UDC 517.9

ON PERIODIC SOLUTIONS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH ARGUMENT DEVIATIONS

ПРО ПЕРІОДИЧНІ РОЗВ'ЯЗКИ СИСТЕМ ЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ВІДХИЛЕННЯМИ АРГУМЕНТУ

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Dedicated to the blessed memory of our teacher A. M. Samoilenko

We discuss the use of periodic successive approximations for the study of the periodic boundary-value problem for a class of linear functional-differential equations. We describe a version involving trigonometric polynomial interpolation version. The application of the technique is shown for a numerical example.

Розглянуто використання періодичних послідовних наближень для вивчення періодичної крайової задачі для класу лінійних функціонально-деференціальних рівнянь. Описано версію, яка має в собі варіант интерполяції тригонометричними поліномами. Застосування методу показано для числового прикладу.

1. Introduction. Techniques based on parametrization and successive approximations and belonging to numerical-analytic methods can be effectively used for the various types of boundary value problems for ordinary and functional differential equations (see, e.g., [1 – 13] and the references therein). This technique belongs to the few approaches that allow both to investigate the existence of a solution and to construct it approximately.

Here, we discuss the method of periodic successive approximations for systems of linear functional differential equations covering systems with multiple argument deviations

$$u_i'(t) = \sum_{j=1}^{n} p_{ij}(t)u_j(\beta_{ij}(t)) + \phi_i\,(t), \qquad t \in [0,T], \quad i = 1,\ldots,n, \tag{1}$$

and describe its version using trigonometric polynomial interpolation. The argument deviations $\beta_{ij} \colon [0,T] \to [0,T],\ i,j,= 1,\ldots,n,$ in (1) are continuous functions, $p_{ij},\ i,j = 1\ldots,n,$ and $\phi_i,$ $i = 1,\ldots,n,$ are continuous. System with argument deviations (1) is a particular case of the system of functional differential equations

$$u_i'(t) = \sum_{j=1}^{n} (l_{ij}u_j)(t) + \phi_i(t), \qquad t \in [0,T]\,, \quad i = 1,\ldots,n, \tag{2}$$

where $l_{ij} \colon C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ are linear bounded operators and $\phi_i,\ i = 1,\ldots,n,$ are continuous. Systems (1), (2) will be considered under the T-periodic boundary conditions

$$u(0) = u(T). \tag{3}$$

We consider the case where the coefficients and the argument deviations are continuous functions. Solutions are sought for in the class of continuously differentiable functions. It is easy to see that system (2) can be rewritten in the form

$$u'(t) = (lu)(t) + \phi(t), \quad t \in [0,T],$$

where $l = \mathrm{col}(l_1, l_2, \ldots, l_n)$ with $l_i \colon C([0,T],\mathbb{R}^n) \to C([0,T],\mathbb{R}),\ i = 1,2,\ldots,n,$ given by the formula

$$l_i u := \sum_{j=1}^{n} l_{ij}u_j, \quad i = 1,2,\ldots,n. \tag{4}$$

For (1), equality (4) means that

$$(l_i u)(t) = \sum_{j=1}^{n} p_{ij}(t)u_j(\beta_{ij}(t)), \qquad i = 1,2,\ldots,n, \quad t \in [0,T].$$

The following notation is used in the sequel. For any vector $u = \mathrm{col}(u_1,..,u_n) \in \mathbb{R}^n$ the absolute value operation is understood componentwise $|u| = \mathrm{col}\,(|u_1|,\ldots,|u_n|)$ and the inequality between vectors are understood also componentwise; $C\,([0,T],\mathbb{R}^n)$ is the Banach space of continuous vector functions $[0,T] \to \mathbb{R}^n$ with the standard uniform norm; $r(Q)$ is the maximal in modulus eigenvalue of a matrix Q; 1_n is the unit matrix of dimension n. For any continuous function $u \colon [a,b] \to \mathbb{R}^n$ and closed interval $J \subseteq [0,T]$, we put

$$\delta_J(u) := \max_{t\in J} u(t) - \min_{t\in J} u(t), \tag{5}$$

where the operators $\max$ and $\min$ are understood in the componentwise sense:

$$\max_{t\in J} u(t) = \mathrm{col}\left(\max_{t\in J} u_1(t),\ldots,\max_{t\in J} u_n(t)\right),$$

etc.

2. Periodic successive approximations. Following [1, 5, 6], introduce the vector of parameters $z = \operatorname{col}(z_1, z_2, \ldots, z_n)$ and formally put

$$z = u(0).$$

This leads us to the problem with two-point linear separated conditions at 0 and T:

$$u'(t) = (lu)(t) + \phi(t), \quad t \in [0, T], \tag{6}$$

$$u(0) = z, \qquad u(T) = z. \tag{7}$$

Instead of (1), (3), we will consider the auxiliary problems (6), (7) keeping z as a free parameter.

To study problem (6), (7), introduce the sequence of functions $\{u_m(\cdot, z) : m \geq 0\}$ by setting

$$u_{m+1}(t, z) := z + \tilde{\phi}(t) + \int_0^t (lu_m(\cdot, z))(s)ds -$$

$$- \frac{t}{T} \int_0^T (lu_m(\cdot, z))(s)ds, \quad t \in [0, T], \quad m \geq 0, \tag{8}$$

$$u_0(\cdot, z) := z,$$

where

$$\tilde{\phi}(t) = \int_0^t \phi(s)ds - \frac{t}{T} \int_0^T \phi(s)ds, \quad t \in [0, T].$$

Here, $z = \operatorname{col}(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ is considered as a vector of parameters. Let us establish the convergence of the sequence of functions (8).

Let $L = (L_{ij})$ be the square matrix with the components equal to the norms of the operators l_{ij} in $C([0, T], \mathbb{R})$: $L_{ij} = \|l_{ij}\|$, $i, j = 1, \ldots, n$,

$$|(l_{ij} u)(t)| \leq L_{ij} \max_{s \in [0,T]} |u(s)|, \quad t \in [0, T], \tag{9}$$

for all $u \in C([0, T], \mathbb{R})$. The matrix is well defined since the operators are bounded. For the system with argument deviations (1),

$$L_{ij} \leq \max_{t \in [0,T]} |p_{ij}(t)|.$$

Theorem 1. *Assume that the spectral radius of L satisfies the inequality*

$$r(L) < \frac{2}{T}. \tag{10}$$

Then for arbitrary fixed $z \in \mathbb{R}^n$ the following assertions are valid:

1. All the functions of sequence (8) are continuously differentiable and satisfy conditions (7).

2. Sequence (8) converges uniformly in $t \in [0, T]$ as $m \to \infty$ to a limit function

$$u_\infty(t, z) = \lim_{m\to\infty} u_m(t, z)\,. \tag{11}$$

3. The limit function $u_\infty(\cdot, z)$ satisfies the T-periodic boundary condition (7):

$$u_\infty(0, z) = z, \qquad u_\infty(T, z) = z.$$

4. The limit function (11) is a unique continuously differentiable solution of the Cauchy problem

$$\begin{aligned} &u'(t) = (lu)(t) + \phi(t) - \frac{1}{T}\Delta(z), \quad t \in [0, T], \\ &x(0) = z, \end{aligned} \tag{12}$$

where

$$\Delta(z) := \int_0^T ((lu_\infty(\cdot, z))(s) + \phi(s))\, ds. \tag{13}$$

5. The estimate

$$|u_\infty(t, z) - u_m(t, z)| \le \frac{T}{4} Q^m (1_n - Q)^{-1} \delta_{[0,T]}(lz + \phi) \tag{14}$$

holds, where

$$Q := \frac{T}{2} L \tag{15}$$

and $\delta_{[0,T]}$ is defined by (5).

Here, l is the operator constructed from $l_{ij},\ i, j = 1, \ldots, n,$ according to (4). In (14), lz stands for its value on the constant vector z.

Lemma 1. *Let $\Lambda \subset \mathbb{R}^k,\ k \ge 1,$ be a closed bounded set and $u \colon [a, b] \times \Lambda \to \mathbb{R}^n$ be a continuous function. Then, for an arbitrary $t \in [a, b]$, the componentwise inequality*

$$\left| \int_0^t \left(u(\tau, \lambda) - \frac{1}{T} \int_0^T u(s, \lambda) ds \right) d\tau \right| \le$$

$$\le \frac{1}{2} \alpha_1(t) \Big(\max_{(s,\lambda)\in[0,T]\times\Lambda} u(s, \lambda) - \min_{(s,p)\in[0,T]\times\Lambda} u(s, \lambda) \Big) \tag{16}$$

holds, where

$$\alpha_1(t) = 2t \left(1 - \frac{t}{T} \right), \quad t \in [0, T]. \tag{17}$$

Proof. For $\Lambda = \varnothing$ (i.e., when u does not depend on the second argument), this statement coincides with [3] (Lemma 3) (see also [5]). The proof is carried out by analogy to [14] and is based on the inequality

$$\left| \int_0^t \left(u(\tau, \lambda) - \frac{1}{T} \int_0^T u(s, \lambda) ds \right) d\tau \right| \le \frac{1}{T} \int_0^t \int_t^T |u(\tau, \lambda) - u(s, \lambda)|\, ds\, d\tau.$$

Estimating the difference under the integral as

$$|u(\tau,\lambda)-u(s,\lambda)| \le \max_{(t,\lambda)\in[0,T]\times\Lambda} u(t,\lambda) - \min_{(t,\lambda)\in[0,T]\times\Lambda} u(t,\lambda),$$

we obtain (16).

Proof of Theorem 1. Let us first establish the estimate

$$\max_{t\in[a,b]} |u_{m+1}(t,z)-u_m(t,z)| \le \frac{T}{4} Q^m \delta_{[0,T]}(lz+\phi) \tag{18}$$

for all $m \ge 0,$ where Q is given by (15).

Introducing the operator

$$(\mathcal{L}y)(t) := \int_a^t y(s)ds - \frac{t-a}{b-a}\int_a^b y(s)ds, \quad t\in[a,b], \tag{19}$$

for $y \in C([0,T],\mathbb{R}^n),$ we can rewrite (8) as

$$u_{m+1} = z + \mathcal{L}(lu_m+\phi) \tag{20}$$

for $m \ge 0$ (we omit the dependence on z for the sake of brevity). In particular,

$$u_1 = z + \mathcal{L}(lu_0+\phi) = z + \mathcal{L}(lz+\phi). \tag{21}$$

By using notation (19) and the equality $\max_{t\in[0,T]} \alpha_1(t) = \dfrac{1}{2}T$ (see (17)), we get from Lemma 1 that the estimate

$$|(\mathcal{L}u)(t)| \le \frac{1}{2}\alpha_1(t)\delta_{[0,T]}(u) \le \frac{T}{4}\delta_{[0,T]}(u), \quad t\in[0,T], \tag{22}$$

holds for any continuous $u.$ Therefore, by (21),

$$|u_1(t)-u_0(t)| = |u_1(t)-z| = |(\mathcal{L}(lz+\phi))(t)| \le \frac{T}{4}\delta_{[0,T]}(lz+\phi),$$

which coincides with (18) for $m=0.$

Assume that estimate (18) is true for $m = m_0 \ge 1.$ In view the linearity of $l,$ (20) yields

$$u_{m+1} - u_m = \mathcal{L}l(u_m - u_{m-1}) \tag{23}$$

for $m \ge 1.$ By using (23) with $m = m_0+1$ and applying inequality (22), we get

$$|u_{m_0+2}(t) - u_{m_0+1}(t)| = |(\mathcal{L}l(u_{m_0+1}-u_{m_0}))(t)| \le$$
$$\le \frac{T}{4}\delta_{[0,T]}(l(u_{m_0+1}-u_{m_0})). \tag{24}$$

According to the definition (5) of $\delta_{[0,T]},$

$$\delta_{[0,T]}(lu) \le 2\max_{t\in[0,T]} |(lu)(t)|$$

for any u and, therefore, (24) gives

$$|u_{m_0+2}(t) - u_{m_0+1}(t)| \leq \frac{T}{2} \max_{t\in[0,T]} |l(u_{m_0+1} - u_{m_0})(t)|. \tag{25}$$

It follows from (9) that the componentwise inequality

$$|(lu)(t)| \leq L \max_{s\in[0,T]} |u(s)|, \quad t \in [0,T], \tag{26}$$

holds for an arbitrary u from $C([0,T],\mathbb{R}^n)$. Using (26) in (25) and recalling that estimate (18) is assumed to be satisfied for $m = m_0$, we get

$$\begin{aligned}|u_{m_0+2}(t) - u_{m_0+1}(t)| &\leq \frac{T}{2} \max_{t\in[0,T]} |l(u_{m_0+1} - u_{m_0})(t)| \leq \\ &\leq \frac{T}{2} L \max_{t\in[0,T]} |u_{m_0+1}(t) - u_{m_0}(t)| \leq \\ &\leq \frac{T}{2} L \frac{T}{4} Q^{m_0} \delta_{[0,T]}(lz+\phi) = \\ &= \frac{T}{4} Q^{m_0+1} \delta_{[0,T]}(lz+\phi),\end{aligned}$$

which means that (18) holds for $m = m_0 + 1$. The uniform convergence of the iterations and estimate (14) then follow from the inequalities

$$\begin{aligned}|u_{m+r}(t) - u_m(t)| &\leq \sum_{j=1}^{r} |u_{m+j}(t) - u_{m+j-1}(t)| \leq \\ &\leq \frac{T}{4} \sum_{j=1}^{r} Q^{m+j-1} \delta_{[0,T]}(lz+\phi) = \\ &= \frac{T}{4} Q^m \sum_{j=1}^{r} Q^{j-1} \delta_{[0,T]}(lz+\phi) \leq \\ &\leq \frac{T}{4} Q^m (1_n - Q)^{-1} \delta_{[0,T]}(lz+\phi)\end{aligned}$$

by passing to the limit as $r \to \infty$. The fact that function (11) is a solution of the Cauchy problem (12) is established by passing to the limit in (8) as $m \to \infty$ and differentiating the resulting equation.

3. Determining equations and their solvability. Similarly to [1], it is shown that the solutions of the original T-periodic boundary value problem (2), (3) are determined by roots of certain equations for z, for which the solution has form $u_\infty(\cdot, z^*)$.

Along with system (6), consider the system with a constant forcing term

$$u'(t) = (lu)(t) + \phi(t) - \frac{1}{T}\mu, \quad t \in [0,T], \tag{27}$$

with the initial conditions

$$u(0) = z, \tag{28}$$

where $\mu = \mathrm{col}\,(\mu_1, \ldots, \mu_n)$ is a control parameter.

Theorem 2. *Let $z \in \mathbb{R}^n$ be fixed. Assume that condition (10) holds. Then, for the solution of the initial value problem (27), (28) to satisfy the parametrized boundary conditions (7) it is necessary and sufficient that the control parameter μ in (27) have the form*

$$\mu = \Delta(z) \tag{29}$$

with $\Delta(z)$ given by (13). Moreover, in this case the solution of (27), (28) coincides with $u_\infty(\cdot, z)$.

The assertion of Theorem 2 is established similarly to the proof of [15] (Theorem 4.2).

Let us consider the function $\Delta : \mathbb{R}^n \to \mathbb{R}^n$ given by formula (13) for all z, where $u_\infty(\cdot, z)$ is the limit function (11). The following statement shows the relation of the limit function $u_\infty(\cdot, z)$ to the solution of the original periodic boundary value problem (1), (3).

Theorem 3. *Assume that condition (10) holds. Then the limit function $u_\infty(\cdot, z)$ is a continuously differentiable solution of the T-periodic boundary value problem (1), (3) if and only if the vector parameter z satisfies the system of n determining equations*

$$\Delta(z) = 0. \tag{30}$$

Proof. It is sufficient to apply Theorem 2 and notice that, for μ of form (29), the differential equation in (27) coincides with (1) if and only if the parameter z satisfies (30).

The difficulty in the realization of this approach is caused by the fact that the analytic construction of the limit function $u^*(\cdot, z)$ is possible in exceptional cases only. However, this obstacle can often be avoided because, as can be shown, it is possible to prove the existence of a solution of the periodic boundary value problem (2), (3) using the properties of a certain approximation $u_m(\cdot, z)$.

For $m \geq 1$, let us define the function $\Delta_m : \mathbb{R}^n \to \mathbb{R}^n$ according to the formula

$$\Delta_m(z) := \int_0^T ((lu_m(\cdot, z))(s) + \phi(s))\, ds \tag{31}$$

for arbitrary $z \in \mathbb{R}^n$. To investigate the solvability of the problem (1), (3), introduce the mth approximate determining system

$$\Delta_m(z) = 0, \tag{32}$$

where the vector function $u_m(\cdot, z)$ is defined by the recurrence relation (8).

4. Existence analysis by using successive approximations. Systems (30) and (32) are close enough to one another for m sufficiently large and one can expect that, under suitable conditions, the solvability of (30) can be deduced from that of (32). In this way, existence results for the periodic boundary value problem (1), (3) can be obtained by studying the solutions of the approximate determining system (32) similarly to [5, 16 – 18].

Note, that unlike (30), the mth approximate determining system (32) involves only the function $u_m : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ known explicitly after m steps of iteration.

Lemma 2. *Under assumption (10), the estimate*

$$|\Delta(z) - \Delta_m(z)| \le \frac{T}{2} Q^{m+1}(1_n - Q)^{-1}\delta_{[0,T]}(lz + \phi) \tag{33}$$

holds for any z *and* $m \ge 0,$ *where* Q *is given by (15).*

Proof. Let us fix an arbitrary z. Using estimate (26), and the linearity of $l,$ equalities (13), (31), (15) and estimates (14), (26), we get

$$|\Delta(z) - \Delta_m(z)| = \left| \int_0^T ((lu_\infty(\cdot, z))(s) - (lu_m(\cdot, z))(s))\, ds \right| \le$$

$$\le \int_0^T |l\,(u_\infty(\cdot, z) - u_m(\cdot, z))\,(s)|\, ds \le$$

$$\le \int_0^T L\, |u_\infty(s, z) - u_m(s, z)|\, ds \le$$

$$\le L\frac{T^2}{4} Q^m (1_n - Q)^{-1}\delta_{[0,T]}(lz + \phi) =$$

$$= \frac{T}{2} Q Q^m (1_n - Q)^{-1}\delta_{[0,T]}(lz + \phi),$$

which yields (33).

According to Theorem 3, the initial values of solutions of problem (2), (3) are determined by the critical points of the vector field Δ. Let us formulate a statement allowing one to prove the existence of critical points of Δ by checking properties of Δ_m for some $m.$

We need a definition from [5]. Let $k \ge 1,\ V \subset \mathbb{R}^k$. We say that functions $g = (g_i)_{i=1}^n :$ $\mathbb{R}^k \to \mathbb{R}^n$ and $h = (h_i)_{i=1}^n : \mathbb{R}^k \to \mathbb{R}^n$ satisfy the relation $g \rhd_V h$ if and only if there exists a function $\nu : V \to \{1, 2, \ldots, n\}$ such that $g_{\nu(z)}(z) > h_{\nu(z)}(z)$ at every point $z \in V$. In other words, $g \rhd_V h$ means that, at every point $z \in V$, at least one component of $g(z)$ is greater than the corresponding component of $h(z).$

Theorem 4. *Assume that inequality (10) holds. Let there exist an* $m \ge 0$ *and an open bounded set* $\Omega \subset \mathbb{R}^n$ *such that, on the boundary* $\partial\Omega$ *of* $\Omega,$ *the mapping* Δ_m *satisfies the condition*

$$|\Delta_m| \rhd_{\partial\Omega} \frac{T}{2} Q^{m+1}(1_n - Q)^{-1}\delta_{[0,T]}(lz + \phi) \tag{34}$$

and, moreover, its Brouwer degree is different from zero:

$$\deg(\Delta_m, \Omega, 0) \ne 0.$$

Then there exists a certain value $z^* \in \Omega$ *such that the function* $u^* = u_\infty(\cdot, z^*)$ *is a continuously differentiable solution of the* T*-periodic boundary value problem (2), (3).*

The proof can be carried by analogy to that of [5] (Theorem 3.28) using Lemma 2. It should be noted that condition (34) involves the terms which are explicitly known after m iterations. In particular, by (5), $\delta_{[0,T]}(lz)$ is the variation of the value of the right-hand side operator l on the constant initial approximation. The value lz is computed using the relation $(lz)(t) = (l1_n)(t)\, z,$ where $l1_n$ is the matrix-valued function whose columns are the values of l on the respective columns of the unit matrix (i. e., $l1_n = [le_1, le_2, \ldots, le_n]$, where e_i stands for the unit vector with 1 at the ith position).

5. Iterations with trigonometric interpolation. Obtaining higher-order approximations according to the above-mentioned scheme requires the analytic computation of iterations involving parameters. For this purpose, it is convenient to use software allowing one to perform symbolic calculations. The analytic computation of functions (8) and, as a consequence, the explicit construction of approximate determining equations (32) may be difficult or impossible if the expression $(lu)(s) + \phi(s)$ in (8) involves complicated terms causing problems with symbolic integration. In order to facilitate the computation of $u_m(\cdot, z),\ m \geq 1,$ one can use a *trigonometric polynomial* version of the iterative scheme (8), in which the results of iteration are replaced by suitable trigonometric interpolation polynomial before passing to the next step. In this case, the iterations have the form of trigonometric polynomials. This scheme is described below.

For a given continuous T-periodic function $y\colon\ [0, T] \to \mathbb{R}$ and a natural $q,$ denote by $\mathcal{T}_q y$ the trigonometric interpolation polynomial of degree q of the form

$$a_0 + a_1 \cos \omega t + b_1 \sin \omega t + \ldots + a_q \cos q\omega t + b_q \sin q\omega t, \tag{35}$$

where $\omega = \dfrac{2\pi}{T}$, such that

$$(\mathcal{T}_q y)(t_i) = y(t_i), \quad i = 0, 1, 2, \ldots, 2q, \tag{36}$$

at the equidistant nodes

$$t_i = \frac{T}{2q+1} i, \quad i = 0, 1, 2, \ldots, 2q. \tag{37}$$

We need some relations between the coefficients and values of the trigonometric polynomial (35). Using the coefficients a_j and b_j, construct the $2q + 1$ dimensional vector

$$\mathcal{T}_q^c y = \operatorname{col}(a_0, a_1, b_1, \ldots, a_q, b_q) \tag{38}$$

and call it the *vector of the coefficients* of the trigonometric polynomial (35). Introduce also the $(2q+1)$-dimensional *vector of the values* of the trigonometric polynomial (35) at the equidistant points (37)

$$\mathcal{T}_q^v y = \operatorname{col}\left(\mathcal{T}_q y(t_0), \mathcal{T}_q y(t_1), \ldots, \mathcal{T}_q y(t_{2q})\right). \tag{39}$$

The vector of coefficients and vector of values are in one-to-one correspondence. There exists a nondegenerate linear transformation realizing the transition from vector (38) to vector (39) and *vice versa*. More precisely, for any $y,$ the vectors of values and coefficients of the corresponding triginometric interpolation polynomial $\mathcal{T}_q = \mathcal{T}_q y$ are related by the equalities

$$\mathcal{T}_q^v = M \mathcal{T}_q^c, \qquad \mathcal{T}_q^c = G \mathcal{T}_q^v. \tag{40}$$

where the elements of the mutually inverse matrices $M=(M_{ij})$ and $G=(G_{ij})$ have the form

$$M_{ij}=\begin{cases}1 & \text{if} \quad j=1,\\ \cos\left((i-1)j\dfrac{\pi}{2q+1}\right) & \text{if} \quad j=2,4,\ldots,2q,\\ \sin\left((i-1)(j-1)\dfrac{\pi}{2q+1}\right) & \text{if} \quad j=3,5,\ldots,2q+1\end{cases}$$

for $i=1,2,\ldots,2q+1$ and

$$G_{ij}=\begin{cases}\dfrac{1}{2q+1} & \text{if} \quad i=1,\\ \dfrac{2}{2q+1}\cos\left(i(j-1)\dfrac{\pi}{2q+1}\right) & \text{if} \quad i=2,4,\ldots,2q,\\ \dfrac{2}{2q+1}\sin\left((i-1)(j-1)\dfrac{\pi}{2q+1}\right) & \text{if} \quad i=3,5,\ldots,2q+1\end{cases} \tag{41}$$

for $j=1,2,\ldots,2q+1$ (see [1, p. 124]).

We extend componentwise the operation and notation $\mathcal{T}_q y(t)$ used in (36) for scalar functions to continuous vector functions $y\colon [0,T]\to\mathbb{R}^n$ by putting

$$\mathcal{T}_q y(t) := \operatorname{col}\big(\mathcal{T}_q y_1(t), \mathcal{T}_q y_2(t), \ldots, \mathcal{T}_q y_n(t)\big). \tag{42}$$

The vector of coefficients and vector of values are extended componetwise for the vector trigonometric polynomials (42) in a similar way,

$$\begin{aligned}\mathcal{T}_q^c y &:= \operatorname{col}(\mathcal{T}_q^c y_1, \mathcal{T}_q^c y_2, \ldots, \mathcal{T}_q^c y_n),\\ \mathcal{T}_q^v y &:= \operatorname{col}(\mathcal{T}_q^v y_1, \mathcal{T}_q^v y_2, \ldots, \mathcal{T}_q^v y_n).\end{aligned} \tag{43}$$

Let us modify the recurrence relation (8) replacing the term $lu_m(\cdot,z)+\phi$ by the corresponding trigonometric interpolation polynomial

$$\mathcal{T}_q\left(lu_m(\cdot,z)+\phi\right)(t) = A_0^m(z)+\sum_{j=1}^{q}(A_j^m(z)\cos j\omega t + B_j^m(z)\sin j\omega t), \quad t\in[0,T], \tag{44}$$

where the coefficients of the polynomials depend on m and on the parameter z:

$$\begin{aligned}A_j^m(z) &= \operatorname{col}(A_{1j}^m(z), A_{2j}^m(z), \ldots, A_{nj}^m(z)), \quad j=0,1,\ldots,q,\\ B_j^m(z) &= \operatorname{col}(B_{1j}^m(z), B_{2j}^m(z), \ldots, B_{nj}^m(z)), \quad j=1,\ldots,q.\end{aligned} \tag{45}$$

Substituting expression (44) into (8) instead of $lu_m(\cdot,z)+\phi$ and renaming $u_m(\cdot,z)$ to $u_m^q(\cdot,z)$, we obtain

$$u_{m+1}^q(t,z) = z+\int\limits_0^t\left(A_0^m(z)+\sum_{j=1}^{q}(A_j^m(z)\cos j\omega s + B_j^m(z)\sin j\omega s)\right)ds-$$

$$-\frac{t}{T}\int_0^T \left(A_0^m(z) + \sum_{j=1}^{q}(A_j^m(z)\cos j\omega s + B_j^m(z)\sin j\omega s)\right)ds,$$

which, after computation, gives the formula

$$u_{m+1}^q(t,z) = z + \frac{1}{\omega}\sum_{j=1}^{q}\frac{1}{j}\left(A_j^m(z)\sin j\omega t - B_j^m(z)\cos j\omega t\right), \tag{46}$$

$$t \in [0,T], \quad m \geq 0.$$

For any $q \geq 1$, formula (46) defines a trigonometric vector polynomial $u_{m+1}^q(\cdot, z)$ of degree q. In particular, all the members of the sequence $\{u_m^q(\cdot, z) : m \geq 0\}$ are continuously differentiable and satisfy conditions (7).

As an approximation to the mth approximate determining system (32) used to check the solvability of system (30), one can use its trigonometric polynomial version obtained by replacing $u_\infty(\cdot, z)$ in (30) by $u_m^q(\cdot, z)$:

$$\int_0^T \left(A_0^m(z) + \sum_{j=1}^{q}(A_j^m(z)\cos j\omega s + B_1^m(z)\sin j\omega s\right)ds = 0. \tag{47}$$

We see that, in fact, (47) means that

$$A_0^m(z) = 0. \tag{48}$$

Under suitable assumptions ensuring that every term inserted into $\mathcal{T}_q$ on the left-hand side of (44) satisfies the Dini-Lipschitz condition (see, e.g. [19, p. 50]), the corresponding trigonometric interpolation polynomials (44) constructed over the equidistant nodes (37) uniformly converge as q grows to ∞.

The coefficients (45) of the trigonometric polynomials can be computed without solving any linear systems of algebraic equations as follows. At every step m of iteration, we first componentwise calculate the values of the function

$$y(t,z) = \left(lu_m^q(\cdot, z)\right)(t) + \phi(t) \tag{49}$$

at nodes (37) and collect them into a vector of type (43):

$$Y^v(z) = \operatorname{col}\left(Y_1^v(z), Y_2^v(z), \ldots, Y_n^v(z)\right), \tag{50}$$

where

$$Y_i^v(z) = \operatorname{col}\left(y_i(t_0,z), y_i(t_1,z), \ldots, y_i(t_{2q},z)\right), \quad i = 1,2,\ldots,n.$$

Function (49) and vectors (50) both depend on the parameter z. Then we componentwise obtain the vectors of coefficients $Y_i^c(z) = \mathcal{T}_q^c y_i(\cdot, z)$,

$$Y_i^c(z) = \operatorname{col}\left(A_{i0}^m, A_{i1}^m, B_{i1}^m, \ldots, A_{iq}^m, B_{iq}^m\right), \quad i = 1,2,\ldots,n,$$

of the corresponding trigonometric polynomials $\mathcal{T}_q y_i(\cdot, z)$ according to (40):

$$Y_i^c(z) = GY_i^v(z), \quad i = 1, 2, \ldots, n, \tag{51}$$

where the matrix G has form (41). Thus, at every step $m,$ the scheme of obtaining approximate solutions consists of:

1. Computation of values (50) at nodes (37).
2. Computation of the coefficients according to formula (51).
3. Finding the roots of the approximate determining equation (48).
4. Substitution of the roots to formula (46).

A rigorous justification of the scheme of periodic successive approximations involving trigonometric interpolation polynomials can be carried out similarly to [20] (Theorem 4.1) in the case of Lagrange polynomial interpolation (this is not treated here).

6. Example. Let us consider the system of differential equations of form (1) on the interval $[0, T],\ T = 1/2,$

$$\begin{aligned}
u_1'(t) &= \frac{1}{2}u_2(t) + \sin(2\pi t)\, u_3\Big(\frac{t}{2}\Big) - \frac{\pi}{4}\sin(4\pi t),\\
u_2'(t) &= u_1(t) + u_3(t^2) + \frac{1}{16}\cos(4\pi t^2) + \left(\frac{\pi}{4} - \frac{1}{16}\right)\cos 4\pi t,\\
u_3'(t) &= u_2(t) + \frac{1}{2}u_3\Big(\frac{t}{2}\Big) + \left(\frac{\pi}{4} - \frac{1}{16}\right)\sin(4\pi t) + \frac{1}{32}\cos 2\pi t
\end{aligned} \tag{52}$$

under the T-periodic boundary conditions

$$\begin{aligned}
u_1(0) &= u_1(T),\\
u_2(0) &= u_2(T),\\
u_3(0) &= u_3(T).
\end{aligned} \tag{53}$$

This system is a particular case of (1), where the coefficients and argument deviations are

$$\begin{aligned}
&p_{11}(t) = 0, \qquad p_{12}(t) = \frac{1}{2}, \qquad p_{13}(t) = \sin 2\pi t,\\
&p_{21}(t) = 1, \qquad p_{22}(t) = 0, \qquad p_{23}(t) = 1,\\
&p_{31}(t) = 0, \qquad p_{32}(t) = 1, \qquad p_{33}(t) = \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{12}(t) = t, \qquad \beta_{13}(t) = t/2,\\
&\beta_{21}(t) = t, \qquad \beta_{23}(t) = t^2,\\
&\beta_{32}(t) = t, \qquad \beta_{33}(t) = t/2.
\end{aligned}$$

One can verify that the triplet of functions

$$u_1^*(t) = \frac{1}{16}\cos(4\pi t),$$
$$u_2^*(t) = \frac{1}{16}\sin(4\pi t), \tag{54}$$
$$u_3^*(t) = -\frac{1}{16}\cos(4\pi t)$$

is a solution of the T-periodic boundary value problem (52), (53).

It is clear that, for this problem,

$$L = \begin{pmatrix} 0 & 1/2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1/2 \end{pmatrix}$$

and, by (15),

$$Q = \begin{pmatrix} 0 & 1/8 & 1/4 \\ 1/4 & 0 & 1/4 \\ 1 & 1/4 & 1/8 \end{pmatrix}.$$

Since $r(Q) \approx 0.417 < 1,$ it follows that condition (10) of Theorem 1 is satisfied.

The sequence of functions (8) for this example is thus uniformly convergent, so we can proceed to constructing the approxiations.

For $m = 0,$ the approximate determining system (48) has the form

$$0.25z_2 + 0.3077683536z_3 = 0,$$
$$0.5z_3 + 0.5z_1 = -0.01792763393, \tag{55}$$
$$0.5z_2 + 0.25z_3 = -0.003125000056$$

and the unique solution of (55) is

$$z_1 \approx -0.044, \qquad z_2 \approx -0.01, \qquad z_3 \approx 0.0085.$$

For obtaining higher approximations, in order to facilitate computation according to the iterative scheme (8), we apply the trigonometric polynomial version in the form (46).

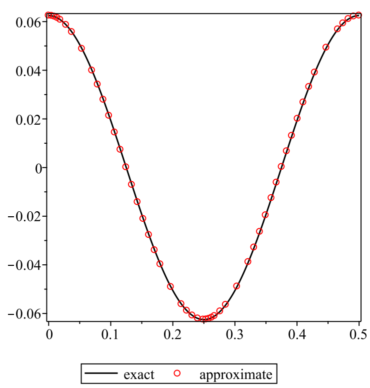
Let us choose $q = 2,$ $\omega = 2\pi/T$ and 5 equidistant points on the interval $[0, 1/2]$:

$$t_0 = 0, \qquad t_1 = \frac{1}{10}, \qquad t_2 = \frac{1}{5}, \qquad t_3 = \frac{3}{10}, \qquad t_4 = \frac{2}{5}.$$

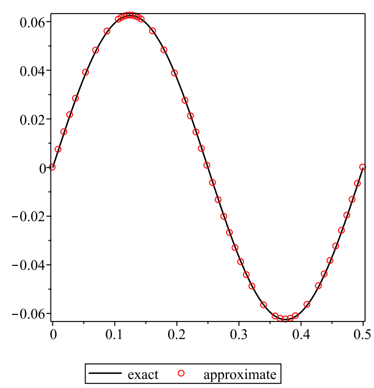
Using (46) and applying Maple 15 for different values of m and solving the approximate determining system of 5 scalar algebraic equations (47), we find the values of introduced parameters $z_1,$ z_2 and $z_3,$ which are presented in Table 1.

Табл. 1. The approximate initial values at several steps of iteration

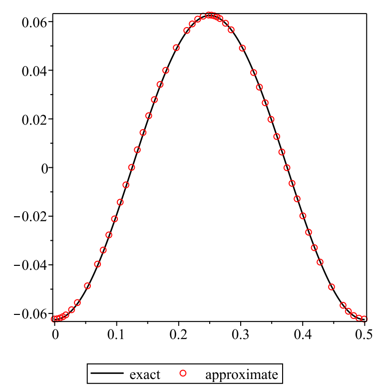
m	z_1	z_2	z_3
0	-0.044404334102	-0.01052453669	0.008549073165
1	0.06082366111	-0.00390838963	-0.06023572474
2	0.06226807361	-0.00042084810	-0.06205671658
3	0.06248688086	$-0.1639537 \cdot 10^{-4}$	-0.06249209425
4	0.06249988743	$-4.5938 \cdot 10^{-7}$	-0.06250010385
5	0.06249998782	$1.837 \cdot 10^{-8}$	-0.06249990547
6	0.06250000014	$-1.63 \cdot 10^{-9}$	-0.06250000510
∞	0.0625	0	-0.0625



(a) First component.



(b) Second component.



(c) Third component.

Fig. 1. The sixth approximation for problem (52), (53).

The graphs of the exact solution (54) and its sixth approximation are shown on Fig. 1. The absolute error of the sixth approximation is shown on Fig. 2.

As is seen from figures, the graph of the exact solution practically coincides with those of its trigonometric approximations. For example, the errors of the first and sixth approximations (i.e., the uniform deviation of the first trigonometric approximation from the exact solution) are

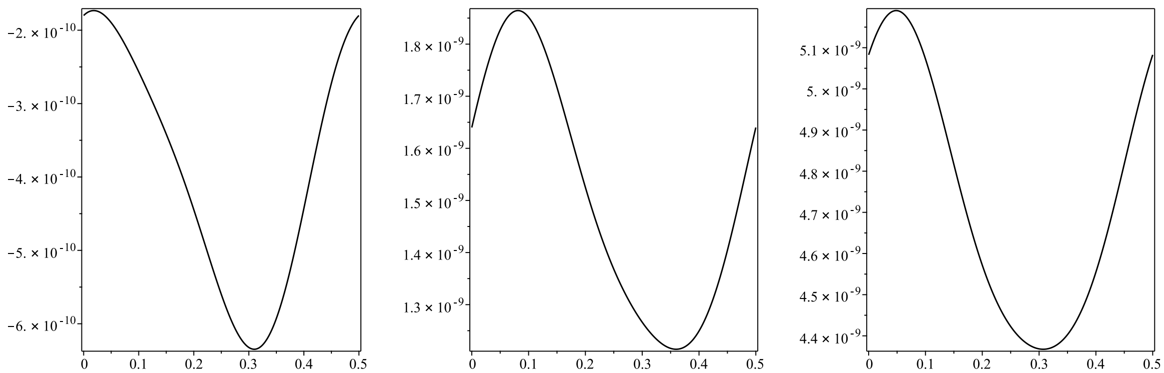
$$|u_1^*(t) - u_{11}^2(t)| \le 0.18 \cdot 10^{-2}, \qquad |u_2^*(t) - u_{12}^2(t)| \le 0.4 \cdot 10^{-2},$$

$$|u_3^*(t) - u_{13}^2(t)| \le 0.24 \cdot 10^{-2}, \qquad |u_1^*(t) - u_{61}^2(t)| \le 6 \cdot 10^{-10},$$

$$|u_2^*(t) - u_{62}^2(t)| \le 2 \cdot 10^{-9}, \qquad |u_3^*(t) - u_{63}^2(t)| \le 5.5 \cdot 10^{-9},$$

where

$$u_{61}^2(t) = 3.5 \cdot 10^{-10} + 0.06249999981 \cos(4\pi t) - 1.14 \cdot 10^{-10} \sin(4\pi t)$$



(a) First component. (b) Second component. (c) Third component.

Fig. 2. The error of the sixth approximation for (52), (53).

$$- 2.163859143 \cdot 10^{-11} \cos(8\pi t) + 2.7410^{-11} \sin(8\pi t),$$

$$u^2_{62}(t) = -1.4964 \cdot 10^{-9} - 1.361 \cdot 10^{-10} \ \cos(4\pi t) + 0.06249999971 \sin(4\pi t)$$

$$+ 2.51 \cdot 10^{-12} \cos(8\pi t) - 4.5770775 \cdot 10^{-11} \sin(8\pi t),$$

$$u^2_{63}(t) = -4.75 \cdot 10^{-9} - 0.06250000033 \cos(4\pi t) - 2.4457 \cdot 10^{-10} \sin(4\pi t)$$

$$- 2.2981539 \cdot 10^{-11} \cos(8\pi t) - 2.4457 \cdot 10^{-10} \sin(8\pi t).$$

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Received 27.12.20