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# BOUNDARY-VALUE PROBLEMS FOR THE SYSTEM OF OPERATOR-DIFFERENTIAL EQUATIONS IN THE BANACH AND HILBERT SPACES*

# КРАЙОВІ ЗАДАЧІ ДЛЯ СИСТЕМИ ОПЕРАТОРНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ У ПРОСТОРАХ БАНАХА ТА ГІЛЬБЕРТА

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The necessary and sufficient conditions for the existence of solutions of linear and nonlinear boundary-value problems in the Hilbert and Banach spaces are obtained. A convergent iterative procedure for finding solutions in the nonlinear case is presented.

Одержано необхідну й достатню умови існування розв’язків лінійної та нелінійної крайових задач у просторах Гільберта й Банаха. Наведено збіжну ітераційну процедури для знаходження розв’язків у нелінійному випадку.

**Introduction.** In this work, we develop constructive methods of analysis of linear and nonlinear boundary-value problems for the operator-differential equations in the Banach and Hilbert spaces. Such problems occupy a central place in the qualitative theory of differential equations [1 – 18]. The specific feature of these problems is that the operator of the linear part of the equation does not have an inverse. These does not allow one to use the traditional methods based on the principles of contracting mappings and a fixed point. For the analysis of a nonlinear system of differential equations, we develop the ideas of the Lyapunov – Schmidt method and efficient methods of perturbation theory with using the theory of generalized inverse [19] and strongly generalized inverse operators [20].

**Statement of the problem.** Consider the following boundary-value problem:

$$\begin{cases} \varphi'(t,\varepsilon) = \varphi(t,\varepsilon) + \psi(t,\varepsilon) + \varepsilon f_1(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon) + g_1(t), \\ \psi'(t,\varepsilon) = \varphi(t,\varepsilon) + \varepsilon f_2(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon) + g_2(t), \quad t \in J, \end{cases} \tag{1}$$

with boundary condition

$$l(\varphi(\cdot,\varepsilon),\psi(\cdot,\varepsilon)) = \alpha, \tag{2}$$

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where $\varphi, \psi \in C^1(J,\mathcal{H})$, $C^1(J,\mathcal{H})$ is the Banach space of continuously differentiable vector-functions on the interval $J \subset \mathbb{R}$ with values in the Hilbert space $\mathcal{H}$; vector-functions $f_1$, $f_2$ are strongly differentiable; $l$ is a linear and bounded operator which translates solutions of (1) into the Hilbert space $\mathcal{H}_1$; vector-functions $g_1(t), g_2(t) \in C(J,\mathcal{H})$. We find the necessary and sufficient conditions of the existence of solutions $\varphi(t,\varepsilon)$, $\psi(t,\varepsilon)$ of the boundary-value problem (1), (2) which for $\varepsilon = 0$ turn in one of solutions of the generating linear boundary-value problem in the following form:

$$\begin{cases} \varphi_0'(t) = \varphi_0(t) + \psi_0(t) + g_1(t), \\ \psi_0'(t) = \varphi_0(t) + g_2(t), \quad t \in J, \end{cases}$$

$$l(\varphi_0(\cdot), \psi_0(\cdot)) = \alpha.$$

At first, we investigate a generating linear case.

**Linear case.** Consider the linear boundary-value problem

$$\begin{cases} \varphi_0'(t) = \varphi_0(t) + \psi_0(t) + g_1(t), \\ \psi_0'(t) = \varphi_0(t) + g_2(t), \quad t \in J, \end{cases} \tag{3}$$

$$l(\varphi_0(\cdot), \psi_0(\cdot)) = \alpha. \tag{4}$$

Denote by $U(t)$ an evolution operator of homogeneous system

$$\begin{cases} \varphi_0'(t) = \varphi_0(t) + \psi_0(t), \\ \psi_0'(t) = \varphi_0(t), \quad t \in J, \end{cases}$$

$$U'(t) = AU(t), \quad U(0) = I,$$

where the matrix operator-valued function has the form

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and the evolution operator $U(t)$ has the form

$$U(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} t^n F_{n+1} & t^n F_n \\ t^n F_n & t^n F_{n-1} \end{pmatrix},$$

where $F_n$ is a Fibonacci sequence:

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 0,$$

or

$$U(t) = \begin{pmatrix} \dfrac{1+\sqrt{5}}{2\sqrt{5}}\, e^{\frac{1+\sqrt{5}}{2}t} + \dfrac{\sqrt{5}-1}{2\sqrt{5}}\, e^{\frac{1-\sqrt{5}}{2}t} & e^{\frac{1+\sqrt{5}}{2}t} - e^{\frac{1-\sqrt{5}}{2}t} \\ e^{\frac{1+\sqrt{5}}{2}t} - e^{\frac{1-\sqrt{5}}{2}t} & \dfrac{2}{5+\sqrt{5}}\, e^{\frac{1+\sqrt{5}}{2}t} + \dfrac{2}{5-\sqrt{5}}\, e^{\frac{1-\sqrt{5}}{2}t} \end{pmatrix}. \tag{5}$$

In this case the set of solutions of the gather (3) has the form

$$\begin{pmatrix}\varphi_0(t,c)\\ \psi_0(t,c)\end{pmatrix} = e^{tA}c + \int\limits_0^t e^{(t-\tau)A}g(\tau)d\tau = \sum_{n=0}^{+\infty}\frac{1}{n!}\begin{pmatrix}t^nF_{n+1}c_1 + t^nF_nc_2\\ t^nF_nc_1 + t^nF_{n-1}c_2\end{pmatrix} +$$

$$+\sum_{n=0}^{+\infty}\frac{1}{n!}\int\limits_0^t \begin{pmatrix}(t-\tau)^nF_{n+1}g_1(\tau) + (t-\tau)^nF_ng_2(\tau)\\ (t-\tau)^nF_ng_1(\tau) + (t-\tau)^nF_{n-1}g_2(\tau)\end{pmatrix}d\tau,$$

where $c = (c_1, c_2)^T,\ c_1, c_2 \in \mathcal{H},\ g(t) = (g_1(t), g_2(t))^T$ (or with using of representation (5)). Subsituting in the boundary condition (4) we obtain the following operator equation:

$$Qc = \alpha - l\int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau, \quad Q = lU(\cdot)\colon \mathcal{H} \to \mathcal{H}_1.$$

By using the theory of strong generalized solutions [21], we can obtain the following result:

$$Qc = \alpha - l\int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau, \qquad Q = lU(\cdot)\colon \mathcal{H}\times\mathcal{H} \to \mathcal{H}_1.$$

**Theorem 1.** *1. (a) Boundary-value problem (3), (4) has strongly generalized solutions if and only if the following condition holds:*

$$\mathcal{P}_{N(\overline{Q}^*)}\left\{\alpha - l\int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau\right\} = 0; \tag{6}$$

*if* $\alpha - l\int_0^{\cdot} U(\cdot)U^{-1}(\tau)f(\tau)d\tau \in R(Q)$, *then generalized solutions will be classical;*

*(b) under condition (6) the set of solutions has the form*

$$\begin{pmatrix}\varphi_0(t,\overline{c})\\ \psi_0(t,\overline{c})\end{pmatrix} = U(t)\mathcal{P}_{N(\overline{Q})}\overline{c} + \overline{(G[g,\alpha])(t)} \quad \forall\,\overline{c}\in\mathcal{H}, \tag{7}$$

*where* $\mathcal{P}_{N(\overline{Q})},\ \mathcal{P}_{N(\overline{Q}^*)}$ *are the orthoprojectors onto the kernel and cokernel of the operator* $\overline{Q}$ *respectively* ($\overline{Q}$ *is the extension of the operator* $Q$ *[20]),*

$$\overline{(G[g,\alpha])(t)} = \int\limits_0^t U(t)U^{-1}(\tau)g(\tau)d\tau + \overline{Q}^+\left\{\alpha - l\int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau\right\}$$

*is a generalized Green's operator,* $\overline{Q}^+$ *is strongly Moore – Penrose pseudoinvertible operator [20];*

*2. (a) boundary-value problem (3), (4) has strongly pseudosolutions if and only if the following condition holds:*

$$\mathcal{P}_{N(\overline{Q}^*)}\left\{\alpha - l\int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)f(\tau)d\tau\right\} \neq 0; \tag{8}$$

*(b) under condition (8) the set of strongly pseudosolutions has the form*

$$\begin{pmatrix}\varphi_0(t,\overline{c})\\ \psi_0(t,\overline{c})\end{pmatrix} = U(t)\mathcal{P}_{N(\overline{Q})}\overline{c} + \overline{(G[g,\alpha])(t)} \quad \forall\, \overline{c}\in\mathcal{H}.$$

**Nonlinear case.** The following statement is hold.

**Theorem 2.** *Suppose that the boundary-value problem (1), (2) has solution which turns in one of solutions of generating boundary-value problem (3), (4) in the form (7) ($\varepsilon = 0$) with an element $\overline{c} = c_0$. Then the element $c_0$ satisfies the following operator equation for generating elements:*

$$F(c) = \mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) f(\tau,\varphi_0(\tau,c),\psi_0(\tau,c),0)d\tau = 0.$$

Here

$$f(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon) = \begin{pmatrix} f_1(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon)\\ f_2(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon)\end{pmatrix}.$$

***Proof.*** If the boundary-value problem (1), (2) has the solution, then from the theorem 1 follows that the following condition is true:

$$\mathcal{P}_{N(\overline{Q}^*)}\left\{\alpha - l\int_0^{\cdot} U(\cdot)U^{-1}(\tau)\left(g(\tau) + \varepsilon f(\tau,\varphi(\tau,\varepsilon),\psi(\tau,\varepsilon),\varepsilon)\right)d\tau\right\} = 0.$$

Since the boundary-value problem (1), (2) has the solution, then by using condition (6) we obtain finally ($\varepsilon \to 0$)

$$\mathcal{P}_{N(\overline{Q}^*)}\left\{l\int_0^{\cdot} U(\cdot)U^{-1}(\tau) f(\tau,\varphi_0(\tau,\overline{c}),\psi_0(\tau,\overline{c}),0)d\tau\right\} = 0.$$

For obtaining the sufficient condition of the existence of solutions we use the following change of variables:

$$\varphi(t,\varepsilon) = \overline{\varphi}(t,\varepsilon) + \varphi_0(t,c_0),$$
$$\psi(t,\varepsilon) = \overline{\psi}(t,\varepsilon) + \psi_0(t,c_0),$$

where the element $c_0$ satisfies the equation for generating elements. We obtain the boundary-value problem

$$\begin{aligned}\overline{\varphi}'(t,\varepsilon) &= \overline{\varphi}(t,\varepsilon) + \overline{\psi}(t,\varepsilon) + \varepsilon f_1(t,\overline{\varphi}(t,\varepsilon)+\varphi_0(t,c_0),\overline{\psi}(t,\varepsilon)+\psi_0(t,c_0),\varepsilon),\\ \overline{\psi}'(t,\varepsilon) &= \overline{\varphi}(t,\varepsilon) + \varepsilon f_2(t,\overline{\varphi}(t,\varepsilon)+\varphi_0(t,c_0),\overline{\psi}(t,\varepsilon)+\psi_0(t,c_0),\varepsilon),\end{aligned} \tag{9}$$

$$l(\overline{\varphi}(\cdot,\varepsilon),\overline{\psi}(\cdot,\varepsilon)) = 0. \tag{10}$$

Suppose that the vector-functions $f_1,\ f_2$ are strongly differentiable in the neighborhood of the generating solution

$$f_1, f_2 \in C^1(\|\varphi-\varphi_0\| \le q_1, \|\psi-\psi_0\| \le q_2),$$

$q_1,\ q_2$ are positive constants.

Use the following expansions:

$$f_1(t,\overline{\varphi}(t,\varepsilon)+\varphi_0(t,c_0),\overline{\psi}(t,\varepsilon)+\psi_0(t,c_0),\varepsilon)$$
$$= f_1(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)+f'_{1\varphi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\varphi}(t,\varepsilon)$$
$$+ f'_{1\psi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\psi}(t,\varepsilon)+\mathcal{R}_1(t,\overline{\varphi}(t,\varepsilon),\overline{\psi}(t,\varepsilon),\varepsilon),$$
$$f_2(t,\overline{\varphi}(t,\varepsilon)+\varphi_0(t,c_0),\overline{\psi}(t,\varepsilon)+\psi_0(t,c_0),\varepsilon)$$
$$= f_2(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)+f'_{2\varphi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\varphi}(t,\varepsilon)$$
$$+ f'_{2\psi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\psi}(t,\varepsilon)+\mathcal{R}_2(t,\overline{\varphi}(t,\varepsilon),\overline{\psi}(t,\varepsilon),\varepsilon),$$

where

$$\mathcal{R}_1(t,0,0,0)=\mathcal{R}'_{1\varphi}(t,0,0,0)=\mathcal{R}'_{1\psi}(t,0,0,0)=0,$$
$$\mathcal{R}_2(t,0,0,0)=\mathcal{R}'_{2\varphi}(t,0,0,0)=\mathcal{R}'_{2\psi}(t,0,0,0)=0.$$

Then we can rewrite the boundary-value problem (9) – (10) in the following form:

$$\begin{aligned}\overline{\varphi}' &= \overline{\varphi}+\overline{\psi}+\varepsilon\left\{f_1+f'_{1\varphi}\overline{\varphi}+f'_{1\psi}\overline{\psi}+\mathcal{R}_1\right\},\\ \overline{\psi}' &= \overline{\varphi}+\varepsilon\left\{f_2+f'_{2\varphi}\overline{\varphi}+f'_{2\psi}\overline{\psi}+\mathcal{R}_2\right\},\end{aligned} \tag{11}$$

$$l\big(\overline{\varphi}(\cdot,\varepsilon),\overline{\psi}(\cdot,\varepsilon)\big)=0. \tag{12}$$

Let

$$F(t,\varepsilon)=\begin{pmatrix} f_1+f'_{1\varphi}\overline{\varphi}+f'_{1\psi}\overline{\psi}+\mathcal{R}_1\\ f_2+f'_{2\varphi}\overline{\varphi}+f'_{2\psi}\overline{\psi}+\mathcal{R}_2\end{pmatrix}.$$

Under condition of solvability [19, 20]

$$\mathcal{P}_{N(\overline{Q}^*)}\left\{l\int_0^{\cdot}U(\cdot)U^{-1}(\tau)F(\tau,\varepsilon)d\tau\right\}=0, \tag{13}$$

the set of solutions of boundary-value problem (11), (12) has the following form:

$$\begin{pmatrix}\overline{\varphi}(t,\overline{c})\\ \overline{\psi}(t,\overline{c})\end{pmatrix}=U(t)\mathcal{P}_{N(\overline{Q})}\overline{c}+\varepsilon\overline{(G[F,0])(t)}\quad \forall\,\overline{c}\in\mathcal{H}.$$

Substituting representation of solutions in the condition (13) we obtain the operator equation

$$B_0\overline{c}=b, \tag{14}$$

where the operator $B_0$ has the form

$$B_0=\mathcal{P}_{N(\overline{Q}^*)}l\int_0^{\cdot}U(\cdot)U^{-1}(\tau)\begin{pmatrix}f'_{1\varphi} & f'_{1\psi}\\ f'_{2\varphi} & f'_{2\psi}\end{pmatrix}U(\tau)\mathcal{P}_{N(\overline{Q})}d\tau,$$

$$b = -\mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix} d\tau -$$

$$- \varepsilon \mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) \begin{pmatrix} f'_{1\varphi} & f'_{1\psi} \\ f'_{2\varphi} & f'_{2\psi} \end{pmatrix} \overline{G[F,0]}(\tau) d\tau.$$

Suppose that the following condition is hold:

$$\mathcal{P}_{N(\overline{B}_0^*)} \mathcal{P}_{N(\overline{Q}^*)} = 0.$$

Then the equation (14) is solvable. One of the solution has the form

$$c = \overline{B}_0^+ b.$$

In such a way we can obtain the following theorem.

**Theorem 3.** *Suppose that the following condition is hold:*

$$\mathcal{P}_{N(\overline{B}_0^*)} \mathcal{P}_{N(\overline{Q}^*)} = 0.$$

*Then for any element $c = c_0 \in \mathcal{H}$ which satisfies the equation for generating elements there exists solution of the boundary-value problem (1), (2). This solution can be found with the iterative procedure*

$$\begin{pmatrix} \overline{\varphi}_{k+1}(t, \overline{c}_k) \\ \overline{\psi}_{k+1}(t, \overline{c}_k) \end{pmatrix} = U(t) \mathcal{P}_{N(\overline{Q})} \overline{c}_k + \overline{h}_{k+1}(t, \varepsilon),$$

$$\overline{c}_k = -\overline{B}_0^+ \mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) \begin{pmatrix} \mathcal{R}_1(\tau, \overline{\varphi}_k, \overline{\psi}_k, \varepsilon) \\ \mathcal{R}_2(\tau, \overline{\varphi}_k, \overline{\psi}_k, \varepsilon) \end{pmatrix} d\tau -$$

$$- \overline{B}_0^+ \mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) \begin{pmatrix} f'_{1\varphi} & f'_{1\psi} \\ f'_{2\varphi} & f'_{2\psi} \end{pmatrix} \overline{h}_k(\tau, \varepsilon) d\tau,$$

$$\overline{h}_{k+1}(t, \varepsilon) = \varepsilon G[f(\cdot, \overline{\varphi}_k + \varphi_0, \overline{\psi}_k + \psi_0, \varepsilon), 0](t),$$

$$\mathcal{R}_1(t, \overline{\varphi}(t,\varepsilon), \overline{\psi}(t,\varepsilon), \varepsilon) = f_1(t, \overline{\varphi}(t,\varepsilon) + \varphi_0(t,c_0), \overline{\psi}(t,\varepsilon) + \psi_0(t,c_0), \varepsilon) -$$

$$- f_1(t, \varphi_0(t,c_0), \psi_0(t,c_0), 0) - f'_{1\varphi}(t, \varphi_0(t,c_0), \psi_0(t,c_0), 0)\overline{\varphi}(t,\varepsilon) -$$

$$- f'_{1\psi}(t, \varphi_0(t,c_0), \psi_0(t,c_0), 0)\overline{\psi}(t,\varepsilon),$$

$$\mathcal{R}_2(t, \overline{\varphi}(t,\varepsilon), \overline{\psi}(t,\varepsilon), \varepsilon) = f_2(t, \overline{\varphi}(t,\varepsilon) + \varphi_0(t,c_0), \overline{\psi}(t,\varepsilon) + \psi_0(t,c_0), \varepsilon) -$$

$$- f_2(t, \varphi_0(t,c_0), \psi_0(t,c_0), 0) - f'_{2\varphi}(t, \varphi_0(t,c_0), \psi_0(t,c_0), 0)\overline{\varphi}(t,\varepsilon) -$$

$$- f'_{2\psi}(t, \varphi_0(t,c_0), \psi_0(t,c_0), 0)\overline{\psi}_k(t,\varepsilon),$$

$$\varphi(t,\varepsilon) = \varphi_0(t,c_0) + \lim_{k\to\infty} \overline{\varphi}_k(t,\varepsilon),$$

$$\psi(t,\varepsilon)=\psi_0(t,c_0)+\lim_{k\to\infty}\overline{\psi}_k(t,\varepsilon).$$

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