators (2.9) are bounded and invertible expressions on $\mathcal{H}^* \times \mathcal{H}$ of Volterra type, whose inverses are given as follows:*

$$\mathbf{\Omega}_\pm^{-1} := \mathbf{1} - \int\limits_\Sigma d\rho(\eta) \int\limits_\Sigma d\rho(\mu) \psi(\eta) \tilde{\Omega}_{x_0}^{-1}(\eta, \mu) \int\limits_{S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\tilde{\varphi}(\mu), (\cdot)],$$

$$\mathbf{\Omega}_\pm^{\circledast,-1} := \mathbf{1} - \int\limits_\Sigma d\rho(\eta) \int\limits_\Sigma d\rho(\mu) \varphi(\eta) \Omega_{x_0}^{\circledast,-1}(\mu, \eta) \int\limits_{S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \overline{Z}^{(m-1),\intercal}[(\cdot), \tilde{\psi}(\mu)], \tag{2.10}$$

where the two sets of functions $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda, \mu \in \Sigma$, are taken arbitrary but fixed.

For the expressions (2.10) to be compatible with mappings (2.8), we must have the following:

$$\psi(\lambda) = \mathbf{\Omega}_\pm^{-1} \cdot \tilde{\psi}(\lambda) = \int\limits_\Sigma d\rho(\eta) \int\limits_\Sigma d\rho(\mu) \tilde{\psi}(\eta) \tilde{\Omega}_x^{-1}(\eta, \mu)] \tilde{\Omega}_{x_0}(\mu, \lambda),$$

$$\varphi(\lambda) = \mathbf{\Omega}_\pm^{\circledast,-1} \cdot \tilde{\varphi}(\lambda) = \int\limits_\Sigma d\rho(\eta) \int\limits_\Sigma d\rho(\mu) \tilde{\varphi}(\eta) \tilde{\Omega}_x^{\circledast,-1}(\mu, \eta) \tilde{\Omega}_{x_0}^{\circledast}(\lambda, \mu),$$

where for any two sets of functions $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda, \mu \in \Sigma$, the next relationship is satisfied:

$$(< \widetilde{\mathrm{L}}^* \widetilde{\varphi}(\lambda), \widetilde{\psi}(\mu) > - < \widetilde{\varphi}(\lambda), \widetilde{\mathrm{L}} \widetilde{\psi}(\mu) >) dx = d(\widetilde{Z}^{(m-1)}[\tilde{\varphi}(\lambda), \widetilde{\psi}(\mu)]),$$

$$\widetilde{Z}^{(m-1)}[\widetilde{\varphi}(\lambda), \widetilde{\psi}(\mu)] = d\widetilde{\Omega}^{(m-2)}[\widetilde{\varphi}(\lambda), \widetilde{\psi}(\mu)],$$

when

$$\widetilde{\mathrm{L}} := \mathbf{\Omega}_\pm \mathrm{L} \mathbf{\Omega}_\pm^{-1}, \ \widetilde{\mathrm{L}}^* := \mathbf{\Omega}_\pm^{\circledast} \mathrm{L}^* \mathbf{\Omega}_\pm^{\circledast,-1}.$$

Moreover, the expressions above for $\mathrm{L} : \mathcal{H} \to \mathcal{H}$ and $\tilde{\mathrm{L}}^* : \mathcal{H}^* \to \mathcal{H}^*$ don't depend on the choice of the indexes below of operators $\mathbf{\Omega}_+$ or $\mathbf{\Omega}_-$ and are in the result differential. Since the last condition properly determines the Delsarte transmutation operators (2.10), we need to state the following theorem.

Theorem 2.1. *The pair* $(\tilde{\mathrm{L}}, \tilde{\mathrm{L}}^*)$ *of the operator expressions* $\tilde{\mathrm{L}} := \mathbf{\Omega}_{\pm}\mathrm{L}\mathbf{\Omega}_{\pm}^{-1}$ *and* $\tilde{\mathrm{L}}^* := \mathbf{\Omega}_{\pm}^{\circledast}\mathrm{L}^*\mathbf{\Omega}_{\pm}^{\circledast,-1}$ *acting on the space* $\mathcal{H}\times\mathcal{H}^*$ *is purely differential for any suitably chosen hypersurfaces* $S_{\pm}(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \in H_{m-1}(M;\mathbb{C})$ *from the homology group* $H_{m-1}(M;\mathbb{C})$.

Proof. For proving the theorem it is necessary to show that the formal pseudodifferential expressions corresponding to the operators $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{L}}^*$ contain no integral elements. Making use of an idea devised in [6, 8], one can formulate the following lemma.

Lemma 2.2. *A pseudodifferential operator* $\mathrm{L} : \mathcal{H} \to \mathcal{H}$ *is purely differential iff the equality*

$$\left(h, \left(\mathrm{L}\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}\right)_{+} f\right) = \left(h, \mathrm{L}_{+}\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} f\right) \tag{2.11}$$

holds for any $|\alpha| \in \mathbb{Z}_+$ *and all* $(h, f) \in \mathcal{H}^* \times \mathcal{H}$, *that is, the condition (2.11) is equivalent to the equality* $\mathrm{L}_+ = \mathrm{L}$, *where, as usual, the sign* "$(\ldots)_+$" *means the purely differential part of the corresponding expression inside the bracket.*

Based now on this lemma and exact expressions of operators (2.9), similarly to calculations done in [8], one shows right away that the operators $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{L}}^*$, which depend, correspondingly, only on both the homological cycles $\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)} \in C_{m-2}(M;\mathbb{C})$ from a simplicial chain complex $\mathcal{K}(M)$, marked by points $x, x_0 \in \mathbb{R}^m$, and on two sets of functions $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda, \mu \in \Sigma$, are purely differential thereby finishing the proof.

The differential-geometric construction suggested above can be nontrivially generalized for the case of $m \in \mathbb{Z}_+$ commuting with each other differential operators on a Hilbert space $\mathcal{H}$ giving rise to a new look at theory of Delsarte transmutation operators based on differential-geometric and topological de Rham – Hodge techniques. These aspects will be discussed in detail in the next chapter below.

3. The general differential-geometric and topological structure of Delsarte transmutation operators. Let $M := \bar{\mathbb{R}}^m$ denote, as before, a suitably compactified metric space of dimension $m = \dim M \in \mathbb{Z}_+$ (without boundary) and define some finite set $\mathcal{L}$ of smooth commuting with each other linear differential operators

$$\mathrm{L}_j(x|\partial) := \sum_{|\alpha|=0}^{n(\mathrm{L}_j)} a_{\alpha}^{(j)}(x)\partial^{|\alpha|}/\partial x^{\alpha} \tag{3.1}$$

with respect to $x \in M$, having Schwatrz coefficients $a_{\alpha}^{(j)} \in \mathcal{S}(M;\mathrm{End}\mathbb{C}^N)$, $|\alpha| = \overline{0, n(\mathrm{L}_j)}$, $n(\mathrm{L}_j) \in \mathbb{Z}_+$, $j = \overline{1,m}$, and acting on the Hilbert space $\mathcal{H} := L_2(M;\mathbb{C}^N)$. It is also assumed that the domains $D(\mathrm{L}_j) := D(\mathcal{L}) \subset \mathcal{H}$, $j = \overline{1,m}$, are dense in $\mathcal{H}$.

Consider now a generalized external antidifferentiation operator $d_{\mathcal{L}} : \Lambda(M;\mathcal{H}) \to \Lambda(M;\mathcal{H})$ acting in the Grassmann algebra $\Lambda(M;\mathcal{H})$ as follows: for any $\beta^{(k)} \in \Lambda^k(M;\mathcal{H})$, $k = \overline{0,m}$,

$$d_{\mathcal{L}}\beta^{(k)} := \sum_{j=1}^{m} dx_j \wedge \mathrm{L}_j(x;\partial)\beta^{(k)} \in \Lambda^{k+1}(M;\mathcal{H}). \tag{3.2}$$

It is easy to see that the operation (3.2) in the case where $\mathrm{L}_j(x;\partial) := \partial/\partial x_j$, $j=\overline{1,m}$, coincides exactly with the standard external differentiation $d=\sum_{j=1}^{m} dx_j\wedge\partial/\partial x_j$ on the Grassmann algebra $\Lambda(M;\mathcal{H})$. Making use of the operation (3.2) on $\Lambda(M;\mathcal{H})$, one can construct the following generalized de Rham co-chain complex:

$$\mathcal{H}\rightarrow\Lambda^0(M;\mathcal{H})\overset{d_{\mathcal{L}}}{\rightarrow}\Lambda^1(M;\mathcal{H})\overset{d_{\mathcal{L}}}{\rightarrow}\ldots\overset{d_{\mathcal{L}}}{\rightarrow}\Lambda^m(M;\mathcal{H})\overset{d_{\mathcal{L}}}{\rightarrow}0. \tag{3.3}$$

The following important property concerning the complex (3.3) holds.

Lemma 3.1. *The co-chain complex (3.3) is exact.*

Proof. It follows easily from the equality $d_{\mathcal{L}}d_{\mathcal{L}}=0$ that holds true due to commutativity of operators (3.1).

Below we will follow the ideas developed in [14]. A differential form $\beta\in\Lambda(M;\mathcal{H})$ will be called $d_{\mathcal{L}}$-closed if $d_{\mathcal{L}}\beta=0$, and a form $\gamma\in\Lambda(M;\mathcal{H})$ will be called $d_{\mathcal{L}}$-homological to zero if there exists a form $\omega\in\Lambda(M;\mathcal{H})$ on M such that $\gamma=d_{\mathcal{L}}\omega$.

Consider now the standard algebraic Hodge star-operation

$$\star:\Lambda^k(M;\mathcal{H})\rightarrow\Lambda^{m-k}(M;\mathcal{H}),\qquad k=\overline{0,m},$$

defined as follows [15]: if $\beta\in\Lambda^k(M;\mathcal{H})$, then the form $\star\beta\in\Lambda^{m-k}(M;\mathcal{H})$ is such that:

i) the $(m-k)$-dimensional volume $|\star\beta|$ of the form $\star\beta$ equals the k-dimensional volume $|\beta|$ of the form β;

ii) the m-dimensional measure $\bar{\beta}^{\intercal}\wedge\star\beta>0$ for a fixed orientation on M.

Define also, on the space $\Lambda(M;\mathcal{H})$, the following natural scalar product: for any $\beta,\gamma\in\in\Lambda^k(M;\mathcal{H})$, $k=\overline{0,m}$,

$$(\beta,\gamma):=\int_M\bar{\beta}^{\intercal}\wedge\star\gamma. \tag{3.4}$$

Using the scalar product (3.4) we can naturally construct the corresponding Hilbert space,

$$\mathcal{H}_{\Lambda}(M):=\overset{m}{\underset{k=0}{\oplus}}\mathcal{H}^k_{\Lambda}(M),$$

well suitable for our further consideration. Notice also here that the Hodge star $\star$-operation satisfies the following easily checked property: for any $\beta,\gamma\in\mathcal{H}^k_{\Lambda}(M)$, $k=\overline{0,m}$,

$$(\beta,\gamma)=(\star\beta,\star\gamma),$$

that is the Hodge operation $\star:\mathcal{H}_{\Lambda}(M)\rightarrow\mathcal{H}_{\Lambda}(M)$ is an isometry and its standard adjoint with respect to the scalar product (3.4) satisfies the condition $(\star)'=(\star)^{-1}$.

Denote by $d'_{\mathcal{L}}$ the formally adjoint expression to the external weak differential operation $d_{\mathcal{L}}:\mathcal{H}_{\Lambda}(M)\rightarrow\mathcal{H}_{\Lambda}(M)$ in the Hilbert space $\mathcal{H}_{\Lambda}(M)$. Making now use of the operations $d'_{\mathcal{L}}$ and $d_{\mathcal{L}}$ in $\mathcal{H}_{\Lambda}(M)$ one can naturally define [15] the generalized Laplace – Hodge operator $\Delta_{\mathcal{L}}:\mathcal{H}_{\Lambda}(M)\rightarrow\mathcal{H}_{\Lambda}(M)$ as

$$\Delta_{\mathcal{L}}:=d'_{\mathcal{L}}d_{\mathcal{L}}+d_{\mathcal{L}}d'_{\mathcal{L}}. \tag{3.5}$$

Take a form $\beta \in \mathcal{H}_\Lambda(M)$ satisfying the equality

$$\Delta_{\mathcal{L}}\beta = 0.$$

Such a form is called *harmonic*. One can also verify that a harmonic form $\beta \in \mathcal{H}_\Lambda(M)$ satisfies simultaneously the following two adjoint conditions:

$$d'_{\mathcal{L}}\beta = 0, \quad d_{\mathcal{L}}\beta = 0, \tag{3.6}$$

easily deducible from (3.5) and (3.6).

It is not hard to check that the following differential operation in $\mathcal{H}_\Lambda(M)$:

$$d^*_{\mathcal{L}} := \star d'_{\mathcal{L}}(\star)^{-1}$$

also defines the usual [12, 13] external antidifferential operation in $\mathcal{H}_\Lambda(M)$. The corresponding co-chain complex dual to (3.3),

$$\mathcal{H} \to \Lambda^0(M;\mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} \Lambda^1(M;\mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} \ldots \xrightarrow{d^*_{\mathcal{L}}} \Lambda^m(M;\mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} 0,$$

is evidently exact too, since the property $d^*_{\mathcal{L}}d^*_{\mathcal{L}} = 0$ holds due to the definition (3.5).

Denote further by $\mathcal{H}^k_{\Lambda(\mathcal{L})}(M)$, $k = \overline{0,m}$, the cohomology groups of $d_{\mathcal{L}}$-closed and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M)$, $k = \overline{0,m}$, the cohomology groups of $d^*_{\mathcal{L}}$-closed differential forms, correspondingly, and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M)$, $k = \overline{0,m}$, the abelian groups of harmonic differential forms from the Hilbert subspaces $\mathcal{H}^k_\Lambda(M)$, $k = \overline{0,m}$. Before formulating next results, define the standard Hilbert – Schmidt rigged chain [10] of positive and negative Hilbert spaces of differential forms,

$$\mathcal{H}^k_{\Lambda,+}(M) \subset \mathcal{H}^k_\Lambda(M) \subset \mathcal{H}^k_{\Lambda,-}(M)$$

and the corresponding rigged chains of Hilbert subspaces of harmonic forms,

$$\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M),$$

and the cohomology groups

$$\begin{gathered} \mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L})}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M), \\ \mathcal{H}^k_{\Lambda(\mathcal{L}^*),+}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*),-}(M), \end{gathered} \tag{3.7}$$

for any $k = \overline{0,m}$. Assume also that the Laplace – Hodge type operator (3.5) is elliptic in $\mathcal{H}^0_\Lambda(M)$. Now by reasonings similar to those in [13, 15] one can formulate the following a little generalized de Rham – Hodge theorem.

Theorem 3.1. *The groups of harmonic forms $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M)$, $k = \overline{0,m}$, are, correspondingly, isomorphic to the cohomology groups $(H^k(M;\mathbb{C}))^\Sigma$, $k = \overline{0,m}$, where $H^k(M;\mathbb{C})$ is the*

k-th cohomology group of the manifold M with complex coefficients, $\Sigma \subset \mathbb{C}^p$ is a set of suitable "spectral" parameters marking the linear space of independent $d^*_{\mathcal{L}}$-closed 0-forms from $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M)$ and, moreover, the following direct sum decompositions hold for any $k = \overline{0,m}$:

$$\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M) \oplus \Delta_{\mathcal{L}}\mathcal{H}^k_{-}(M) = \mathcal{H}^k_{\Lambda,-}(M) = \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M) \oplus d_{\mathcal{L}}\mathcal{H}^{k-1}_{\Lambda,-}(M) \oplus d'_{\mathcal{L}}\mathcal{H}^{k+1}_{\Lambda,-}(M).$$

Another variant of the statement similar to the one above was formulated in [14] and reads as the following generalized de Rham–Hodge theorem.

Theorem 3.2 [14]. *The generalized cohomology groups $\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M), k = \overline{0,m}$, are isomorphic, correspondingly, to the cohomology groups $(H^k(M;\mathbb{C}))^{\Sigma}$, $k = \overline{0,m}$.*

A proof of this theorem is based on some special sequence [14] of differential Lagrange type identities. Define the following closed subspace:

$$\mathcal{H}^*_0 := \{\varphi^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M) : d^*_{\mathcal{L}}\varphi^{(0)}(\lambda) = 0,\ \varphi^{(0)}(\lambda)|_{\Gamma} = 0,\ \lambda \in \Sigma\} \tag{3.8}$$

for some smooth $(m-1)$-dimensional hypersurface $\Gamma \subset M$ and $\Sigma \subset (\sigma(\mathcal{L}) \cap \bar{\sigma}(\mathcal{L}^*)) \times \times\Sigma_{\sigma} \subset \mathbb{C}^p$, where $\mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M)$ is, as above, a suitable Hilbert–Schmidt rigged [10, 11] zero-order cohomology group Hilbert space from the chain given by (3.7), $\sigma(\mathcal{L})$ and $\sigma(\mathcal{L}^*)$ are, correspondingly, mutual spectra of the sets of operators $\mathcal{L}$ and $\mathcal{L}^*$. Thereby, the dimension $\dim \mathcal{H}^*_0 = \operatorname{card} \Sigma$ is assumed to be known.

The next lemma stated by I.V. Skrypnik holds and is fundamental for the proof.

Lemma 3.2 [14, 16–18]. *There exists a set of differential $(k+1)$-forms $Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \in \in \Lambda^{k+1}(M;\mathcal{H})$, $k = \overline{0,m}$, and a set of k-forms $Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \in \Lambda^k(M;\mathcal{H})$, $k = \overline{0,m}$, parametrized by a set $\Sigma \ni \lambda$ and semilinear in $(\varphi^{(0)}(\lambda), \psi^{(k)}) \in \mathcal{H}^*_0 \times \mathcal{H}^k_{\Lambda,-}(M)$, such that*

$$Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] = dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}]$$

for all $k = \overline{0,m}$ and $\lambda \in \Sigma$.

Proof. A proof is based on the following generalized Lagrange type identity that holds for any pair $(\varphi^{(0)}(\lambda), \psi^{(k)}) \in \mathcal{H}^*_0 \times \mathcal{H}^k_{\Lambda,-}(M)$:

$$\begin{aligned} 0 &= < d^*_{\mathcal{L}}\varphi^{(0)}(\lambda), \star(\psi^{(k)} \wedge \overline{\gamma}) > := \\ &:= < \star d'_{\mathcal{L}}(\star)^{-1}\varphi^{(0)}(\lambda), \star(\psi^{(k)} \wedge \bar{\gamma}) > = \\ &= < d'_{\mathcal{L}}(\star)^{-1}\varphi^{(0)}(\lambda), \psi^{(k)} \wedge \bar{\gamma} > = < (\star)^{-1}\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)} \wedge \bar{\gamma} > + \\ &\quad + Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \wedge \bar{\gamma} = \\ &= < (\star)^{-1}\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)} \wedge \bar{\gamma} > + dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \wedge \bar{\gamma}, \end{aligned} \tag{3.9}$$

where $Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M;\mathbb{C})$, $k = \overline{0,m}$, and $Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \in \Lambda^{k-1}(M;\mathbb{C})$, $k = \overline{0,m}$, are some semilinear differential forms parametrized by a parameter $\lambda \in \Sigma$, and $\bar{\gamma} \in \Lambda^{m-k-1}(M;\mathbb{C})$ is an arbitrary constant $(m-k-1)$-form. Thereby, the semilinear differential

k-forms $Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M;\mathbb{C})$, $k = \overline{0,m}$, and the k-forms $Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \in \in \Lambda^k(M;\mathbb{C})$, $k = \overline{0,m}$, $\lambda \in \Sigma$, constructed above exactly constitute those searched for in the lemma.

Based now on Lemma 3.2 one can construct the cohomology group isomorphism claimed in Theorem 3.1 formulated above. Namely, following [14, 16, 19], let us take some simplicial [13] partition $\mathcal{K}(M)$ of the manifold M and introduce linear mappings $B_\lambda^{(k)} : \mathcal{H}^k_{\Lambda,-}(M) \to \to C_k(M;\mathbb{C}))$, $k = \overline{0,m}$, $\lambda \in \Sigma$, where $C_k(M;\mathbb{C})$, $k = \overline{0,m}$, are, as before, free abelian groups over the field $\mathbb{C}$ generated, correspondingly, by all k-chains of simplexes $S^{(k)} \in C_k(M;\mathbb{C})$, $k = \overline{0,m}$, from the simplicial complex $\mathcal{K}(M)$ as follows:

$$B_\lambda^{(k)}(\psi^{(k)}) := \sum_{S^{(k)} \in C_k(M;\mathbb{C}))} S^{(k)} \int_{S^{(k)}} Z^{(k)} \left[\varphi^{(0)}(\lambda), \psi^{(k)}\right] \tag{3.10}$$

with $\psi^{(k)} \in \mathcal{H}^k_\Lambda(M)$, $k = \overline{0,m}$. We have the following theorem.

Theorem 3.3 [14, 16–18]. *The set of operations (3.10) parametrized by $\lambda \in \Sigma$ realizes the cohomology groups isomorphism formulated in the Theorem 3.2.*

Proof. A proof of this theorem can be obtained by passing in (3.10) to the corresponding cohomology, and homology groups of M, $\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)$ and $H_k(M;\mathbb{C})$, for every $k = \overline{0,m}$. By taking an element $\psi^{(k)} := \psi^{(k)}(\mu) \in \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)$, $k = \overline{0,m}$, and solving the equation $d_{\mathcal{L}}\psi^{(k)}(\mu) = 0$, where $\mu \in \Sigma_k$ is some set of the related "spectral"parameters marking elements of the subspace $\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)$, one easily finds from (3.10) and the identity (3.9) that

$$dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = 0$$

for all pairs $(\lambda, \mu) \in \Sigma \times \Sigma_k$, $k = \overline{0,m}$. This, in particular, means due to the Poincare lemma [12, 13] that there exist differential $(k-1)$-forms $\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi(\mu)] \in \Lambda^{k-1}(M;\mathbb{C})$, $k = \overline{0,m}$, such that

$$Z^{(k)}[\varphi^{(0)}(\lambda), \psi(\mu)] = d\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi(\mu)]$$

for all pairs $(\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)$ parametrized by $(\lambda, \mu) \in \Sigma \times \Sigma_k$, $k = \overline{0,m}$. As a result of passing in the right-hand side of (3.10) to the homology groups $H_k(M;\mathbb{C})$, $k = = \overline{0,m}$, one gets, due to the standard Stokes theorem [12], that the mappings

$$\hat{B}_\lambda^{(k)} : \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M) \rightleftarrows H_k(M;\mathbb{C})$$

are isomorphisms for every $\lambda \in \Sigma$. Making further use of the Poincare duality [13] between the homology groups $H_k(M;\mathbb{C})$, $k = \overline{0,m}$, and the cohomology groups $H^k(M;\mathbb{C})$, $k = \overline{0,m}$, correspondingly, one finally obtains the statement claimed in Theorem 3. 3, that is,

$$\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M) \simeq (H^k(M;\mathbb{C}))^\Sigma.$$

Assume now that $M := \mathrm{T}^r \times \bar{\mathbb{R}}^s$, $\dim M = s + r \in \mathbb{Z}_+$, and $\mathcal{H} := L_2(\mathrm{T}^r; L_2(\mathbb{R}^s;\mathbb{C}^N))$, where $\mathrm{T}^r := \underset{j=1}{\overset{r}{\times}} \mathrm{T}_j$, $\mathrm{T}_j := [0, T_j) \subset \mathbb{R}_+$, $j = \overline{1,r}$, and put

$$d_{\mathcal{L}} := \sum_{j=1}^{2} dt_j \wedge \mathrm{L}_j, \ \ \mathrm{L}_j(t;x|\partial) := \partial/\partial t_j - L_j(t;x|\partial),$$

where

$$L_j(t;x|\partial) = \sum_{|\alpha|=0}^{n(L_j)} a_\alpha^{(j)}(t;x)\partial^{|\alpha|}/\partial x^\alpha, \ j = \overline{1,r},$$

are differential operations parametrically dependent on $t \in \mathrm{T}^r$ and defined on dense subspaces $D(\mathrm{L}_j) = D(\mathcal{L}) \subset L_2(\mathbb{R}^s;\mathbb{C}^N)$, $j = \overline{1,r}$. It is also assumed that the operators $\mathrm{L}_j : \mathcal{H} \to \mathcal{H}$, $j = \overline{1,r}$, commute with each other.

Take now a fixed pair $(\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx) \in \mathcal{H}_0^* \times \mathcal{H}^s_{\Lambda(\mathcal{L}),-}(M)$, parametrized by elements $(\lambda,\mu) \in \Sigma\times\Sigma$, such that Theorem 3. 3 and the Stokes theorem [12, 13] would imply the equality

$$B_\lambda^{(s)}(\psi^{(0)}(\mu)dx) = S_{t;x}^{(s)} \int\limits_{\partial S_{t;x}^{(s)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \tag{3.11}$$

where $S_{t;x}^{(s)} \in H_s(M;\mathbb{C})$ is some arbitrary but fixed element parametrized by an arbitrarily chosen point $(t;x) \in M \cap S_{t;x}^{(s)}$. Consider the next integral expressions

$$\Omega_{(t;x)}(\lambda,\mu) := \int\limits_{\partial S_{t;x}^{(s)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx],$$

$$\Omega_{(t_0;x_0)}(\lambda,\mu) := \int\limits_{\partial S_{t_0;x_0}^{(s)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx],$$

where the point $(t_0;x_0) \in M \cap S_{t_0;x_0}^{(s)}$ is taken fixed, $\lambda,\mu \in \Sigma$, and interpret them as the corresponding kernels [10] of the integral invertible operators of Hilbert–Schmidt type, $\Omega_{(t;x)}$, $\Omega_{(t_0;x_0)} : L_2^{(\rho)}(\Sigma;\mathbb{C}) \to L_2^{(\rho)}(\Sigma;\mathbb{C})$, where ρ is some Borel measure on the parameter sets Σ. It is also assumed above that the boundaries $\partial S_{t;x}^{(s)} := \sigma_{t;x}^{(s-1)}$ and $\partial S_{t_0;x_0}^{(s)} := \sigma_{t_0;x_0}^{(s-1)}$ are taken to be homological to each other as $(t;x) \to (t_0;x_0) \in M$. Define now the expressions

$$\mathbf{\Omega}_\pm : \psi^{(0)}(\eta) \to \tilde{\psi}^{(0)}(\eta)$$

for $\psi^{(0)}(\eta)dx \in \mathcal{H}^s_{\Lambda(\mathcal{L}),-}(M)$ and some $\tilde{\psi}^{(0)}(\eta)dx \in \mathcal{H}^s_{\Lambda,-}(M)$, where, by definition,

$$\begin{aligned}\tilde{\psi}^{(0)}(\eta) &:= \psi^{(0)}(\eta)\cdot\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)} = \\ &= \int\limits_\Sigma d\rho(\mu)\int\limits_\Sigma d\rho(\xi)\psi^{(0)}(\mu)\Omega^{-1}_{(t;x)}(\mu,\xi)\Omega_{(t_0;x_0)}(\xi,\eta)\end{aligned} \tag{3.12}$$

for any $\eta \in \Sigma$, which is motivated by the expression (3.11). Suppose now that the elements (3.12) are the ones that are related to some another Delsarte transformed cohomology group $\mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, that is, we have the condition

$$d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\eta)dx = 0 \tag{3.13}$$

for $\tilde{\psi}^{(0)}(\eta)dx \in \mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, $\eta \in \Sigma$, and some new external antidifferentiation operation in $\mathcal{H}_{\Lambda,-}(M)$,

$$d_{\tilde{\mathcal{L}}} := \sum_{j=1}^{m} dx_j \wedge \tilde{\mathrm{L}}_j(t;x|\partial),\ \tilde{\mathrm{L}}_j(t;x|\partial) := \partial/\partial t_j - \tilde{L}_j(t;x|\partial),$$

where the expressions

$$\tilde{L}_j(t;x|\partial) = \sum_{|\alpha|=0}^{n(L_j)} \tilde{a}^{(j)}_\alpha(t;x)\partial^{|\alpha|}/\partial x^\alpha,\ j = \overline{1,r},$$

are differential operations in $L_2(\mathbb{R}^s;\mathbb{C}^N)$ parametrically dependent on $t \in \mathrm{T}^r$. Suppose now that

$$\tilde{\mathrm{L}}_j := \mathbf{\Omega}_\pm \mathrm{L}_j \mathbf{\Omega}_\pm^{-1} \tag{3.14}$$

for each $j = \overline{1,r}$, where $\mathbf{\Omega}_\pm : \mathcal{H} \to \mathcal{H}$ are the corresponding Delsarte transmutation operators generated by some elements $S_\pm(\sigma^{(s-1)}_{x;t}, \sigma^{(s-1)}_{x_0;t_0}) \in C_s(M;\mathbb{C})$ related naturally with the boundaries $\partial S^{(s)}_{x;t} = \sigma^{(s-1)}_{x;t}$ and $\partial S^{(s)}_{x_0;t_0} = \sigma^{(s-1)}_{x_0;t_0}$ homological to each other. Since all of the operators $\mathrm{L}_j : \mathcal{H} \to \mathcal{H}$, $j = \overline{1,r}$, commute, the same property also holds for the transformed operators (3.14), that is, $[\tilde{\mathrm{L}}_j, \tilde{\mathrm{L}}_k] = 0$, $k,j = \overline{0,m}$. The latter, due to (3.14), is evidently equivalent to the following general expression:

$$d_{\tilde{\mathcal{L}}} = \mathbf{\Omega}_\pm d_{\mathcal{L}} \mathbf{\Omega}_\pm^{-1}. \tag{3.15}$$

For the condition (3.15) and (3.13) to be satisfied, let us consider the expressions

$$\tilde{B}^{(s)}_\lambda(\tilde{\psi}^{(0)}(\eta)dx) = S^{(s)}_{t;x}\tilde{\Omega}_{(t;x)}(\lambda,\eta),$$

corresponding to (3.11) and related to the corresponding external differentiation (3.15), where $S^{(s)}_{t;x} \in C_s(M;\mathbb{C})$ and $(\lambda,\eta) \in \Sigma \times \Sigma$. Assume further that there are also mappings

$$\mathbf{\Omega}^\circledast_\pm : \varphi^{(0)}(\lambda) \to \tilde{\varphi}^{(0)}(\lambda),$$

where $\mathbf{\Omega}^\circledast_\pm : \mathcal{H}^* \to \mathcal{H}^*$ are some operators associated (but not necessary adjoint!) to the corresponding Delsarte transmutation operators $\mathbf{\Omega}_\pm : \mathcal{H} \to \mathcal{H}$ and satisfying the standard relationships $\tilde{\mathrm{L}}^*_j := \mathbf{\Omega}^\circledast_\pm \mathrm{L}^*_j \mathbf{\Omega}^{\circledast,-1}_\pm$, $j = \overline{1,r}$. The proper Delsarte type operators $\mathbf{\Omega}_\pm : \mathcal{H}^0_{\Lambda(\mathrm{L}),-}(M) \to \mathcal{H}^0_{\Lambda(\mathrm{L}),-}(M)$ are related to two different realizations of the action (3.12) if the necessary conditions

$$d_{\mathrm{L}}\tilde{\psi}^{(0)}(\eta)dx = 0,\ d^*_{\mathrm{L}}\tilde{\varphi}^{(0)}(\lambda) = 0, \tag{3.16}$$

are satisfied and mean, evidently, that the embeddings $\tilde{\varphi}^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\mathrm{L}^*),-}(M)$, $\lambda \in \Sigma$, and $\tilde{\psi}^{(0)}(\eta)dx \in \mathcal{H}^s_{\Lambda(\mathrm{L}),-}(M)$, $\eta \in \Sigma$, hold. Now we need to formulate a lemma that is important for the conditions (3.16) to hold.

Lemma 3.3. *The invariance property*

$$\tilde{Z}^{(s)} = \Omega_{(t_0;x_0)}\Omega^{-1}_{(t;x)}Z^{(s)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)} \tag{3.17}$$

holds for any $(t;x)$ *and* $(t_0;x_0) \in M$.

As a result of (3.17) and the symmetry invariance between the cohomology spaces $\mathcal{H}^0_{\Lambda(\mathrm{L}),-}(M)$ and $\mathcal{H}^0_{\Lambda(\mathrm{L}),-}(M)$, one obtains the following pairs of related mappings:

$$\begin{aligned} \psi^{(0)} &= \tilde{\psi}^{(0)}\tilde{\Omega}^{-1}_{(t;x)}\tilde{\Omega}_{(t_0;x_0)}, \ \varphi^{(0)} = \tilde{\varphi}^{(0)}\tilde{\Omega}^{\circledast,-1}_{(t;x)}\tilde{\Omega}^{\circledast}_{(t_0;x_0)}, \\ \tilde{\psi}^{(0)} &= \psi^{(0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)}, \ \tilde{\varphi}^{(0)} = \varphi^{(0)}\Omega^{\circledast,-1}_{(t;x)}\Omega^{\circledast}_{(t_0;x_0)}, \end{aligned} \tag{3.18}$$

where the integral operator kernels in $L_2^{(\rho)}(\Sigma;\mathbb{C}) \otimes L_2^{(\rho)}(\Sigma;\mathbb{C})$ are defined by

$$\tilde{\Omega}_{(t;x)}(\lambda,\mu) := \int\limits_{\sigma^{(s)}_{t;x}} \tilde{\Omega}^{(s-2)}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx],$$

$$\tilde{\Omega}^{\circledast}_{(t;x)}(\lambda,\mu) := \int\limits_{\sigma^{(s)}_{t;x}} \bar{\tilde{\Omega}}^{(s-2),\intercal}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx].$$

for all $(\lambda,\mu) \in \Sigma\times\Sigma$. This allows to find proper Delsarte transmutation operators ensuring the pure differential nature of the transformed expressions (3.14).

Note also that here, due to (3.17) and (3.18), we have

$$\Omega_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)} + \tilde{\Omega}_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)} = 0 \tag{3.19}$$

for any $(t_0;x_0)$ and $(t;x) \in M$, which means that $\tilde{\Omega}_{(t_0;x_0)} = -\Omega_{(t_0;x_0)}$.

One can now define, similarly to (3.8), three additional subspaces

$$\begin{aligned} \mathcal{H}_0 &:= \{\psi^{(0)}(\mu) \in \mathcal{H}^0_{\Lambda(\mathrm{L}),-}(M) : d_{\mathrm{L}}\psi^{(0)}(\mu) = 0, \ \ \psi^{(0)}(\mu)|_{\Gamma} = 0, \ \mu \in \Sigma\}, \\ \tilde{\mathcal{H}}_0 &:= \{\tilde{\psi}^{(0)}(\mu) \in \mathcal{H}^0_{\Lambda(\tilde{\mathrm{L}}),-}(M) : d_{\tilde{\mathrm{L}}}\tilde{\psi}^{(0)}(\mu) = 0, \ \ \tilde{\psi}^{(0)}(\mu)|_{\tilde{\Gamma}} = 0, \ \mu \in \Sigma\}, \\ \tilde{\mathcal{H}}^*_0 &:= \{\tilde{\varphi}^{(0)}(\eta) \in \mathcal{H}^0_{\Lambda(\mathrm{L}^*),-}(M) : d^*_{\tilde{\mathrm{L}}}\tilde{\psi}^{(0)}(\eta) = 0, \ \ \tilde{\varphi}^{(0)}(\eta)|_{\tilde{\Gamma}} = 0, \ \eta \in \Sigma\}, \end{aligned} \tag{3.20}$$

that are closed and dense in $\mathcal{H}^0_{\Lambda,-}(M)$, where Γ and $\tilde{\Gamma} \subset M$ are some smooth $(s-1)$-dimensional hypersurfaces. Construct the actions

$$\mathbf{\Omega}_{\pm} : \psi^{(0)} \to \tilde{\psi}^{(0)} := \psi^{(0)}\Omega^{-1}_{(t;x)}\Omega_{(t;x)}, \quad \mathbf{\Omega}^{\circledast}_{\pm} : \varphi^{(0)} \to \tilde{\varphi}^{(0)} := \varphi^{(0)}\Omega^{\circledast,-1}_{(t;x)}\Omega^{\circledast}_{(t_0;x_0)} \tag{3.21}$$

on arbitrary but fixed pairs of elements $(\varphi^{(0)}(\lambda), \psi^{(0}(\mu)) \in \mathcal{H}^*_0 \times \mathcal{H}_0$, parametrized by the set Σ, where, by the definition, it is necessary that all obtained pairs $(\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx)$, $\lambda,\mu \in \Sigma$, would belong to $\mathcal{H}^0_{\Lambda(\mathrm{L}^*),-}(M)\times\mathcal{H}^s_{\Lambda(\mathrm{L}),-}(M)$. Note also that the related operator property (3.19) can be compactly written as follows:

$$\tilde{\Omega}_{(t;x)} = \tilde{\Omega}_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)} = -\Omega_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)}.$$

Construct now from the expressions (3.21) the following operator kernels in the Hilbert space $L_2^{(\rho)}(\Sigma;\mathbb{C})\otimes L_2^{(\rho)}(\Sigma;\mathbb{C})$:

$$\begin{aligned}\Omega_{(t;x)}(\lambda,\mu)-\Omega_{(t_0;x_0)}(\lambda,\mu) &= \int_{\partial S^{(s)}_{t;x}}\Omega^{(s-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]- \\ &\quad -\int_{\partial S^{(s)}_{t_0;x_0}}\Omega^{(s-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx] = \\ &= \int\limits_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0})} d\Omega^{(s-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx] = \\ &= \int\limits_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0})} Z^{(s)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx],\end{aligned}$$

and, similarly,

$$\begin{aligned}\Omega^{\circledast}_{(t;x)}(\lambda,\mu)-\Omega^{\circledast}_{(t_0;x_0)}(\lambda,\mu) &= \int\limits_{\partial S^{(s)}_{t;x}}\bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]- \\ &\quad -\int\limits_{\partial S^{(s)}_{t_0;x_0}}\bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx] = \\ &= \int\limits_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0})} d\bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx] = \\ &= \int\limits_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0})} \bar{Z}^{(s-1),\intercal}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx],\end{aligned} \tag{3.22}$$

where $\lambda,\mu\in\Sigma$, and, by the definition, the s-dimensional surfaces $S^{(s)}_{+}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0}))$ and $S^{(s)}_{-}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0})\subset M$ are spanned smoothly without self-intersection between two homological cycles $\sigma^{(s-1)}_{t;x}=\partial S^{(s)}_{t;x}$ and $\sigma^{(s-1)}_{t_0;x_0}:=\partial S^{(s)}_{t_0;x_0}\in C_{s-1}(M;\mathbb{C})$ in such a way that the boundary $\partial(S^{(s)}_{+}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0})\cup S^{(s)}_{-}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0}))=\varnothing$. Since the integral operator expressions $\Omega_{(t_0;x_0)}$, $\Omega^{\circledast}_{(t_0;x_0)}: L_2^{(\rho)}(\Sigma;\mathbb{C})\to L_2^{(\rho)}(\Sigma;\mathbb{C})$ are, evidently, constant and assumed to be invertible, for a fixed point $(t_0;x_0)\in M$ in order to extend the actions given by (3.21) to the whole Hilbert space $\mathcal{H} \times \mathcal{H}^*$, one can apply the classical approach of variation of constants, making use of the expressions (3.22). As a result, we easily obtain the following Delsarte transmutation integral operator expressions for fixed pairs $(\varphi^{(0)}(\xi), \psi^{(0)}(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0, \lambda, \mu \in \Sigma$:

$$\begin{aligned} \mathbf{\Omega}_{\pm} &= \mathbf{1} - \int\limits_{\Sigma\times\Sigma} d\rho(\xi)d\rho(\eta)\tilde{\psi}(x;\xi)\Omega^{-1}_{(t_0;x_0)}(\xi,\eta) \int\limits_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0})} Z^{(s)}[\varphi^{(0)}(\eta),\cdot], \\ \mathbf{\Omega}^{\circledast}_{\pm} &= \mathbf{1} - \int\limits_{\Sigma\times\Sigma} d\rho(\xi)d\rho(\eta)\tilde{\varphi}(x;\eta)\Omega^{\circledast,-1}_{(t_0;x_0)}(\xi,\eta) \int\limits_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{t;x},\sigma^{(s-1)}_{t_0;x_0})} \bar{Z}^{(s),\intercal}[\cdot,\psi^{(0)}(\xi)dx], \end{aligned} \tag{3.23}$$

which are bounded invertible integral operators of Volterra type on the whole space $\mathcal{H} \times \times\mathcal{H}^*$. Applying the same arguments as in Section 1, one can also show that the correspondingly transformed sets of operators $\tilde{\mathrm{L}}_j := \mathbf{\Omega}_{\pm}\mathrm{L}_j\mathbf{\Omega}_{\pm}^{-1}$, $j = \overline{1,r}$, and $\tilde{\mathrm{L}}_k^* := \mathbf{\Omega}_{\pm}^{\circledast}\mathrm{L}_k^*\mathbf{\Omega}_{\pm}^{\circledast,-1}$, $k = \overline{1,r}$, are also purely differential. Thereby, one can formulate the following final theorem.

Theorem 3.4. *The expressions (3.23) are bounded invertible Delsarte transmutation integral operators of Volterra type onto $\mathcal{H} \times \mathcal{H}^*$, transforming, correspondingly, given commuting sets of operators L_j, $j = \overline{1,r}$, and their formally adjoint L_k^*, $k = \overline{1,r}$, into the pure differential sets of operators $\tilde{\mathrm{L}}_j := \mathbf{\Omega}_{\pm}\mathrm{L}_j\mathbf{\Omega}_{\pm}^{-1}$, $j = \overline{1,r}$, and $\tilde{\mathrm{L}}_k^* := \mathbf{\Omega}_{\pm}^{\circledast}\mathrm{L}_k^*\mathbf{\Omega}_{\pm}^{\circledast,-1}$, $k = \overline{1,r}$. Moreover, the suitably constructed closed subspaces $\mathcal{H}_0 \subset \mathcal{H}$ and $\tilde{\mathcal{H}}_0 \subset \mathcal{H}$ such that $\mathbf{\Omega} : \mathcal{H}_0 \rightleftarrows \tilde{\mathcal{H}}_0$, strongly depend on the topological structure of the generalized cohomology groups $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M)$ and $\mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}),-}(M)$ that are parametrized by elements $S^{(s)}_{\pm}(\sigma^{(s-1)}_{t;x}, \sigma^{(s-1)}_{t_0;x_0}) \in C_s(M;\mathbb{C})$.*

Suppose now that all of the differential operators $L_j := L_j(x|\partial)$, $j = \overline{1,r}$, considered above don't depend on the variable $t \in \mathrm{T}^r \subset \mathbb{R}^r_+$. Then, evidently, one can take

$$\mathcal{H}_0 := \{\psi^{(0)}_\mu(\xi) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : L_j\psi^{(0)}_\mu(\xi) = \mu_j\psi^{(0)}_\mu(\xi),\ \ j = \overline{1,r},$$
$$\psi^{(0)}_\mu(\xi)|_\Gamma = 0,\ \mu := (\mu_1,...,\mu_r) \in (\sigma(\tilde{L}) \cap \bar{\sigma}(L^*)),\ \xi \in \Sigma_\sigma\},$$

$$\tilde{\mathcal{H}}_0 := \{\tilde{\psi}^{(0)}_\mu(\xi) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : \tilde{L}_j\tilde{\psi}^{(0)}_\mu(\xi) = \mu_j\tilde{\psi}^{(0)}_\mu(\xi),\ \ j = \overline{1,r},$$
$$\tilde{\psi}^{(0)}_\mu(\xi)|_{\tilde{\Gamma}} = 0,\ \mu := (\mu_1,...,\mu_r) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*),\ \xi \in \Sigma_\sigma\},$$

$$\mathcal{H}_0^* := \{\varphi^{(0)}_\lambda(\eta) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : L_j\varphi^{(0)}_\lambda(\eta) = \bar{\lambda}_j\varphi^{(0)}_\lambda(\eta),\ \ j = \overline{1,r},$$
$$\varphi^{(0)}_\lambda(\eta)|_\Gamma = 0,\ \lambda := (\lambda_1,...,\lambda_r) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*),\ \eta \in \Sigma_\sigma\},$$

$$\tilde{\mathcal{H}}_0^* := \{\tilde{\varphi}^{(0)}_\lambda(\eta) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : \tilde{L}_j\tilde{\varphi}^{(0)}_\lambda(\eta) = \bar{\lambda}_j\tilde{\varphi}^{(0)}_\lambda(\eta),\ \ j = \overline{1,r},$$
$$\tilde{\varphi}^{(0)}_\lambda(\eta)|_{\tilde{\Gamma}} = 0,\ \lambda := (\lambda_1,\ldots,\lambda_r) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*),\ \eta \in \Sigma_\sigma\}$$

and construct the corresponding Delsarte transmutation operators

$$\mathbf{\Omega}_{\pm} = \mathbf{1} - \int\limits_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) d\rho_{\Sigma_\sigma}(\eta) \times$$

$$\times \int\limits_{S_{\pm}^{(s)}(\sigma_x^{(s-1)},\sigma_{x_0}^{(s-1)})} dx \qquad \tilde{\psi}_\lambda^{(0)}(\xi)\Omega_{(x_0)}^{-1}(\lambda;\xi,\eta)\bar{\varphi}_\lambda^{(0),\intercal}(\eta)(\cdot)$$

and

$$\mathbf{\Omega}_{\pm}^{\circledast} = \mathbf{1} - \int\limits_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) d\rho_{\Sigma_\sigma}(\eta) \times$$

$$\times \int\limits_{S_{\pm}^{(s)}(\sigma_x^{(s-1)},\sigma_{x_0}^{(s-1)})} dx \qquad \tilde{\varphi}_\lambda^{(0)}(\xi)\bar{\Omega}_{(x_0)}^{\intercal,-1}(\lambda;\xi,\eta) \times \bar{\psi}_\lambda^{(0),\intercal}(\eta)(\cdot), \tag{3.24}$$

acting already on the Hilbert space $L_2(\mathbb{R}^s;\mathbb{C}^N)$, where, for any $(\lambda;\xi,\eta) \in (\sigma(\tilde{L})\cap\bar{\sigma}(L^*))\times\Sigma_\sigma^2$, kernels

$$\Omega_{(x_0)}(\lambda;\xi,\eta) := \int\limits_{\sigma_{x_0}^{(s-1)}} \Omega^{(s-1)}[\varphi_\lambda^{(0)}(\xi), \psi_\lambda^{(0)}(\eta)dx],$$

$$\Omega_{(x_0)}^{\circledast}(\lambda;\xi,\eta) := \int\limits_{\sigma_{x_0}^{(s-1)}} \bar{\Omega}^{(s-1),\intercal}[\varphi_\lambda^{(0)}(\xi), \psi_\lambda^{(0)}(\eta)dx]$$

belong to $L_2^\rho(\not\mathscr{k}_\sigma;\mathbb{C}) \times L_2^\rho(\not\mathscr{k}_\sigma;\mathbb{C})$ for every $\lambda \in \sigma(\tilde{L})\cap\bar{\sigma}(L^*)$ considered as a parameter. Moreover, since $\partial\mathbf{\Omega}_\pm/\partial t_j = 0$, $j = \overline{1,r}$, one easily gets a set of differential expressions, since

$$\mathcal{R}(\tilde{L}) := \{\tilde{L}_j(x|\partial) := \mathbf{\Omega}_\pm L_j(x|\partial)\mathbf{\Omega}_\pm^{-1} : j = \overline{1,r}\},$$

which is a ring of differential operators that commute with each other and act in $L_2(\mathbb{R}^s;\mathbb{C}^N)$, generated by the corresponding initial ring $\mathcal{R}(L)$.

Thus we have described above the ring $\mathcal{R}(\tilde{L})$ of multidimensional differential operators that commute with each other, which is generated by the initial ring $\mathcal{R}(L)$. This problem in the one-dimensional case was treated in detail before and effectively solved in [20, 21] by means of algebraic-geometric methods and inverse spectral transform techniques. Our approach gives another approach to this problem in multidimension and is of special interest due to its clear and explicit dependence on dimension of the differential operators.

4. A special case: soliton theory aspect. ***4.1.*** Consider our generalized de Rham–Hodge theory of a commuting set $\mathcal{L}$ of differential operators on a Hilbert space $\mathcal{H} := L_2(\mathrm{T}^2;H)$, $H := L_2(\mathbb{R}^s;\mathbb{C}^N)$, for the special case when $M := \mathrm{T}^2\times\bar{\mathbb{R}}^s$ and

$$\mathcal{L} := \{\mathrm{L}_j := \partial/\partial t_j - L_j(t;x|\partial) : t\in\mathrm{T}^2 := \mathrm{T}_1\times\mathrm{T}_2,\ t_j \in\ \mathrm{T}_j := [0,T_j) \subset \mathbb{R}_+,\ j = \overline{1,2}\}, \tag{4.1}$$

where, by definition,

$$L_j(t;x|\partial) := \sum_{|\alpha|=0}^{n(L_j)} a_\alpha^{(j)}(t;x)\partial^{|\alpha|}/\partial x^\alpha,$$

with the coefficients $a_\alpha^{(j)} \in C^1(\mathrm{T}^2; S(\mathbb{R}^s; \mathrm{End}\,\mathbb{C}^N))$, $\alpha \in \mathbb{Z}_+^s$, $|\alpha| = \overline{0, n(L_j)}$, $j = \overline{1,2}$. The corresponding scalar product is given now by

$$(\varphi, \psi) := \int_{\mathrm{T}^2} dt \int_{\mathbb{R}^s} dx < \varphi, \psi >$$

for any pair $(\varphi, \psi) \in H^* \times H$. The corresponding external differential is

$$d_{\mathcal{L}} := \sum_{j=1}^{2} dt_j \wedge \mathrm{L}_j + \sum_{i=1}^{s} dx_j \wedge c_j \mathbf{1},$$

where one assumes that for all $t \in \mathrm{T}^2$ and $x \in \mathbb{R}^s$ the commutator

$$[\mathrm{L}_1, \mathrm{L}_2] = 0.$$

This means, obviously, that the corresponding generalized de Rham–Hodge co-chain complexes

$$\begin{aligned} \mathcal{H} &\rightarrow \Lambda^0(M;\mathcal{H}) \overset{d_{\mathcal{L}}}{\rightarrow} \Lambda^1(M;\mathcal{H}) \overset{d_{\mathcal{L}}}{\rightarrow} \ldots \overset{d_{\mathcal{L}}}{\rightarrow} \Lambda^m(M;\mathcal{H}) \overset{d_{\mathcal{L}}}{\rightarrow} 0, \\ \mathcal{H} &\rightarrow \Lambda^0(M;\mathcal{H}) \overset{d_{\mathcal{L}}^*}{\rightarrow} \Lambda^1(M;\mathcal{H}) \overset{d_{\mathcal{L}}^*}{\rightarrow} \ldots \overset{d_{\mathcal{L}}^*}{\rightarrow} \Lambda^m(M;\mathcal{H}) \overset{d_{\mathcal{L}}^*}{\rightarrow} 0 \end{aligned}$$

are exact. Define now, due to (3.8) and (3.20), the closed subspaces $H_0^{\circledast}$ and $H_0 \subset H_-$ as follows:

$$\begin{aligned} \mathcal{H}_0 &:= \{\psi^{(0)}(\lambda;\eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M) : \partial\psi^{(0)}(\lambda;\eta)/\partial t_j = \\ &= L_j(t;x|\partial)\psi^{(0)}(\lambda;\eta),\ j = \overline{1,2}, \\ \psi^{(0)}(\lambda;\eta)|_{t=t_0} &= \psi_\lambda(\eta) \in H_-,\ \psi^{(0)}(\lambda;\eta)|_\Gamma = 0, \\ (\lambda;\eta) &\in \Sigma \subset (\sigma(L) \cap \bar{\sigma}(L^*)) \times \Sigma_\sigma\}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \mathcal{H}_0^* &:= \{\varphi^{(0)}(\lambda;\eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M) : -\partial\varphi^{(0)}(\lambda;\eta)/\partial t_j = \\ &= L_j(t;x|\partial)\varphi^{(0)}(\lambda;\eta),\ j = \overline{1,2}, \\ \varphi^{(0)}(\lambda;\eta)|_{t=t_0} &= \varphi_\lambda(\eta) \in H_-,\ \varphi^{(0)}(\lambda;\eta)|_\Gamma = 0, \\ (\lambda;\eta) &\in \Sigma \subset (\sigma(L) \cap \bar{\sigma}(L^*)) \times \Sigma_\sigma\} \end{aligned}$$

for some hypersurface $\Gamma \subset M$ and a "spectral"degeneration set $\Sigma_\sigma \in \mathbb{C}^{p-1}$. By means of subspaces (4.2) one can now proceed to the construction of Delsarte transmutation operators $\mathbf{\Omega}_\pm : H \to H$ in the general form like (3.24) with the kernels $\Omega_{(t_0;x_0)}(\lambda;\xi,\eta) \in L_2^\rho(\Sigma_\sigma;\mathbb{C}) \otimes \otimes L_2^\rho(\Sigma_\sigma;\mathbb{C})$ for every $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$. The are defined by

$$\Omega_{(t_0;x_0)}(\lambda;\xi,\eta) := \int\limits_{\sigma^{(s-1)}_{t_0;x_0}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda;\xi),\psi^{(0)}(\lambda;\eta)dx],$$

$$\Omega^{\circledast}_{(t_0;x_0)}(\lambda;\xi,\eta) := \int\limits_{\sigma^{(s-1)}_{t_0;x_0}} \bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda;\xi),\psi^{(0)}(\lambda;\eta)dx]$$

for all $(\lambda;\xi,\eta) \in (\sigma(L) \cap \bar{\sigma}(L^*)) \times \Sigma_\sigma^2$. As a result one gets for $\rho := \rho_\sigma \odot \rho_{\Sigma_\sigma^2}$ the integral expressions

$$\begin{aligned}
\mathbf{\Omega}_\pm &= \mathbf{1} - \int\limits_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta)\times \\
&\times \int\limits_{S_\pm^{(s)}(\sigma^{(s-1)}_{t_0;x},\sigma^{(s-1)}_{t_0;x_0})} dx \quad \tilde{\psi}^{(0)}(\lambda;\xi)\Omega^{-1}_{(t_0;x_0)}(\lambda;\xi,\eta)\bar{\varphi}^{(0),\intercal}(\lambda;\eta)(\cdot), \\
\mathbf{\Omega}^{\circledast}_\pm &= \mathbf{1} - \int\limits_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta)\times \\
&\times \int\limits_{S_\pm^{(s)}(\sigma^{(s-1)}_{t_0;x},\sigma^{(s-1)}_{t_0;x_0})} dx \quad \tilde{\varphi}^{(0)}_\lambda(\xi)\bar{\Omega}^{\intercal,-1}_{(t_0;x_0)}(\lambda;\xi,\eta)\times\bar{\psi}^{(0),\intercal}(\lambda;\eta)(\cdot),
\end{aligned} \tag{4.3}$$

where $S_+^{(s)}(\sigma^{(s-1)}_{t_0;x},\sigma^{(s-1)}_{t_0;x_0}) \in C_s(M;\mathbb{C})$ is some smooth s-dimensional surface between two homological cycles $\sigma^{(s-1)}_{t_0;x}$ and $\sigma^{(s-1)}_{t_0;x_0} \in K(M)$ and $S_-^{(s)}(\sigma^{(s-1)}_{t_0;x},\sigma^{(s-1)}_{t_0;x_0}) \in C_s(M;\mathbb{C})$ is its smooth counterpart such that $\partial(S_+^{(s)}(\sigma^{(s-1)}_{t_0;x},\sigma^{(s-1)}_{t_0;x_0}) \cup S_-^{(s)}(\sigma^{(s-1)}_{t_0;x},\sigma^{(s-1)}_{t_0;x_0})) = \varnothing$. Following the results of Section 3 one can construct from (4.3) the corresponding factorized Fredholm operators $\mathbf{\Omega}$ and $\mathbf{\Omega}^{\circledast} : H \to H$, $H = L_2(\mathbb{R};\mathbb{C}^N)$ as follows:

$$\mathbf{\Omega} := \mathbf{\Omega}_+^{-1}\mathbf{\Omega}_-, \quad \mathbf{\Omega}^{\circledast} := \mathbf{\Omega}_+^{\circledast-1}\mathbf{\Omega}_-^{\circledast}.$$

It is also important to notice here that the kernels $\hat{K}_\pm(\mathbf{\Omega})$ and $\hat{K}_\pm(\mathbf{\Omega}^{\circledast}) \in H_- \otimes H_-$ satisfy exactly the generalized [10] determining equations in the following tensor form:

$$(\tilde{\mathcal{L}} \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega})(\mathbf{1}\otimes\mathcal{L}^*)\hat{K}_\pm(\mathbf{\Omega}),$$

$$(\tilde{\mathcal{L}}^* \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega}^{\circledast}) = (\mathbf{1}\otimes\mathcal{L})\hat{K}_\pm(\mathbf{\Omega}^{\circledast}).$$

Since, evidently, $\operatorname{supp}\hat{K}_+(\mathbf{\Omega})\cap\operatorname{supp}\hat{K}_-(\mathbf{\Omega}) = \varnothing$ and $\operatorname{supp}\hat{K}_+(\mathbf{\Omega}^{\circledast})\cup\operatorname{supp}\hat{K}_-(\mathbf{\Omega}^{\circledast}) = \varnothing$, one deduces from the results in [22, 23] that the corresponding Gelfand – Levitan – Marchenko type equations

$$\hat{K}_+(\mathbf{\Omega}) + \hat{\Phi}(\mathbf{\Omega}) + \hat{K}_+(\mathbf{\Omega}) * \hat{\Phi}(\mathbf{\Omega}) = \hat{K}_-(\mathbf{\Omega}),$$

$$\hat{K}_+(\mathbf{\Omega}^{\circledast}) + \hat{\Phi}(\mathbf{\Omega}^{\circledast}) + \hat{K}_+(\mathbf{\Omega}^{\circledast}) * \hat{\Phi}(\mathbf{\Omega}^{\circledast}) = \hat{K}_-(\mathbf{\Omega}^{\circledast}),$$

where, by definition, $\mathbf{\Omega} := \mathbf{1} + \hat{\Phi}(\mathbf{\Omega})$, $\mathbf{\Omega}^{\circledast} := 1 + \hat{\Phi}(\mathbf{\Omega}^{\circledast})$, can be solved [22] in the space $\mathcal{B}_\infty^\pm(H)$ for the kernels $\hat{K}_\pm(\mathbf{\Omega})$ and $\hat{K}_\pm(\mathbf{\Omega}^{\circledast}) \in H_-\otimes H_-$ that depend parametrically on $t \in \mathrm{T}^2$. Thereby, the Delsarte transformed differential operators $\tilde{\mathrm{L}}_j : \mathcal{H} \to \mathcal{H}$, $j = \overline{1,2}$, will, evidently, commute with each other too, satisfying the following operator relations:

$$\tilde{\mathrm{L}}_j = \partial/\partial t_j - \mathbf{\Omega}_\pm L_j \mathbf{\Omega}_\pm^{-1} - (\partial\mathbf{\Omega}_\pm/\partial t_j)\mathbf{\Omega}_\pm^{-1} := \partial/\partial t_j - \tilde{L}_j, \tag{4.4}$$

where the operator expressions for $\tilde{L}_j : H \to H$, $j = \overline{1,2}$, are purely differential. The latter property makes it possible to construct some nonlinear, in general, partial differential equations for the coefficients of differential operators (4.4) and solve them by means of standard procedures either using the inverse spectral transform [3, 20] or the Darboux – Backlund [5] transformation, producing a wide class of exact soliton like solutions. Another not simple and very interesting aspect of the approach suggested in this paper concerns regular algorithms of treating differential operator expressions depending on a "spectral"parameter $\lambda \in \mathbb{C}$, which was just very recently discussed in [23].

5. Conclusion. Consider a differential operator $\mathrm{L} : \mathcal{H} \to \mathcal{H}$ in the form (2.1) assume that its spectrum is known. By means of the general form of the Delsarte transmutation operators (3.23) one can construct a more complicated transformed differential operator $\tilde{\mathcal{L}} := \mathbf{\Omega}_\pm\mathcal{L}\mathbf{\Omega}_\pm^{-1}$ on $\mathcal{H}$ with a different spectrum. These Delsarte transformed operators can be effectively used for both studying spectral properties of differential operators [3, 4, 10, 11, 24] and constructing a wide class of nontrivial differential operators with a prescribed spectrum as it was done [3, 20] in one dimension.

As was shown before in [6, 24] for the two-dimensional Dirac and three-dimensional perturbed Laplace operators, the kernels of the corresponding Delsarte transmutation operator satisfy some special linear integral equations of Fredholm type, called the Gelfand – Levitan – Marchenko equations, which are very important for solving the corresponding inverse spectral problem and have many applications in modern mathematical physics. Such equations can be naturally constructed for our multidimensional case too, thereby making it possible to pose the corresponding inverse spectral problem for describing a wide class of multidimensional operators with a priori given spectral characteristics. The mentioned problem appears (see [10]) to be strongly related to that of a spectral representation of kernels commuting in some sense with a given pair of differential operators. Also, similar to [6, 25], one can use such results for studying the so-called completely integrable nonlinear evolution equations, especially for constructing by means of special Darboux type transformations [5, 7] their exact solutions like solitons and many others. Such an activity is now in progress and the corresponding results will be published later.

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