Nonlinear Oscillations, Vol. 4, No. 3, 2001

APPLICATION OF INVERSE SCATTERING TRANSFORM TO THE PROBLEMS OF GENERALIZED AMPLITUDE MODULATION OF WAVES

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The notion of the generalized amplitude modulation of oscillations and waves is introduced. The inverse Scattering Transform Method is used to investigate the problem of generalized amplitude modulation for the Korteweg – de Vries equation. Some theorems on these problems are presented.

AMS Subject Classification: 34, 35Q58

1. Introduction and Definitions

Some applications of the Inverse Scattering Transform (IST) are represented. The IST was discovered in the papers of the outstanding scientists: M.D. Kruskal, C.S. Gardner, J.M. Greene, R.M. Miura, P.D. Lax, V.E. Zakharov, A.B. Shabat, S.P. Novikov, V.O. Marchenko, L.D.Faddeev, and others (see [1 – 8]).

The Korteweg – de Vries (KdV) equation has been shown to describe the asimptotic development of small — but finite amplitude shallow — water waves, hydromagnetic waves in a cold plasma, ion-acoustic waves, acoustic waves in an anharmonic crystal (see [1, 2]). In this paper, the author describes a new field of applications for KdV-equation using the IST. These new applications lie in some problems formulated by analogy with the problems of radio communications, mathematical theory of communication, electronics. Our basic assumptions are connected with the notion of a generalized amplitude modulation for the KdV equation in spaces with weak dispertion.

In this paper we deal with application of the IST to finding a solution of a Cauchy problem for the KdV equation with initial conditions that satisfy the condition of generalized amplitude modulation.

The notion of amplitude modulation of oscillations and waves has been introduced in electrical engineering, radio communication, electronics, mathematical theory of communication. The simplest case deals with the notion of amplitude modulation for a harmonic oscillation.

Definition 1. *Let $s(x) = A\sin(\omega x + \varphi)$ be a harmonic oscillation with constant amplitude A. The amplitude modulation of the oscillation $s(x)$ with a function $m(x) \geq 0$ consists of forming the product $s_m(x)$,*

$$s_m(x) = Am(x)\sin(\omega x + \varphi), \tag{1}$$

where the amplitude function $Am(x)$ for $s_m(x)$ is already a variable and is specified depending on the applications.

Existence of periodic solutions of the Korteweg – de Vries equation has been proved in 1974 by S.P.Novikov, V.O. Marchenko, and P.D. Lax, see [1, 3].

Apart from periodic and oscillation solutions, there are soliton and N-soliton solutions of the Korteweg – de Vries equation. Therefore, the generalized amplitude modulation using solitons and other decreasing on $\pm\infty$ functions (waves) is a vital and actual problem.

Definition 2. *Let $s(x)$ be a bounded and Stepanov oscillating solution of some differential equation on the axis $-\infty < x < \infty$, and let $f(x)$ be a function decreasing on $\pm\infty$ and satisfying the following condition:*

$$\int_{-\infty}^{\infty} (1 + |x|)|f(x)|dx < \infty. \tag{2}$$

A generalized amplitude modulation of $s(x)$ using the function $f(x)$, by definition, is the product

$$s_f(x) = f(x)s(x).$$

2. Statement of the Problem

Let

$$v_t = 6vv_x - v_{xxx} \tag{3}$$

be the Korteweg – de Vries (KdV) equation. Let the initial condition function, $v(x, 0)$, be given by product

$$v(x, 0) = f(x)s(x). \tag{4}$$

The aim of this paper is to solve the Cauchy problem (3), (4) by using the Inverse Scattering Transform (IST) Method. Our basic assumption is that the initial function $v(x, 0)$ is a generalized amplitude modulation as in Definition 2.

Our remark to this problem is the following. V.E.Zakharov and A.B.Shabat (see [7]) have solved the nonlinear Schrödinger equation by using the IST. The soliton they have constructed is an envelope of waves in nonlinear optics (of course, we are interested only in stable amplitude modulations for the nonlinear Schrödinger equation).

We shall be working under the following convention. Let $v(x, t)$ be a solution of the initial Cauchy problem (3), (4). We shall say that the problem of generalized amplitude modulation has a positive solution, if $v(x, t)$ satisfies condition (2) or (5) for fixed finite t, $0 \leq t < \infty$.

3. New Results Obtained by the Author (a Lemma and Theorems)

Let $L = -d^2/dx^2 + q(x)$ and $L : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$. We will suppose that the domain of definition of the operator L is sufficient for L to be selfadjoint, $L = L^*$.

Lemma 1. *Let the following assumptions 1) – 3) hold.*

1) *The function $q(x)$ has three derivatives and let the condition*

$$\int_{-\infty}^{\infty} (1+|x|) \left| q^{(k)}(x) \right| dx < \infty, \quad k = 0, 1, 2, 3, \tag{5}$$

hold;

2) *The operator $L = -d^2/dx^2 + q(x)$, $L : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$ has an eigenfunction $f_1(x)$ corresponding to an eigenvalue λ_1, $Lf_1(x) = \lambda_1 f_1(x)$, where $f_1 \in L_2(-\infty, \infty)$ and $\lambda_1 < 0$;*

3) *Let a number $\lambda_2 > 0$ be an eigenvalue of the continuous spectrum of the operator L with a corresponding eigenfunction $f_2(x)$ and let $\lambda_2 = d|\lambda_1|$, where $d >> 1$.*

Then the function $v(x) = f_1(x) f_2(x)$ is a solution of the equation

$$-\varphi'''(x) + 2 \left(q(x) \frac{d}{dx} + \frac{d}{dx} q(x) \right) \varphi(x) = 4\mu\varphi'(x), \tag{6}$$

where the number $\mu \neq 0$ is obtained via λ_1 and λ_2, and $v \in L_2(-\infty, \infty)$.

Theorem 1. *Let the initial function*

$$v(x, 0) = f_1(x) f_2(x) \tag{7}$$

in Cauchy problem (3), (4) be a generalized amplitude modulation (see Definition 2) and let $v(x, 0) = f_1(x) f_2(x)$ be a solution of equation (6) under the conditions of Lemma. Then there exists a solution of the Cauchy problem (3), (4), $v(x, t)$, and satisfies the following conditions:

$$\max_{0 \le t \le T} \int_{-\infty}^{\infty} (1+|x|) \left| v^{(i)}(x, t) \right| dx < \infty, \quad i = 0, 1, 2, 3, \tag{8}$$

for all $T > 0$, $T \neq \infty$.

As a consequence of Theorem 1, we have a positive solution of the problem of generalized amplitude modulation with the condition

$$\int_{-\infty}^{\infty} (1+|x|) \, |v(x, t)| \, dx < \infty \tag{9}$$

for all $t > 0$, $t \neq \infty$, with the initial generalized amplitude modulation $v(x, 0)$, $t = 0$, and $v(x, t), t \geq 0, \ t \neq \infty$.

Remark 1. The manifold of KdV-solutions has a certain parametric property. We can now formulate this parametric property using invariance of the Korteweg – de Vries (KdV) equation (3) under the Galilei transformation (for the Galilei transformation for the KdV equation, see [2, 5]). If $v(x, t)$ is a solution of the KdV equation (3), then

$$u(x, t, \tau) = v(x - 6\tau t, t) - \tau, \quad \tau \text{ is a parameter,} \tag{10}$$

is also a solution of equation (3). If $\tau \neq 0$, the solution $u(x,t,\tau)$ does not satisfy the property (8). But the first term $v(x-6\tau t,t)$ does satisfy (8). We conclude also that the KdV model has the property of the phase-parametric modulation of the solution $u(x,t,\tau)$ in the variable x via the phase function $\varphi(t,\tau) = -6\tau t$ for evolution in time t and $\tau \neq 0$.

Now suppose that the conditions in the following Definition 3 hold, for Theorem 2 and Theorem 3, instead of the previous Definition 2.

Definition 3. *Let $s(x)$ be a periodic three times differentiable function defined on the whole real axis and let $f(x)$ be a three times differentiable function satisfying the condition*

$$\int_{-\infty}^{\infty} (1+|x|)\left|f^{(i)}(x)\right| dx < \infty, \quad i = 0,1,2,3. \tag{11}$$

Let $s_f(x)$ be defined by $s_f(x) = f(x)s(x)$. The function $s_f(x)$ is said to be a generalized amplitude modulation of $s(x)$ by using the function $f(x)$.

In the following Theorem 2 we will touch only the case of the oscillating function $f(x) = \varphi(x)$ with the compact support.

Theorem 2. *Let the following assumptions 1) – 3) hold.*

1) *Let $s(x)$ be a periodic three times differentiable function defined on the whole real axis;*

2) *Let $\varphi(x)$ be a three times differentiable oscillating function with the compact support on the real axis;*

3) *Let the frequency of oscillation, ω_φ, of $\varphi(x)$ on its support and the frequency ω_s of the function $s(x)$ be so that the inequality $\omega_\varphi << \omega_s$ holds.*

Then there exists a solution $v(x,t)$ of the Cauchy problem (3), (4) satisfying the following condition (8),

$$\max_{0\le t\le T} \int_{-\infty}^{\infty} (1+|x|)\left|v^{(i)}(x,t)\right| dx < \infty, \quad i = 0,1,2,3,$$

for all $T > 0$, $T \neq \infty$.

Remark 2. The condition $\omega_\varphi << \omega_s$ in 3) of Theorem 2 is essential. This condition guarantees a nonresonant generalized amplitude modulation of oscillations. Theorem 2 is very important in applications. The following example can be useful in applications. Let $\varphi_0(x)$ belong to $C_0^\infty(\mathbf{R})$ and let $f(x)$ be a periodic three times differentiable function on the whole real axis. Let

$$\varphi(x) = \varphi_0(x)f(x)$$

satisfy condition 3) of Theorem 2. Then for the initial conditions

$$v(x,0) = \varphi_0(x)f(x)s(x) = \varphi(x)s(x),$$

the conclusion of Theorem 2 is true.

Theorem 3. *Let $s(x)$ be a periodic three times differentiable function on the whole real axis. Let $f(x)$ be a three times differentiable function and let $f(x)$ satisfy the assumption (11) (see Definition 3). Then, for the initial data $v(x,0) = f(x)s(x)$, the solution $v(x,t)$ of the Cauchy problem (3), (4) exists and satisfies the condition (8) for all $T > 0$, $T \neq \infty$.*

Let conditions of Theorem 3 hold. Then for the solution $v(x,t)$ of the KdV-equation above, Remark 1 holds true.

The main result of paper [2] as well as previous to [2] numerical computations are that the solution of an initial-value problem for the KdV-equation (3) may be an N-soliton solution. In the following Theorem 4, we shall be using the factorization condition of reflectionless potentials instead of the condition of the generalized amplitude modulation. For the definition of reflectionless potentionals, see [1–4, 9].

Theorem 4. *Let $V(x,0) = f_1(x)f_2(x)$ be a reflectionless potential of a selfadjoint Schrödinger operator $L : L_2(\mathbf{R}) \to L_2(\mathbf{R})$. Then*

1) *the function $V(x,0)$ satisfies the condition*

$$\int_{-\infty}^{\infty} e^{\varepsilon|x|}|V(x,0)|dx < \infty \qquad \text{for some } \varepsilon > 0;$$

2) *the solution $V(x,t)$ of the Cauchy problem (3), (4) corresponding to the initial function $V(x,0) = f_1(x)f_2(x)$ is an N-soliton solution with the following representation:*

$$\begin{aligned} V(x,t) &= -2\frac{\partial^2}{\partial x^2}\ln\Delta(x,t), \\ \Delta(x,t) &= \mathrm{Det}\left(\delta_{kl} + m_k^2(0)\frac{e^{-(\kappa_k+\kappa_l)x}e^{8\kappa_k^3 t}}{\kappa_k+\kappa_l}\right), \end{aligned}$$

where $\{\kappa_l > 0, m_k > 0\}$ is reflectionless scattering data of the Schrödinger operator $L : L_2(\mathbf{R}) \to L_2(\mathbf{R})$ with the potential $V(x,0)$;

3) *$V(x,t)$ satisfies the condition*

$$\int_{-\infty}^{\infty} e^{\varepsilon|x|}|V(x,t)|dx < \infty, \quad \varepsilon > 0$$

for all fixed $t \in [0,\infty)$;

4) *the solution $V(x,t)$ of the initial Cauchy problem (3), (4) has the representation:*

$$V(x,t) = -2\left[\Delta(x,t)_x^{-1}\sum_{l=1}^{n}\tilde{\Delta}_l(x,t) + \Delta(x,t)^{-1}\sum_{l=1}^{n}\tilde{\Delta}_l(x,t)'_x\right],$$

where $\tilde{\Delta}_l(x,t) = \Delta_l(x,t)\exp\{-\kappa_l x\}$, by the definition, is the determinant obtained from the determinant $\Delta(x,t)$ by substituting the derivative of l-column instead of the l-column.

Remark 3. The proof of Theorem 4 is given as a consequence of [3] and [4, 9].

Remark 4. This paper belongs to the field of mathematics. The obtained results could have very interesting application in problems of media with weak dispersion. Phenomena of stable behaviour of the solution of the Korteweg – de Vries equation (3) with generalized amplitude modulation was mathematically substantiated in this article. On the problem of stability of solitons, also see paper [8].

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Received 29.12.2001