UDC 517.555

ON BANACH SPACES AND FRÉCHET SPACES OF LAPLACE – STIELTJES INTEGRALS

ПРО ПРОСТОРИ БАНАХА І ФРЕШЕ ІНТЕГРАЛІВ ЛАПЛАСА – СТІЛТЬЄСА

A. O. Kuryliak, M. M. Sheremeta

Ivan Franko Nat. Univ. Lviv,
Universytetska str., 1, Lviv, 79000, Ukraine
e-mail: andriykuryliak@gmail.com
sheremeta.m.m@gmail.com

We investigate spaces of Laplace – Stieltjes integrals $I(\sigma)=\int_0^\infty f(x)e^{x\sigma}dF(x)$, where $\sigma\in\mathbb{R}$, the function F is nonnegative nondecreasing unbounded and right continuous on $[0,+\infty)$ and the function f is real-valued on $[0,+\infty)$. This integral is a generalization of the Dirichlet series $D(\sigma)=\sum_{n=1}^{\infty}d_ne^{\lambda_n\sigma}$ with nonnegative exponents λ_n increasing to $+\infty$ if $F(x)=n(x)=\sum_{\lambda_n\le x}1$, and $f(x)=d_n$ for $x=\lambda_n$ and $f(x)=0$ for $x\ne\lambda_n$.

For a positive continuous function h on $[0,+\infty)$ that increases to $+\infty$, by LS_h we denote a class of integrals I such that $|f(x)|\exp\{xh(x)\}\to 0$ as $x\to+\infty$ and define $\|I\|_h = \sup\{|f(x)|\exp\{xh(x)\}\colon x\ge 0\}$. We prove that if $F\in V$ and $\ln F(x)=o(x)$ as $x\to+\infty$, then $(LS_h,\|\cdot\|_h)$ is a nonuniformly convex Banach space. Other properties of the space LS_h and its dual space are also studied. As a consequence, for Banach spaces of Laplace–Stieltjes integrals of a finite generalized order, results are obtained. Some results are refined in the case where $I(\sigma)=D(\sigma)$.

In addition, for fixed $\varrho<+\infty$, let $\overline{S}_\varrho$ be a class of entire Dirichlet series $D(\sigma)$ such that their generalized order $\varrho_{\alpha,\beta}[D]:=\limsup_{\sigma\to+\infty}\dfrac{\alpha(\ln M(\sigma,D))}{\beta(\sigma)}\le\varrho$, where $M(\sigma,D)=\sum_{n=1}^{\infty}|d_n|e^{\sigma\lambda_n}$ and the functions α and β are positive continuous on $[x_0,+\infty)$ and increasing to $+\infty$. For $q\in\mathbb{N}$, let

$$\|D\|_{\varrho;q}=\sum_{n=1}^{\infty}|d_n|\exp\left\{\lambda_n\beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho+1/q}\right)\right\},\qquad d(D_1,\,D_2)=\sum_{q=1}^{\infty}\frac{1}{2^q}\frac{\|D_1-D_2\|_{\varrho;q}}{1+\|D_1-D_2\|_{\varrho;q}}.$$

The space with the metric d is denoted by $\overline{S}_{\varrho,d}$. We prove that $\overline{S}_{\varrho,d}$ is a Fréchet space under certain conditions on the functions α and β and the sequence (λ_n).

Досліджено простори інтегралів Лапласа – Стілтьєса $I(\sigma)=\int_0^\infty f(x)e^{x\sigma}dF(x)$, де $\sigma\in\mathbb{R}$, функція F невід'ємна, неспадна, необмежена й неперервна праворуч на $[0,+\infty)$, а функція f дійснозначна на $[0,+\infty)$. Цей інтеграл є узагальненням ряду Діріхле $D(\sigma)=\sum_{n=1}^{\infty}d_ne^{\lambda_n\sigma}$ з невід'ємними зростаючими до $+\infty$ показниками λ_n, якщо виберемо $F(x)=n(x)=\sum_{\lambda_n\le x}1$, а $f(x)=d_n$, якщо $x=\lambda_n$, і $f(x)=0$, якщо $x\ne\lambda_n$.

Для додатної неперервної на $[0,+\infty)$ зростаючої до $+\infty$ функції h через LS_h позначено клас таких інтегралів I, що $|f(x)|\exp\{xh(x)\}\to 0$ при $x\to+\infty$, і означено $\|I\|_h=\sup\big\{|f(x)|\exp\{xh(x)\}:x\ge 0\big\}$. Доведено, що коли $F\in V$ і $\ln F(x)=o(x)$ при $x\to+\infty$, тоді $(LS_h,\|\cdot\|_h)$ є нерівномірно опуклим простором Банаха. Вивчено інші властивості простору LS_h та його дуального простору. Як наслідок, отримано твердження про простори Банаха інтегралів Лапласа – Стілтьєса скінченного узагальненого порядку. Уточнено деякі результати у випадку, коли $I(\sigma) = D(\sigma)$.

Окрім цього, для фіксованого $\varrho < +\infty$ нехай $\overline{S}_{\varrho}$ — клас цілих рядів Діріхле $D(\sigma)$, для яких узагальнений порядок $\varrho_{\alpha,\beta}[D] := \limsup_{\sigma\to+\infty} \dfrac{\alpha(\ln M(\sigma, D))}{\beta(\sigma)} \le \varrho$, де $M(\sigma, D) = \sum_{n=1} |d_n| e^{\sigma\lambda_n}$, а функції $\alpha,\ \beta$ є додатними, неперервними й зростаючими до $+\infty$ на $[x_0, +\infty)$. Для $q \in \mathbb{N}$ нехай

$$\|D\|_{\varrho;q} = \sum\nolimits_{n=1}^{\infty} |d_n| \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\}, \quad d(D_1,\, D_2) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|D_1 - D_2\|_{\varrho;q}}{1 + \|D_1 - D_2\|_{\varrho;q}}.$$

Простір із метрикою d позначено через $\overline{S}_{\varrho,d}$. Доведено, що $\overline{S}_{\varrho,d}$ є простором Фреше за певних умов на функції $\alpha,\ \beta$ і послідовність (λ_n).

1. Introduction. Let V be a class of nonnegative nondecreasing unbounded continuous on the right functions F on $[0, +\infty)$. We assume that a real-valued function f on $[0, +\infty)$ is such that the Lebesgue – Stieltjes integral $\int_0^A f(x)e^{x\sigma} dF(x)$ exists for every $A \in [0, +\infty)$ and $\sigma \in \mathbb{R}$. The integral

$$I(\sigma) = \int_0^{\infty} f(x)e^{x\sigma} dF(x), \quad \sigma \in \mathbb{R}, \tag{1}$$

is called of Laplace – Stieltjes [1, p. 7]. Integral (1) is a direct generalization of the ordinary Laplace integral $I_o(\sigma) = \int_0^{\infty} f(x)e^{x\sigma} dx$ and of Dirichlet series

$$D(\sigma) = \sum_{n=1}^{\infty} d_n e^{\lambda_n \sigma} \tag{2}$$

with nonnegative exponents $\lambda_n,\ 0 \le \lambda_n \uparrow +\infty$ as $n \to \infty$, if we choose $F(x) = n(x) = \sum_{\lambda_n \le x} 1,\ f(x) = d_n$ if $x = \lambda_n$ and $f(x) = 0$ if $x \ne \lambda_n$ (see also [2 – 4]);

$$M(\sigma, I) := \int_0^{\infty} |f(x)| e^{x\sigma} dF(x), \quad \sigma \in \mathbb{R}, \tag{3}$$

and $\mu(\sigma, I) := \max\{|f(x)|e^{x\sigma} : x \ge 0\},\ \sigma \in \mathbb{R}$, be the maximum of the integrand. It is clear, that if $f(x) \ge 0$ for all $x \ge 0$, then $M(\sigma, I) = I(\sigma)$, and asymptotic properties of integrals of such a kind are studied in [1]. If σ_M is the abscissa of convergence of the integral (3) and σ_μ is the abscissa of maximum of the integrand, then $\sigma_M \ge \sigma_\mu$ [1, p. 13] provided $\ln F(x) = o(x)$ as $x \to +\infty$, [1, p. 8],

$$\sigma_\mu = \liminf_{x\to+\infty} \frac{1}{x} \ln \frac{1}{|f(x)|}.$$

From hence it follows that if $\ln F(x) = o(x)$ as $x \to +\infty$ and

$$\frac{1}{x} \ln \frac{1}{|f(x)|} \to +\infty, \quad x \to +\infty, \tag{4}$$

then $\sigma_M = +\infty$, i.e., integral (1) converges for all $\sigma \in \mathbb{R}$.

Let h be a positive continuous function on $[0,+\infty)$ increasing to $+\infty$. Here we will study the properties of the integrals (1) for which

$$|f(x)|\exp\{xh(x)\}\to 0,\quad x\to+\infty. \tag{5}$$

2. Banach spaces of Laplace – Stieltjes integrals. At first we remark that if $\ln F(x)=o(x)$ as $x\to+\infty$, then the integral

$$J_h(\sigma)=\int\limits_0^\infty e^{-xh(x)}e^{x\sigma}dF(x) \tag{6}$$

converges for all $\sigma\in(-\infty,+\infty)$.

By LS_h we denote a class of integrals (1) with real-valued functions f such that (5) holds. On LS_h we define operations

$$(I_1+I_2)(x)=\int\limits_0^\infty (f_1(x)+f_2(x))e^{x\sigma}dF(x)$$

and

$$(\lambda I)(\sigma)=\int\limits_0^\infty \lambda f(x)e^{x\sigma}dF(x),$$

where $I_j(\sigma)=\displaystyle\int_0^\infty f_j(x)e^{x\sigma}dF(x)$ for $j=1,2,$ and let

$$\|I\|_h=\sup\left\{|f(x)|\exp\{xh(x)\}:x\ge 0\right\}.$$

Under these operations LS_h becomes a normed linear space.

Theorem 2.1. *If $F\in V$ and $\ln\,F(x)=o(x)$ as $x\to+\infty$, then $(LS_h,\|\cdot\|_h)$ is nonuniformly convex Banach space.*

Proof. Let (I_p) be a Cauchy sequence in $LS_h,\ I_p(\sigma)=\displaystyle\int_0^\infty f_p(x)e^{x\sigma}dF(x)$. Then in view of (5) $|f_p(x)|\exp\{xh(x)\}\to 0$ as $x\to+\infty$ for each p, and for a given $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that $\|I_p-I_q\|_h<\varepsilon$ for all $p\ge n_0$ and $q\ge n_0$, i.e.,

$$\sup\left\{|f_p(x)-f_q(x)|\exp\{xh(x)\}:x\ge 0\right\}<\varepsilon.$$

Thus, $|f_p(x)\exp\{xh(x)\}-f_q(x)\exp\{xh(x)\}|<\varepsilon$ for all $x\ge 0,\ p\ge n_0$ and $q\ge n_0$. This shows that $(f_p(x)\exp\{xh(x)\})$ is Cauchy sequence in $\mathbb{R}$, so it converges to $f_0(x)\exp\{xh(x)\}$ (say) as $p\to\infty$. Since

$$|f_0(x)|\exp\{xh(x)\}\le|f_0(x)\exp\{xh(x)\}-f_p(x)|\exp\{xh(x)\}|+$$
$$+|f_p(x)|\exp\{xh(x)\}\to 0,\quad x\to+\infty,$$

the integral $I_0(\sigma)=\displaystyle\int_0^\infty f_0(x)e^{x\sigma}dF(x)$ belongs to LS_h. Also we have

$$\|I_p-I_0\|_h=\sup\left\{|f_p(x)-f_0(x)|\exp\{xh(x)\}:x\ge 0\right\}\to 0,\quad p\to\infty,$$

i.e., the $(LS_h,\|\cdot\|_h)$ is complete and, thus, a Banach space.

Now we choose numbers $0 < a_n < b_n < c_n < +\infty$ such that

$$\int\limits_0^{a_n} e^{-xh(x)} e^{x\sigma} dF(x) > 0, \qquad \int\limits_{b_n}^{c_n} e^{-xh(x)} e^{x\sigma} dF(x) > 0$$

and put

$$J_n(\sigma) = \int\limits_0^{a_n} e^{-xh(x)} e^{x\sigma} dF(x)$$

and

$$J_n^*(\sigma) = \int\limits_0^{a_n} e^{-xh(x)} e^{x\sigma} dF(x) + \int\limits_{b_n}^{c_n} e^{-xh(x)} e^{x\sigma} dF(x).$$

Then $J_n \in LS_h,\ J_n^* \in LS_h,\ \|J_n\|_h = 1,\ \|J_n^*\|_h = 1,\ \|J_n^* + J_n\|_h = 2$, and $\|J_n^* - J_n\|_h = 1 \not\to 0$, i.e., the space $(LS_h, \|\cdot\|_h)$ is nonuniformly convex (see, for example, [5, p. 183].

The following statement concerns for (I_m) uniform convergence.

Proposition 2.1. *Let $F \in V$ and $\ln F(x) = o(x)$ as $x \to +\infty$. If $(I_m) \subset LS_h$ converges to $I \in LS_h$ by $\|\cdot\|_h$, then $I_m(\sigma)$ converges uniformly to $I(\sigma)$ over compact subset of $\mathbb{R}$.*

Proof. If $\|I_m - I\|_h < \varepsilon$ for every $\varepsilon > 0$ and all $m \ge m_0(\varepsilon)$, then

$$\sup\{|f_m(x) - f_0(x)| \exp\{xh(x)\} : x \ge 0\} < \varepsilon$$

and, thus, $|f_m(x) - f_0(x)| \exp\{xh(x)\} < \varepsilon$ for every $\varepsilon > 0$, all $m \ge m_0(\varepsilon)$, and all $x \ge 0$. Therefore, if $m \ge m_0(\varepsilon)$ and $\sigma \in [\sigma_1, \sigma_2]$, then, in view of the condition $\ln F(x) = o(x)$ as $x \to +\infty$, we have

$$|I_m(\sigma) - I(\sigma)| \le \int\limits_0^{\infty} |f_m(x) - f(x)| e^{xh(x)} e^{-xh(x)} e^{x\sigma} dF(x) <$$

$$< \varepsilon \int\limits_0^{\infty} e^{-xh(x)} e^{x\sigma} dF(x) \le \varepsilon \int\limits_0^{\infty} e^{-x(h(x)-\sigma_2)} dF(x) \le$$

$$\le \varepsilon \int\limits_0^{\infty} F(x) de^{-x(h(x)-\sigma_2)} \le \varepsilon \int\limits_0^{\infty} e^{K_1 x} de^{-x(h(x)-\sigma_2)} \le K_2 \varepsilon,$$

where $K_j = \text{const} > 0$. Hence it follows that $I_m(\sigma)$ converges uniformly to $I(\sigma)$ on $[\sigma_1, \sigma_2]$.

Remark 2.1. The converse statement to Proposition 2.1 is false. Indeed, let for every $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$

$$F(x) = \begin{cases} 0, & 1 \le x < 2, \\ n, & 2n \le x < 2(n+1), \end{cases} \qquad f_m(x) = \begin{cases} \alpha_{m,n} > 0, & x = 2n-1, \\ 0, & x \ne 2n-1. \end{cases}$$

Then for all $m \in \mathbb{Z}_+$

$$I_m(\sigma) = \int\limits_0^\infty f_m(x)e^{x\sigma}dF(x) = \sum_n f_m(2n)e^{2n\sigma} = 0,$$

i.e., $I_m(\sigma) \to I_0(\sigma)$ as $m \to \infty$ for all $\sigma \in [\sigma_1, \sigma_2]$. On the other hand, if $\alpha_{m,1} - \alpha_{0,1} \geq \eta > 0$ for all $m \geq 1$, then

$$\|I_m - I_0\|_h = \sup\{|f_m(x) - f_0(x)|\exp\{xh(x)\} : x \geq 0\} \geq$$
$$\geq |f_m(1) - f_0(1)|\exp\{h(1)\} =$$
$$= \|\alpha_{m,1} - \alpha_{0,1}\|\exp\{h(1)\} \geq \eta\exp\{h(1) > 0.$$

Let us turn to the study of dual spaces. For $(LS_h, \|\cdot\|_h)$ by LS_h^* we denote the dual space, i.e., LS_h^* is the family of all continuous linear functionals on $(LS_h, \|\cdot\|_h)$.

Let $\Lambda(I) = \int_1^\infty f(x)g(x)dF(x)$, where g is a real-valued function on $(1, +\infty)$ such that $\int_1^\infty |f(x)g(x)|dF(x) < +\infty$. Then Λ is a linear functional and the following statement is true.

Proposition 2.2. *In order that $\Lambda \in LS_h^*$, it is sufficient that*

$$\int\limits_1^\infty |g(x)|\exp\{-xh(x)\}dF(x) < +\infty. \qquad (7)$$

Proof. By definition we have

$$\|\Lambda(I)\|_h = \sup\{|\Lambda(I)| : \|I\|_h \leq 1\} \leq$$
$$\leq \sup\left\{\int\limits_0^\infty |f(x)|e^{xh(x)}|g(x)|e^{-xh(x)}dF(x) : \sup_{x>0}|f(x)|e^{-xh(x)} \leq 1\right\} \leq$$
$$\leq \int\limits_1^\infty |g(x)|e^{-xh(x)}dF(x) < +\infty,$$

i.e., $\Lambda \in LS_h^*$.

Conjecture. *Every bounded linear functional Λ defined for $I \in (LS_h, \|\cdot\|_h)$ is of the form*

$$\Lambda(I) = \int\limits_1^\infty f(x)g(x)dF(x), \qquad I(\sigma) = \int\limits_0^\infty f(x)e^{x\sigma}dF(x).$$

Below we prove this conjecture for Dirichlet series (2), but first we present some corollaries.

Let Ω be a class of positive unbounded functions Φ on $(-\infty, +\infty)$ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty, +\infty)$. For $\Phi \in \Omega$ let φ be the inverse function to Φ' and

$$\Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}$$

be the function associated with Φ in the sense of Newton. It is known [1, p. 30] that if $\Phi \in \Omega$, then $\ln \mu(\sigma, I) \le \Phi(\sigma)$ for all $\sigma \ge \sigma_0$ if and only if $\ln |f(x)| \le -x\Psi(\varphi(x))$ for all $x \ge x_0$. In view of this statement in [6] it is introduced a class LS_Φ of integrals (1) with real-valued functions f such that $|f(x)| \exp\{x\Psi(\varphi(x))\} \to 0$ as $x \to +\infty$ and on LS_Φ it is defined

$$\|I\|_\Phi = \sup\{|f(x)| \exp\{x\Psi(\varphi(x))\} : x \ge 0\}.$$

If we choose $h(x) = \Psi(\varphi(x))$ then $LS_h = LS_\Phi$, and from above we get the following statement [6].

Corollary 2.1. *Let $\Phi \in \Omega$, $F \in V$ and $\ln F(x) = o(x)$ as $x \to +\infty$. Then $(LS_\Phi, \|\cdot\|_\Phi)$ is a nonuniformly convex Banach space. If $(I_m) \subset LS_\Phi$ converges to $I \in LS_\Phi$ by $\|\cdot\|_\Phi$ then $I_m(\sigma)$ converges uniformly to $I(\sigma)$ over compact subset of $\mathbb{R}$. If*

$$\int\limits_1^\infty |g(x)| \exp\{-x\Psi(\varphi(x))\} dF(x) < +\infty,$$

then the functional $\Lambda(I) = \int_1^\infty f(x)g(x)dF(x)$ belongs to the dual space LS_Φ^*.

Now we consider Banach spaces of Laplace – Stieltjes integrals of finite generalized order. For this purpose, by L we denote a class of continuous nonnegative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \ge 0$ for $x \le x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \le x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each fixed $c \in (0, +\infty)$, i.e., α is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

If $\alpha \in L$ and $\beta \in L$, then the value

$$\varrho_{\alpha,\beta}[I] := \limsup_{\sigma\to+\infty} \frac{\alpha(\ln M(\sigma, I))}{\beta(\sigma)}$$

is called the generalized (α, β)-order of I. In order to have a formula for the finding $\varrho_{\alpha,\beta}[I]$ we say [1, p. 21] that $|f|$ has regular variation in regard to F if there exist $a \ge 0$, $b \ge 0$, and $h > 0$ such that $\int_{x-a}^{x+b} |f(t)|dF(t) \ge h|f(x)|$ for all $x \ge a$. We put

$$\kappa_{\alpha,\beta}[f] := \limsup_{x\to+\infty} \frac{\alpha(x)}{\beta\left(\dfrac{1}{x} \ln \dfrac{1}{|f(x)|}\right)}.$$

Then the following statement is true [1] [p. 77 – 81].

Lemma 2.1. *Let the functions $\alpha \in L_{si}$ and $\beta \in L^0$ are continuously differentiable and $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for every $c \in (0, +\infty)$. Suppose that $F \in V$ and $\ln F(x) = o\left(x\beta^{-1}(c\alpha(x))\right)$ as $x \to +\infty$ for every $c \in (0, +\infty)$. Then $\varrho_{\alpha,\beta}[I] \le \kappa_{\alpha,\beta}[f]$. If, moreover, $|f|$ has regular variation in regard to F, then $\varrho_{\alpha,\beta}[I] = \kappa_{\alpha,\beta}[f]$.*

We need also the following lemma [7].

Lemma 2.2. *If* $\beta \in L$ *and*

$$B(\delta) = \limsup_{x\to+\infty} \frac{\beta((1+\delta)x)}{\beta(x)}, \quad \delta > 0,$$

then in order that $\beta \in L^0$, *it is necessary and sufficient that* $B(\delta) \to 1$ *as* $\delta \to +0$.

From Lemma 2.2 it follows that if $\beta \in L^0$, then for every fixed $t > 0$ there exists $\tau = \tau(t) > 0$ such that

$$\beta^{-1}((1+t)x) > (1+\tau)\beta^{-1}(x), \quad x > x_0, \tag{8}$$

because if there exists a $(x_k) \uparrow +\infty$ such that $\beta^{-1}((1+t)x_k) \le (1+o(1))\beta^{-1}(x_k)$ as $k \to \infty$, then for $y_k = \beta^{-1}(x_k)$ we have $(1+t)\beta(y_k) \le \beta((1+o(1))y_k)$ as $k \to \infty$, that is impossible.

If $\kappa_{\alpha,\beta}[f] < +\infty$, then

$$\ln|f(x)| \le -x\beta^{-1}\left(\frac{\alpha(x)}{\kappa_{\alpha,\beta}[f]+o(1)}\right) \quad \text{as} \quad x \to +\infty.$$

Therefore, for every $\kappa \in (\kappa_{\alpha,\beta}[f], +\infty)$ in view (8) we get

$$|f(x)|\exp\left\{x\beta^{-1}\left(\frac{\alpha(x)}{\kappa}\right)\right\} \le \exp\left\{-x\left(\beta^{-1}\left(\frac{\alpha(x)}{\kappa_{\alpha,\beta}[I]+o(1)}\right) - \beta^{-1}\left(\frac{\alpha(x)}{\kappa}\right)\right)\right\} \to 0 \tag{9}$$

as $x \to +\infty$. By $LS_{(\alpha,\beta;\kappa)}$ we denote a class of integrals (1) with real-valued functions f such that $\kappa_{\alpha,\beta}[f] < \kappa$. In view of (9) on $LS_{(\alpha,\beta;\kappa)}$ we can define

$$\|I\|_{(\alpha,\beta;\kappa)} = \sup\left\{|f(x)|\exp\left\{x\beta^{-1}(\alpha(x)/\kappa)\right\} : x \ge 0\right\}$$

and by using the results from Section 2, it is easy to prove the following statement.

Corollary 2.2. *Let* $\alpha \in L$, $\beta \in L^0$, $F \in V$ *and* $\ln F(x) = o(x)$ *as* $x \to +\infty$. *Then* $(LS_{(\alpha,\beta;k)}, \|\cdot\|_{(\alpha,\beta;k)})$ *is a nonuniformly convex Banach space for each* $\kappa \in (\kappa_{\alpha,\beta}[I], +\infty)$. *If* $(I_m) \subset LS_{(\alpha,\beta;k)}$ *converges to* $I \in LS_{(\alpha,\beta;k)}$ *by* $\|\cdot\|_{(\alpha,\beta;k)}$, *then* $I_m(\sigma)$ *converges uniformly to* $I(\sigma)$ *over compact subset of* $\mathbb{R}$. *If*

$$\int_1^\infty |g(x)|\exp\{-x\beta^{-1}(\alpha(x)/k))\}dF(x) < +\infty,$$

then the functional $\Lambda(I) = \int_1^\infty f(x)g(x)dF(x)$ *belongs to the dual space* $LS^*_{(\alpha,\beta;k)}$.

To be completed, we consider the case when $I(\sigma) = D(\sigma)$ is Dirichlet series (2) with real-valued coefficients d_n. Suppose that this series absolutely converges for all $\sigma \in (-\infty, +\infty)$ and say that $D \in S_h$ if $|d_n|\exp\{\lambda_n h(\lambda_n)\} \to 0$ as $n \to \infty$. If we put

$$\|D\|_h = \sup\{|d_n|\exp\{\lambda_n h(\lambda_n)\} : n \ge 1\},$$

then from Theorem 2.1 we obtain the following statement.

Corollary 2.3. *If* $\ln n(x) = o(x)$ *as* $x \to +\infty$, *then* $(S_h, \|\cdot\|_h)$ *is a nonuniformly Banach space.*

The following statement completes Proposition 2.1.

Proposition 2.3. *If* $\ln n(x) = o(x)$ *as* $x \to +\infty$, *then in order that* $(D_m) \subset S_h$ *converges to* $D \in S_h$ *by* $\|\cdot\|_h$ *it is necessary and sufficient that* $D_m(\sigma)$ *converges uniformly to* $D(\sigma)$ *over each compact subset of* $\mathbb{R}$.

Proof. The necessity follows from Proposition 2.1. Conversely, let $D_m(\sigma)$ converges uniformly to $D(\sigma)$ over each $B = [\sigma_1, \sigma_2]$, where $D_m(\sigma) = \sum_{n=1}^{\infty} d_{m,n} \exp\{\lambda_n \sigma\}$. Then $|d_{m,n} - - d_n| \exp\{\lambda_n \sigma_1\} < \varepsilon$ for all $n \geq 1$ and all $m \geq m_0 = m_0(\varepsilon, \sigma_1)$, whence

$$|d_{m,n} - d_n| \exp\{\lambda_n h(\lambda_n)\} \leq \varepsilon \exp\{-\lambda_n(\sigma_1 - h(\lambda_n))\}.$$

Choosing $\sigma_1 = h(\lambda_n)$ from hence we get $|d_{m,n} - d_n| \exp\{\lambda_n h(\lambda_n)\} \leq \varepsilon$ for every ε, all $n \geq 1$ and all $m \geq m_0^* = m_0^*(\varepsilon, n)$, i.e., $\|D_m - D\|_h \to 0$ as $m \to \infty$.

For $D \in (S_h, \|\cdot\|_h)$ by S_h^* we denote the dual space and put $\Lambda(D) = \sum_{n=1}^{\infty} d_n g_n$, where g_n is real-valued sequence such that

$$\sum_{n=1}^{\infty} |g_n| \exp\{-\lambda_n h(\lambda_n)\} < +\infty. \tag{10}$$

Then Λ is a linear functional and we will prove our conjecture for $I(\sigma) = D(\sigma)$.

Proposition 2.4. *Every bounded linear functional* Λ *defined for* $(S_h, \|\cdot\|_h)$ *is of the form*

$$\Lambda(D) = \sum_{n=1}^{\infty} d_n g_n, \qquad D(\sigma) = \sum_{n=1}^{\infty} d_n \exp\{\lambda_n \sigma\}, \tag{11}$$

where g_n *is real-valued sequence satisfying (10).*

Proof. We use a method from [8]. As in the proof of Proposition 2.2 in view of (10) now we have

$$\sum_{n=1}^{\infty} |d_n g_n| \leq \sup_{n \geq 1} |d_n| e^{\lambda_n h(\lambda)} \sum_{n=1}^{\infty} |g_n| e^{-\lambda_n h(\lambda_n)} =$$

$$= \|D\|_h \sum_{n=1}^{\infty} |g_n| e^{-\lambda_n h(\lambda_n)} < +\infty,$$

i.e., Λ is a well defined functional on $(S_h, \|\cdot\|_h)$. Moreover,

$$|\Lambda(D)| \leq \|D\|_h \sum_{n=1}^{\infty} |g_n| e^{-\lambda_n h(\lambda_n)},$$

whence

$$\|\Lambda\|_h \leq \sum_{n=1}^{\infty} |g_n| \exp\{-\lambda_n h(\lambda_n)\}. \tag{12}$$

Conversely, at first we remark that if $D \in (S_h, \|\cdot\|_h)$ and $D_m(\sigma) = \sum_{n=1}^{m} d_n e^{\lambda_n \sigma}$, then $\|D_m - D\|_h = \sup_{n>m} |d_n| \exp\{-\lambda_n h(\lambda_n)\} \to 0$ as $m \to \infty$ and by Proposition 2.1 $D_m(\sigma)$

converges uniformly to $D(\sigma)$ over each compact subset of $\mathbb{R}$. Therefore, if $\Lambda \in S_h^*$ and we define $\Lambda\left(e^{\sigma\lambda_n}\right) = g_n$ for each n, then

$$\Lambda(D) = \Lambda\left(\lim_{m\to\infty}\sum_{n=1}^{m} d_n \exp\{\sigma\lambda_n\}\right) = \lim_{m\to\infty}\sum_{n=1}^{m} d_n\Lambda(\exp\{\sigma\lambda_n\}) = \sum_{n=1}^{\infty} d_n g_n.$$

Now we show that $\sum_{n=1}^{\infty}|d_n g_n| \le \|\Lambda\|_h$ so that $\sum_{n=1}^{\infty}|d_n g_n| < +\infty$. For this goal take $p \in \mathbb{N}$ and let $d_n = \exp\{-\lambda_n h(\lambda_n)\}\operatorname{sgn}(g_n)$ for $1 \le n \le p$ and $d_n = 0$ for $n > p$. If we define $D(\sigma) = \sum_{n=1}^{\infty} d_n \exp\{\lambda_n\sigma\}$, then obviously $D \in S_h$ and $\|D\|_h = 1$. Hence,

$$|\Lambda(D)| = \left|\sum_{n=1}^{p}\exp\{-\lambda_n h(\lambda_n)\}\operatorname{sgn}(g_n)\Lambda(\exp\{\sigma\lambda_n\})\right| = \sum_{n=1}^{p}|g_n|\exp\{-\lambda_n h(\lambda_n)\}$$

and $|\Lambda(D)| \le \|\Lambda\|_h\|D\|_h = \|\Lambda\|_h$, so that

$$\sum_{n=1}^{p}|g_n|\exp\{-\lambda_n h(\lambda_n)\} \le \|\Lambda\|_h$$

and

$$\sum_{n=1}^{\infty}|g_n|\exp\{-\lambda_n h(\lambda_n)\} = \sup_p\sum_{n=1}^{p}|g_n|\exp\{-\lambda_n h(\lambda_n)\} \le \sup_p\|\Lambda\|_h = \|\Lambda\|_h. \tag{13}$$

Inequalities (12) and (13) together show that $\sum_{n=1}^{\infty}|g_n|\exp\{-\lambda_n h(\lambda_n)\} = \|\Lambda\|_h$.

3. Fréchet spaces of entire Dirichlet series of finite generalized order. In the spaces of entire Dirichlet series of finite generalized order it is possible to introduce other metric. Recall that if $\alpha \in L,\ \beta \in L$ and Dirichlet series (2) is entire, then

$$\varrho_{\alpha,\beta}[D] := \limsup_{\sigma\to+\infty}\frac{\alpha(\ln M(\sigma, D))}{\beta(\sigma)},$$

where $M(\sigma, D) = \sum_{n=1}^{\infty}|d_n|e^{\sigma\lambda_n}$, is called [9, 10] the generalized (α,β)-order of D. From Lemma 2.1 the next well-known [9, 10] lemma follows (see also [1, p. 21]).

Lemma 3.1. *Let the functions α and β satisfy the conditions of Lemma 2.2. If* $\ln n = o\left(\lambda_n\beta^{-1}(c\alpha(\lambda_n))\right)$ *as $n\to\infty$ for each $c \in (0,+\infty)$, then*

$$\varrho_{\alpha,\beta}[F] = \limsup_{n\to\infty}\frac{\alpha(\lambda_n)}{\beta\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|d_n|}\right)}.$$

For fixed $\varrho < +\infty$ by $\overline{S}_\varrho$ we denote a class of entire Dirichlet series (2), for what $\varrho_{\alpha,\beta}[D] \le \varrho$. Then by Lemma 3.1

$$|d_n| \le \exp\left\{-\lambda_n\beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + o(1)}\right)\right\}, \quad n\to\infty. \tag{14}$$

Using an idea of the paper [11] (see also [12, 13]), for $q \in \mathbb{N}$ we define

$$\|D\|_{\varrho;q} = \sum_{n=1}^{\infty} |d_n| \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\}.$$

In view of (14) and (8) as above we have

$$|d_n| \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\} \le$$
$$\le \exp\left\{-\lambda_n \left(\beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + o(1)}\right) - \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right)\right\} \le$$
$$\le \exp\left\{-\xi\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + o(1)}\right)\right\}, \quad n \to \infty,$$

for some $\xi = \xi(q) > 0$. Thus, since $\ln n = o\left(\lambda_n \beta^{-1}\left(c\alpha(\lambda_n)\right)\right)$ as $n \to \infty$ for each $c \in (0, +\infty)$, $\|D\|_{\varrho;q}$ exists for each $q \in \mathbb{N}$ and it easily to check that $\|D\|_{\varrho;q}$ is a norm on $\overline{S}_{\varrho}$.

Clearly, $\|D\|_{\varrho;q} \le \|D\|_{\varrho;q+1}$. Therefore [14], the family $\|D\|_{\varrho;q} : q \in \mathbb{N}$ induces on $\overline{S}_{\varrho}$ a unique topology such that $\overline{S}_{\varrho}$ becomes a local convex vector space and this topology is given by the metric d, where

$$d(D_1, D_2) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|D_1 - D_2\|_{\varrho;q}}{1 + \|D_1 - D_2\|_{\varrho;q}}. \tag{15}$$

The space with the metric d we denote by $\overline{S}_{\varrho,d}$.

Theorem 3.1. *If the functions α, β and the sequence (λ_n) satisfy the conditions of Lemma 3.1, then $\overline{S}_{\varrho,d}$ is a Fréchet space.*

Proof. It is sufficient to show that $\overline{S}_{\varrho,d}$ is complete. Let therefore (D_j) is d-Cauchy sequence in $\overline{S}_{\varrho,d}$ and so far for a given $\varepsilon > 0$ there corresponds a $m_j = m_j(\varepsilon)$ such that $\|D_j - D_k\|_{\varrho;q} < \varepsilon$ for all $j,\ k \ge j_0$ and $q \in \mathbb{N}$; consequently for these $j,\ k,$ and q we have

$$\sum_{n=1}^{\infty} \left|d_n^{(j)} - d_n^{(k)}\right| \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\} < \varepsilon, \tag{16}$$

i.e., $\left|d_n^{(j)} - d_n^{(k)}\right| < \varepsilon$ and $\left(d_n^{(j)}\right)_{j \ge 1}$ is a Cauchy sequence. Therefore, $d_n^{(j)} \to d_n$ as $j \to \infty$. Letting $k \to \infty$ in (16) one has for $j \ge j_0$

$$\sum_{n=1}^{\infty} \left|d_n^{(j)} - d_n\right| \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\} < \varepsilon, \tag{17}$$

and consequently taking $j = j_0$ in (17) we get for a fixed q

$$\sum_{n=1}^{\infty} \left|d_n^{(j_0)} - d_n\right| \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\} < \varepsilon,$$

whence in view of (14) with $d_n^{(j_0)}$ instead d_n we obtain

$$|d_n| \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\} \le \left|d_n^{(j_0)}\right| \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\} + \varepsilon \le$$

$$\le \exp\left\{-\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + o(1)}\right)\right\} \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\} + \varepsilon \le$$

$$\le 2\varepsilon,$$

i.e.,

$$\limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\dfrac{1}{\lambda_n} \ln \dfrac{1}{|d_n|}\right)} \le$$

$$\le \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\dfrac{1}{\lambda_n} \ln \dfrac{1}{2\varepsilon} + \beta^{-1}\left(\dfrac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right)} = \varrho + \frac{1}{q}.$$

By Lemma 3.1 in view of the arbitrariness of q from hence we get $\varrho_{\alpha,\beta}[D] \le \varrho$. Thus, by using (17), again we see that $\|D_j - D\|_{\varrho;q} < \varepsilon$ for $j \ge j_0$.

For $\overline{S}_{\varrho,d}$ by $\overline{S}^*_{\varrho,d}$ we denote the dual space. The following analogue of Proposition 2.4 is true.

Proposition 3.1. *Let the functions α, β and the sequence (λ_n) satisfy the conditions of Lemma 3.1. Then every continuous linear functional Λ on $\overline{S}_{\varrho,d}$ is of form (11) if and only if for all $n \in \mathbb{N}$ and $q \in \mathbb{N}$*

$$|g_n| \le K \exp\left\{\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right\}, \quad K = \text{const} > 0. \tag{18}$$

Proof. Let $\Lambda \in \overline{S}^*_{\varrho,d}$, clearly means if $D_m \to D$ in $\overline{S}_{\varrho,d}$, then $\Lambda(D_m) \to \Lambda(D)$.

Now let d_n satisfy (14) and $D_m(s) = \sum_{n=1}^m d_n e^{s\lambda_n}$. Then we claim that $D_m \to D$ in $\overline{S}_{\varrho,d}$ (observe that $D_m \in \overline{S}_{\varrho,d}$). To ascertain this, it is sufficient to prove that $D_m \to D$ under the norm $\|\cdot\|_{\varrho,q}$ for every $q \in \mathbb{N}$.

So let q be fixed integer. Choose $\varepsilon \in (0,\, 1/q)$. Then by using (8) we can determine an integer $m = m(\varepsilon)$ such that

$$|d_n| \le \exp\left\{-\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + \varepsilon}\right)\right\}, \quad n \ge m+1,$$

and it follows as above that

$$\|D_m - D\|_{\varrho,q} = \left\|\sum_{n=m+1}^{\infty} d_n e^{s\lambda_n}\right\|_{\varrho,q} \le$$

$$\le \sum_{n=m+1}^{\infty} \exp\left\{-\lambda_n\left(\beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + \varepsilon}\right) - \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right)\right\} \to 0, \quad m \to \infty,$$

and this ascertains our claim. From hence and the continuity of Λ we have $\lim_{m\to\infty}\Lambda(D_m) = \Lambda(D)$ in the topology given by d.

Note that $\Lambda(D_m)=\sum_{n=1}^{m} d_n g_n$, where $g_n=\Lambda\big(e^{\sigma\lambda_n}\big)$.

Since Λ is continuous on $\left(\overline{S}_{\varrho,d}, \|\cdot\|_{\varrho,q}\right)$, for each $q\in\mathbb{N}$ there exists a $K>0$ (independent of q) such that $|g_n|=\left|\Lambda\left(e^{\sigma\lambda_n}\right)\right|\le K\left\|\left(e^{\sigma\lambda_n}\right)\right\|_{\varrho,q}$ and so, by using the definition of the norm $\left\|\left(e^{\sigma\lambda_n}\right)\right\|_{\varrho,q}$, we get (18).

To prove the other part, let now g_n satisfies (18). Then

$$|\Lambda(D)|\le K\sum_{n=1}^{\infty}|g_n|\exp\left\{\lambda_n\beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho+1/q}\right)\right\},\quad q\in\mathbb{N},$$

and so $|\Lambda(D)|\le K\|\Lambda(D)\|_{\varrho,q}$ for all $q\in\mathbb{N}$. Therefore, $\Lambda\in\left(\overline{S}_{\varrho,d}, \|\cdot\|_{\varrho,q}\right)^*$ for all $q\in\mathbb{N}$. Since $\|D\|_{\varrho;q}\le\|D\|_{\varrho;q+1}$, from (15) it follows that $\overline{S}^*_{\varrho,d}=\bigcup_{q\ge1}(\overline{S}_{\varrho,d}, \|\cdot\|_{\varrho,q})^*$. Thus, $\Lambda\in\overline{S}^*_{\varrho,d}$.

References

1. M. M. Sheremeta, *Asymptotical behaviour of Laplace – Stieltjes integrals*, Mathematical Studies Monograph Series, **15**, VNTL Publishers, Lviv (2010).
2. O. B. Skaskiv, *On some relations between the maximum modulus and the maximal term of an entire Dirichlet series* (in Russian), Math. Notes., **66**, № 1-2, 223 – 232 (1999).
3. I. Ye. Ovchar, O. B. Skaskiv, *Some analogue of the Wiman inequality for the Laplace integrals on a small parameter* (in Ukrainian), Carpathian Math. Publ., **5**, № 2, 305 – 309 (2013).
4. A. O. Kuryliak, I. E. Ovchar, O. B. Skaskiv, *Wiman's inequality for Laplace integrals*, Int. J. Math. Anal. (Ruse), **8**, № 5-8, 381 – 385 (2014).
5. V. A. Trenogin, *Functional analysis* (in Russian), Nauka, Moscow (1980).
6. M. M. Sheremeta, V. S. Dobushowskyy, A. O.Kuryliak, *On a Banach space of Laplace – Stieltjes integrals*, Mat. Stud., **48**, № 2, 143 – 149 (2017).
7. M. M. Sheremeta, *On two classes of positive functions and the belonging to them of main characteristics of entire functions*, Mat. Stud., **19**, № 1, 75 – 82 (2003).
8. O. P. Juneja, B. L. Srivastava, *On a Banach space of a class of Dirichlet series*, Indian J. Pure Appl. Math., **12**, № 4, 521 – 529 (1981).
9. Ja. D. Pyanylo, M. M. Sheremeta, *On the growth of entire functions given by Dirichlet series*, Izv. Vysš. Učebn. Zaved. Matematika, №10 (161), 91 – 93 (1975) (in Russian).
10. M. M. Sheremeta, *Entire Dirichlet series*, ISDO, Kiev (1993) (in Ukrainian).
11. T. Husain, P. K. Kamthan, *Spaces of entire functions represented by Dirichlet series*, Collect. Math, **9**, № 3, 203 – 216 (1968).
12. A. Kumar, G. S. Srivastava, *Spaces of entire functions of slow growth represented by Dirichlet series*, Portugal. Math., **51**, № 1, 3 – 11 (1994).
13. S. I. Fedynyak, *A space of entire Dirichlet series*, Carpathian Math. Publ., **5**, № 2, 336 – 340 (2013) (in Ukrainian).
14. T. Husain, *The open mapping and closed graph theorems in topological vector spaces*, Clarendon Press, Oxford (1965).

Received 12.05.21,
after revision — 29.06.21