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BOUNDED SOLUTIONS OF EVOLUTIONARY EQUATIONS. I

ОБМЕЖЕНІ РОЗВ'ЯЗКИ ЕВОЛЮЦІЙНИХ РІВНЯНЬ. I

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We obtain necessary and sufficient conditions for the existence of solutions of differential equations bounded on the whole axis in the Fréchet and Banach spaces and a criterion for the existence of almost periodic solutions. Examples of differential equations that illustrate the proposed theory are presented.

Одержано необхідні й достатні умови існування обмежених на всій осі розв'язків диференціальних рівнянь у просторах Фреше та Банаха, а також критерій існування майже періодичних розв'язків. Наведено приклади диференціальних рівнянь, які ілюструють отриману теорію.

One of the central questions of the qualitative theory of differential equations is a question of the behaviour of solutions at infinity. A class of systems whose solutions can both fall towards zero with exponential speed and increase indefinitely are analyzed with the help of exponential dichotomy concept on all axes and semi-axes of the considered system. Bounded solutions of such systems on the entire axis in the finite-dimensional case were considered by O. Perron, A. Meizel [1] and also W. Coppel [2], R. Sacker, G. Sell [3 – 5], Yu. Mitropolsky, A. Samoilenko, V. Kulik [6], and it can be read about infinite-dimensional Banach spaces in the monographs of M. Krein [7], Yu. Daletsky and M. Krein [8], H. Massera and H. Schaffer [9], F. Hartman [10], I. Chueshov [11]. In these papers, the given task was explored under the assumption of an exponential dichotomy on the whole axis of the corresponding homogeneous equation.

In K. Palmer's articles [12, 13] the condition of an exponential dichotomy on the whole axis of a homogeneous differential system was weakened with the replacement of the condition of exponential dichotomy on the semi-axes, and for the first time, the noetherianity of the corresponding operator at the solution of the task about the bounded on the whole axis solutions

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was proved. This idea was further developed in the paper [14], where using generalized-invertible operators and pseudoinverse by Moore – Penrose matrices the problem of the existence of bounded on the entire axis solutions for linear and nonlinear perturbations of the system was investigated. These and other results can be found in the monographs of A. Boichuk and A. Samoilenko [15, 16].

The concept of exponential dichotomy for evolution equations with unbounded operator coefficients was studied, in particular, in the famous monograph of D. Henry [17] and in the article written by G. Rodriguez, J. Filho [18], where under the assumption of an exponential dichotomy on the semi-axes of the corresponding homogeneous operator equation Fredholm analogue of alternative is proved. The study of noetherianity of the corresponding differential equation with unbounded coefficients operator is presented in the article by A. Baskakov [19 – 22], Yu. Latushkin, Yu. Tomilov [23]. It should be noted also works of [24 – 33].

This paper presents a number of results in regards to the existence of bounded solutions of differential equations, which the author received over the past years and new results under the condition of exponential dichotomy, ν-, μ-dichotomy [34, 35] of the corresponding homogeneous equation. The examples of applications of the theory to the study of the specific differential equations and systems of differential equations in Banach and Frechet spaces and general locally-convex topological spaces are given. Also, some results from the theory of almost periodic solutions of operator-differential equations are given.

Bounded solutions of linear differential equations in Banach space. In this part, we explore the question of the existence of the solutions limited on all axis of linear and nonlinear differential equations with unlimited and unbounded operator coefficients in Banach spaces under the condition of exponential dichotomy on the axes of the corresponding homogeneous equation.

Linear equations with bounded operator. The main results from the theory of unlimited solutions of differential equations in Banach space with a limited operator coefficient are given in the right part of the equation. The main part of these results were obtained in the papers [36 – 41].

Statement of the problem and basic concepts. In Banach space $\mathbf{B}$, we consider the differential equation

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t), \tag{1}$$

where the vector function $f(t)$ maps elements of $\mathbb{R}$ space into the elements of Banach space $\mathbf{B}$, $f(t) \in BC(\mathbb{R}, \mathbf{B})$

$$BC(\mathbb{R}, \mathbf{B}) := \left\{ f(\cdot) : \mathbb{R} \to \mathbf{B}, \ \ f(\cdot) \in C(\mathbb{R}, \mathbf{B}), \ \ |||f||| = \sup_{t \in \mathbb{R}} \|f(t)\|_{\mathbf{B}} < \infty \right\},$$

$BC(\mathbb{R}, \mathbf{B})$ is a Banach space of continuos and bounded on $\mathbb{R}$ functions; operator valued function $A(t)$ is strongly continuous with the corresponding norm $|||A||| = \sup_{t \in \mathbb{R}} \|A(t)\| < \infty$, and the solution $x = x(t)$ of the equation

$$x(t) = x_0 + \int\limits_{t_0}^{t} (A(s)x(s) + f(s))ds,$$

is continuously differentiable at each point $t \in \mathbb{R}$ is continuously differentiable at each point (1) everywhere on $\mathbb{R}$. It is necessary to find the solution $x(t)$ of the equation (1) in the space of Banach functions $BC^1(\mathbb{R}, \mathbf{B})$, that are continuously differentiable and bounded together with the derivative on $\mathbb{R}$. Assume that the homogeneous operator equation

$$\frac{dx(t)}{dt} = A(t)x(t), \tag{2}$$

is exponentially dichotomous on the semi-axes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors P and Q respectively, which means that there are projectors $P(P^2 = P)$ and $Q(Q^2 = Q)$, constants $k_{1,2} \geq 1$ and $\alpha_{1,2} > 0$ such that

$$\left\|U(t)PU^{-1}(s)\right\| \leq k_1 e^{-\alpha_1(t-s)}, \quad t \geq s,$$

$$\left\|U(t)(I-P)U^{-1}(s)\right\| \leq k_1 e^{-\alpha_1(s-t)}, \quad s \geq t, \quad \text{for any} \quad t, s \in \mathbb{R}_+,$$

and

$$\left\|U(t)QU^{-1}(s)\right\| \leq k_2 e^{-\alpha_2(t-s)}, \quad t \geq s,$$

$$\left\|U(t)(I-Q)U^{-1}(s)\right\| \leq k_2 e^{-\alpha_2(s-t)}, \quad s \geq t, \quad \text{for any} \quad t, s \in \mathbb{R}_-,$$

where $U(t) = U(t, 0)$ — evolutionary operator [8] of the equation (2) such that

$$\frac{dU(t)}{dt} = A(t)U(t), \quad U(0) = I,$$

— identity operator.

The following result is valid for the equation (1) [36].

Theorem 1. *Suppose that the homogeneous equation (2) is exponentially dichotomous on the semi-axes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors P and Q respectively. If the operator*

$$D = P - (I - Q) \colon \mathbf{B} \to \mathbf{B}, \tag{3}$$

which maps Banach space $\mathbf{B}$ to itself, is a generalized-invertible operator then:

(i) so that there are solutions of equations (1), bounded on the whole axis, it is necessary and sufficient that the vector function $f(t) \in BC(\mathbb{R}, \mathbf{B})$ satisfies the condition

$$\int\limits_{-\infty}^{+\infty} H(t)f(t)dt = 0, \tag{4}$$

where

$$H(t) = P_{N(D^*)}QU^{-1}(t) = P_{N(D^*)}(I-P)U^{-1}(t);$$

(ii) if condition (4) is satisfied, then the solutions of equation (1), bounded on the whole axis look as follows:

$$x_0(t, c) = U(t)PP_{N(D)}c + (G[f])(t) \quad \forall c \in \mathbf{B}, \tag{5}$$

where $(G[f])(t)$ [36] is a generalized Green's operator of the bounded on the whole axis solutions problem, D^- is generalized-invertible to the operator D, projectors $P_{N(D)} = I - D^-D$ and $P_{N(D^*)} = I - DD^-$, c is an arbitrary element of Banach space $\mathbf{B}$.

Remark 1. I. Recall that if the operator D is generalized-invertible [15], then:

(i) operator D is normally solvable ($\overline{R(D)} = R(D) = \operatorname{Im} D$);

(ii) the subspace $N(D) = \ker D$ has a direct complement in $\mathbf{B}$;

(iii) the subspace $R(D)$ has a direct complement in $\mathbf{B}$.

If these conditions hold, then there is always a generalized-invertible operator D^- for operator D.

II. In the finite-dimensional case, when $\mathbf{B} = \mathbb{R}^n$ and operator D is $(n \times n)$-dimensional matrix, properties (i), (ii), (iii) are always satisfied (since subspaces $N(D)$ and $N(D^*) = \operatorname{coker} D$ are finite-measurable, and therefore complementary and the operator D has closed set of values). Note that the condition (4) is equivalent to orthogonality inhomogeneities of equation (1) to the solutions of the corresponding homogeneous conjugate equation. In such situation, we obtain the well-known lemma of K. Palmer [12].

III. Application of the theory of generalized-inverse and pseudoinverse operators allow us to obtain previously known results and the new facts regarding our initial problem. If we consider the equation in $\mathbb{R}^n$ under the assumption that the corresponding linear homogeneous equation is exponentially dichotomous on the axes, then the operator

$$(Lx)(t) = \dot{x}(t) - A(t)x(t),$$

can only be Noetherian [14]. And if we consider the equation in Banach space $\mathbf{B}$, the initial problem would have much more cases of solutions. It follows from the theorem 1 that the operator L can be (according to the classification of S. Krein [42]):

(i) normally-solvable operator $\left(\overline{\operatorname{Im} L} = \operatorname{Im} L\right)$;

(ii) d-normal $\left(\overline{R(L)} = R(L), \dim \operatorname{coker} L < \infty\right)$;

(iii) n-normal $\left(\overline{R(L)} = R(L), \dim \ker L < \infty\right)$;

(iv) Noetherian $(\operatorname{ind} L = \dim \ker L - \dim \operatorname{coker} L < \infty)$;

(v) Fredholm $(\operatorname{ind} L = 0)$.

Linear and nonlinear equations with unbounded operator. Let us show, that the same results can be derived for unbounded operators. In Banach space $\mathbf{B}$ we consider the homogeneous differential equation

$$\frac{dx(t)}{dt} = A(t)x(t), \quad t \in J, \tag{6}$$

where for each $t \in J \subset \mathbb{R}$, the unbounded operator $A(t)$ is closed with a dense domain $D(A(t)) = D \subset \mathbf{B}$, which doesn't depend on t.

Remark 2. Since an arbitrary bounded operator is closed, it follows that the equation (6) is more general than the equation (2).

Let us remind a couple of classical concepts of the theory of semigroups of operators and the related with them facts.

Definition 1 [18; 43, p. 237]. *The set of bounded and linear operators* $\{T(t,s) \mid t \geq s;\ t,s \in$ $\in J\}$ *in Banach Space* $\mathbf{B}$ *is called an evolution operator if conditions:*

(i) $T(s,s) = I,\ s \in J$;

(ii) $T(t,\sigma)T(\sigma,s) = T(t,s),\ t \geq \sigma \geq s$ *in* J;

if $T(t,s)$ *additionally satisfies the condition*

(iii) $x \in \mathbf{B}$ *mapping* $(t,s) \mapsto T(t,s)x$ *continuous* $t \geq s$,
then $T(t,s)$ *is strongly continuous evolutionary operator (or a set of strongly continuous evolutionary operators).*

If the Cauchy problem generated by the equation (6) and by the initial condition $x(s,s,x_0) = x_0 \in D$ is uniformly correct [43], then we can determine for $t \geq s$ on J linear operator $T(t,s)\colon D \to \mathbf{B}$ by the rule

$$T(t,s)x_0 = x(t,s,x_0).$$

This result was proved by S. Krein [43, p. 237 – 239].

Statement 1. *Suppose that the Cauchy problem for the equation (6) is uniformly correct. Then the set of linear operators* $\{T(t,s),\ t \geq s,\ t,s \in J\}$, *which is defined above, is a strongly continuous evolutionary operator. In addition,* $T(t,s)D \subset D$ *and, if* $x \in D$, *then* $T(t,s)x$ *is differentiable with respect to the variable* t *and*

$$\frac{\partial}{\partial t} T(t,s)x = A(t)T(t,s)x, \quad t \geq s, \quad t,s \in J.$$

If the mapping $t \mapsto A(t)$ *is strongly continuous, then* $T(t,s)x$ *is differentiable for* $s < t$ *on* J *for all* $x \in D$ *and*

$$\frac{\partial}{\partial s} T(t,s)x = -T(t,s)A(s)x.$$

In this case, it is said, that $T(t,s)$ is evolutionary operator, *associated* with the equation (6).

For a fixed $t \geq s$ the operator $T(t,s)$ will be bounded linear operator and since the set D is dense in $\mathbf{B}$, it can be extended to the whole space $\mathbf{B}$ by continuity, which is assumed. The extension of the evolutionary operator to the whole space will be denoted in the same way.

Definition 2 [17, p. 245]**.** *Evolutionary operator* $\{T(t,s) \mid t \geq s;\ t,s \in J\}$ *admits an exponential dichotomy on* J, *if there exist projector-valued operator function* $\{P(t) \mid t \in J\}$ *in* $\mathcal{L}(\mathbf{B})$ *and the real constants* $\alpha > 0$ *and* $M \geq 1$ *such that:*

(i) $T(t,s)P(s) = P(t)T(t,s),\ t \geq s$*;*

(ii) restriction $T(t,s) \restriction_{N(P(s))},\ t \geq s$ *of the operator* $T(t,s)$ *to the kernel* $N(P(s))$ *of the projector* $P(s)$ *is an isomorphism from* $N(P(s))$ *to* $N(P(t))$, $T(s,t)$ *we define as inverse*

$$T(s,t) = (T(t,s) \restriction_{N(P(s))})^{-1}\colon N(P(t)) \to N(P(s));$$

(iii) $\|T(t,s)P(s)\| \leq Me^{-\alpha(t-s)},\ t \geq s$*;*

(iv) $\|T(t,s)(I - P(s))\| \leq Me^{-\alpha(s-t)},\ s \geq t.$

One of the most interesting cases is the case of the exponential dichotomy on the semi-axes $\mathbb{R}_s^- = (-\infty; s]$, $\mathbb{R}_s^+ = [s; \infty)$ (in this case, the projector-valued functions, which are defined on the semi-axes, we will denote by $P_-(t)$ for all $t \geq s$, and $P_+(t)$ for all $t \leq s$, with constants M_1, α_1 and M_2, α_2 respectively).

Using these concepts, we obtain the necessary and sufficient conditions for the existence of the weak bounded solutions for an inhomogeneous equation

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t), \quad t \in J, \tag{7}$$

where $f \in BC(J,\mathbf{B}) = \{f\colon J \to \mathbf{B}; f \text{ continuous and bounded}\}$. Boundedness we understand in the sense that $|||f||| = \sup_{t\in J} \|f(t)\| < \infty$. The unbounded operator $A(t)$ is closed with the dense domain $D(A(t)) = D \subset \mathbf{B}$, which does not depend on t.

Definition 3 [18]. *Let $f: J \to \mathbf{B}$ be a continuous vector function, and let $T(t,s)$ be a strongly continuous evolutionary operator on J, which is associated with (6) u is called weak (generalized) solution on J of inhomogeneous equation (7), if $u: J \to \mathbf{B}$ continuous and it satisfies the equality*

$$u(t) = T(t,\tau)u(\tau) + \int\limits_{\tau}^{t} T(t,s)f(s)ds, \quad t \geq \tau, \tag{8}$$

for each $\tau \in J$.

The main result of this part has the following form.

Theorem 2. *Let $\{T(t,s) \mid t \geq s \in \mathbb{R}\}$ be a strongly continuous evolutionary operator associated with the equation (6). Assume that the following conditions hold:*

(i) $T(t,s)$ admits an exponential dichotomy on the semi-axes $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ with projector-valued operator functions $P_+(t)$ and $P_-(t)$, respectively;

(ii) operator $D = P_+(0) - (I - P_-(0))$ is generalized invertible.

Then:

(i) so that there to exist weak solutions of the equation (7), bounded on the whole axis it is necessary and sufficient that the vector function $f \in BC(\mathbb{R}, \mathbf{B})$ satisfies the condition

$$\int\limits_{-\infty}^{+\infty} H(t)f(t)dt = 0, \tag{9}$$

where $H(t) = P_{N(D^*)}P_-(0)T(0,t)$;

(ii) if condition (9) holds, then the weak solutions of the equation (7) have the following form:

$$x_0(t,c) = T(t,0)P_+(0)P_{N(D)}c + (G[f])(t,0) \quad \forall c \in \mathbf{B}, \tag{10}$$

where

$$(G[f])(t,s) = \begin{cases} \displaystyle\int_s^t T(t,\tau)P_+(\tau)f(\tau)d\tau - \int_t^{+\infty} T(t,\tau)(I - P_+(\tau))f(\tau)d\tau + \\ \displaystyle\quad +T(t,s)P_+(s)D^-\left[\int_s^{\infty} T(s,\tau)(I - P_+(\tau))f(\tau)d\tau + \right. \\ \displaystyle\quad \left. + \int_{-\infty}^{s} T(s,\tau)P_-(\tau)f(\tau)d\tau\right], \quad t \geq s, \\ \displaystyle\int_{-\infty}^t T(t,\tau)P_-(\tau)f(\tau)d\tau - \int_t^{s} T(t,\tau)(I - P_-(\tau))f(\tau)d\tau + \\ \displaystyle\quad +T(t,s)(I - P_-(s))D^-\left[\int_s^{\infty} T(s,\tau)(I - P_+(\tau))f(\tau)d\tau + \right. \\ \displaystyle\quad \left. + \int_{-\infty}^{s} T(s,\tau)P_-(\tau)f(\tau)d\tau\right], \quad s \geq t, \end{cases}$$

is generalized Green operator of the problem of the solutions bounded on the whole axis:

$$(G[f])(0+,0) - (G[f])(0-,0) = -\int\limits_{-\infty}^{+\infty} H(t)f(t)dt,$$

$$\mathcal{L}(G[f])(t,0) = f(t), \quad t \in \mathbb{R},$$

and

$$(\mathcal{L}x)(t) := \frac{dx(t)}{dt} - A(t)x(t),$$

D^- *— generalized-invertible to the operator* D*;* $P_{N(D)} = I - D^-D,\ P_{N(D^*)} = I - DD^-$ *— projectors [44] onto the kernel and cokernel of the operator* D*.*

Remark 3. The same theorem remains correct even if the evolutionary operator $T(t,s)$ admits an exponential dichotomy on the $\mathbb{R}^+_s$ and $\mathbb{R}^-_s$ axes. Further in this paper (for the reader's convenience) the case $s=0$ is considered.

Proof. Solutions of equation (7), bounded on the semi-axes $\mathbb{R}^+_0$ and $\mathbb{R}^-_0$, have the following form:

$$x(t,\xi_1,\xi_2) = \begin{cases} T(t,0)P_+(0)\xi_1 - \int_t^{+\infty} T(t,\tau)(I-P_+(\tau))f(\tau)d\tau + \\ \quad + \int_0^t T(t,\tau)P_+(\tau)f(\tau)d\tau, \quad t \geq 0, \\ T(t,0)(I-P_-(0))\xi_2 + \int_{-\infty}^t T(t,\tau)P_-(\tau)f(\tau)d\tau - \\ \quad - \int_t^0 T(t,\tau)(I-P_-(\tau))f(\tau)d\tau, \quad t \leq 0. \end{cases} \tag{11}$$

Indeed,

$$\|T(t,0)P_+(0)\xi_1\| \leq \|T(t,0)P_+(0)\|\|\xi_1\| \leq M_1e^{-\alpha_1 t}\|\xi_1\|,$$

and

$$\frac{\partial(T(t,0)P_+(0)\xi_1)}{\partial t} = A(t)T(t,0)P_+(0)\xi_1.$$

Thus, the expression $T(t,0)P_+(0)\xi_1$ defines all bounded on $\mathbb{R}^+_0$ solutions of the homogeneous equation (6).

We now prove the boundedness of one of the integrals defined in (11).

$$\left\| \int\limits_t^{+\infty} T(t,\tau)(I-P_+(\tau))f(\tau)d\tau \right\| \leq \int\limits_t^{+\infty} \|T(t,\tau)(I-P_+(\tau))\|\|f(\tau)\|d\tau \leq$$

$$\leq \sup_{t\in\mathbb{R}} \|f(t)\| \int\limits_t^{+\infty} M_1 e^{\alpha_1(t-\tau)}d\tau = \frac{M_1}{\alpha_1}|||f||| < \infty.$$

The boundedness of other integrals is checked in the same way. Based on the following equation:

$$\frac{\partial\left(-\int_t^{+\infty} T(t,\tau)\left(I-P_+(\tau)\right)f(\tau)d\tau + \int_0^t T(t,\tau)P_+(\tau)f(\tau)d\tau\right)}{\partial t} =$$

$$= T(t,t)(I - P_+(t))f(t) - A(t)\int\limits_t^{+\infty} T(t,\tau)(I - P_+(\tau))f(\tau)d\tau +$$

$$+ T(t,t)P_+(t)f(t) + A(t)\int\limits_0^t T(t,\tau)P_+(\tau)f(\tau)d\tau =$$

$$= f(t) + A(t)\left\{-\int\limits_t^{+\infty} T(t,\tau)(I - P_+(\tau))f(\tau)d\tau + \int\limits_0^t T(t,\tau)P_+(\tau)f(\tau)d\tau\right\},$$

we see that the expression (11) really defines all bounded solutions of the equation (7) on the semi-axes.

In order for the expression (11) to define weak bounded solutions along the entire axis, it is necessary and sufficient for the following condition to hold:

$$x(0+,\xi_1,\xi_2) = x(0-,\xi_1,\xi_2).$$

This condition is equivalent to the solvability of the operator equation

$$P_+(0)\xi_1 - \int\limits_0^{+\infty} T(0,\tau)(I - P_+(\tau))f(\tau)d\tau = (I - P_-(0))\xi_2 + \int\limits_{-\infty}^0 T(0,\tau)P_-(\tau)f(\tau)d\tau. \qquad (12)$$

If ξ_1 and ξ_2 are the solutions of equation (12), then using them in (11) gives us a weak solution of the equation (7) on the whole axis. Let us show that if conditions of the theorem hold, then set of weak and bounded on the whole axis solutions of (7) can be found as

$$x(t,\xi) = \begin{cases} T(t,0)P_+(0)\xi - \int_t^{+\infty} T(t,\tau)(I - P_+(\tau))f(\tau)d\tau + \\ \quad + \int_0^t T(t,\tau)P_+(\tau)f(\tau)d\tau, \quad t \geq 0, \\ T(t,0)(I - P_-(0))\xi + \int_{-\infty}^t T(t,\tau)P_-(\tau)f(\tau)d\tau - \\ \quad - \int_t^0 T(t,\tau)(I - P_-(\tau))f(\tau)d\tau, \quad t \leq 0. \end{cases} \qquad (13)$$

This means that cardinality of weak bounded solutions (11) and (13) coincide and the elements ξ_1 and ξ_2 can be chosen as $\xi = \xi_1 = \xi_2$. Clearly that any bounded solution of the form (13) is contained in the set of weak bounded solutions (11) under the conditions of solvability.

Let us rewrite the equation (12) in the following form:

$$P_+(0)\xi_1 = (I - P_-(0))\xi_2 + g, \qquad (14)$$

where

$$g = \int\limits_{-\infty}^0 T(0,\tau)P_-(\tau)f(\tau)d\tau + \int\limits_0^{+\infty} T(0,\tau)(I - P_+(\tau))f(\tau)d\tau.$$

By using that $P_+^2(0) = P_+(0),$ we obtain for each element $\xi_1: P_+^2(0)\xi_1 = P_+(0)\xi_1.$ Putting that in (14) we obtain that

$$P_+(0)((I - P_-(0))\xi_2 + g) = (I - P_-(0))\xi_2 + g$$

or

$$P_+(0)(I - P_-(0))\xi_2 - (I - P_-(0))\xi_2 = g - P_+(0)g.$$

Now, by using the fact that $(I - P_-(0))^2 = I - P_-(0),$ we obtain the operator equation

$$P_+(0)(I - P_-(0))\xi_2 - (I - P_-(0))^2\xi_2 = g - P_+(0)g,$$

which can be rewritten in the following way:

$$D(I - P_-(0))\xi_2 = (I - P_+(0))g. \tag{15}$$

By the condition of the theorem, the operator D is normally solvable (moreover, generalized-invertible), therefore [15], the following condition will be necessary and sufficient: $P_{N(D^*)}(I - P_-(0))g = 0$ (according to [44] $P_{N(D^*)}$ can be derived by using the equality $P_{N(D^*)} = I - DD^-$). Since $P_{N(D^*)}D = 0$ (this follows, for example, from the fact that

$$P_{N(D^*)}D = \left(I - DD^-\right) D = D - DD^-D = D - D = 0),$$

or, what is the same, $P_{N(D^*)}\left(P_+(0) - (I - P_-(0))\right) = 0$, then $P_{N(D^*)}(I - P_+(0)) = P_{N(D^*)}P_-(0).$) Based on this, we find that $P_{N(D^*)}g$:

$$P_{N(D^*)} \int_{-\infty}^{0} T(0,\tau)P_-(\tau)f(\tau)d\tau + P_{N(D^*)} \int_{0}^{+\infty} T(0,\tau)(I - P_+(\tau))f(\tau)d\tau =$$

$$= P_{N(D^*)} \left(P_-^2(0) \int_{-\infty}^{0} T(0,\tau)f(\tau)d\tau + (I - P_+(0))^2 \int_{0}^{+\infty} T(0,\tau)f(\tau)d\tau \right) =$$

$$= P_{N(D^*)}P_-(0) \left(\int_{-\infty}^{0} P_-(0)T(0,\tau)f(\tau)d\tau + \int_{0}^{+\infty} (I - P_+(0))T(0,\tau)f(\tau)d\tau \right) =$$

$$= P_{N(D^*)}P_-(0)g = P_{N(D^*)}(I - P_+(0))g = 0.$$

The condition $P_{N(D^*)}g = 0$ is necessary and sufficient for the solvability of the equation $D\xi = g$ or $P\xi = (I - Q)x + g,$ which is the equation (14), assuming that $\xi = \xi_1 = \xi_2.$ Therefore, it is proved that the set of bounded solutions (11) is a subset of the set of bounded solutions (13) and, therefore, they are equal. After all that we get that the condition of the existence of bounded on the whole axis solutions of equation (7) is equivalent to the solvability of the operator equation

$$D\xi = \int_{0}^{\infty} T(0,\tau)(I - P_+(\tau))f(\tau)d\tau + \int_{-\infty}^{0} T(0,\tau)P_-(\tau)f(\tau)d\tau. \tag{16}$$

Since the operator D is generalized-invertible, then the equation (16) has the solutions if and only if

$$P_{N(D^*)}\left\{\int\limits_0^{\infty} T(0,\tau)(I-P_+(\tau))f(\tau)d\tau + \int\limits_{-\infty}^{0} T(0,\tau)P_-(\tau)f(\tau)d\tau\right\} = 0.$$

If this condition holds, then the equation (16) has a set of solutions

$$\xi = D^-\left(\int\limits_0^{\infty} T(0,\tau)(I-P_+(\tau))f(\tau)d\tau + \int\limits_{-\infty}^{0} T(0,\tau)P_-(\tau)f(\tau)d\tau\right) + P_{N(D)}c,$$

where c is an arbitrary element of Banach space $\mathbf{B}$. Substituting the obtained solutions in (11) we obtain (10).

Let us prove the property of the generalized Green operator with respect to the jump at 0, by using the notation

$$(G[f])(0+0)-(G[f])(0-0) = -\int\limits_0^{\infty} T(0,\tau)(I-P_+(\tau))f(\tau)d\tau + P_+(0)D^-g-$$

$$-\int\limits_{-\infty}^{0} T(0,\tau)P_-(\tau)f(\tau)d\tau - (I-P_-(0))D^-g =$$

$$= -g + P_+(0)D^-g - D^-g + P_-(0)D^-g =$$

$$= (P_+(0)-I+P_-(0))D^-g - g$$

$$= DD^-g - g = -\left(I-DD^-\right)g =$$

$$= -P_{N(D^*)}g = -\int\limits_{-\infty}^{+\infty} H(t)f(t)dt.$$

The second property of the theorem is verified by a direct substitution of the Green operator in the equation (7).

The theorem is proved.

To show the relationship between the proved statement and the results of [18] we formulate a lemma.

Lemma 1. *If the projectors $P_+(0)$ and $P_-(0)$ commute ($[P_+(0), P_-(0)] = 0$), then the operator $D = P_+(0) - (I - P_-(0))$ always has a generalized-inverse, which is equal to the operator D.*

Proof. It is clear that the operator D is bounded. First let us find

$$D^2 = (P_+(0)-I+P_-(0))(P_+(0)-I+P_-(0))$$

$$= P_+^2(0) - P_+(0) + P_-(0)P_+(0)-$$

$$- P_+(0) + I - P_-(0) + P_+(0)P_-(0) - P_-(0) + P_+^2(0) =$$
$$= 2P_+(0)P_-(0) - P_+(0) - P_-(0) + I.$$

Then

$$D^3 = D^2 D = (2P_+(0)P_-(0) - P_+(0) - P_-(0) + I)(P_+(0) - I + P_-(0)) =$$
$$= 2P_+(0)P_-(0)P_+(0) - P_+(0)^2 - P_-(0)P_+(0) + P_+(0) - 2P_+(0)P_-(0)+$$
$$+ P_+(0) + P_-(0) - I + 2P_+(0)P_-(0)^2 - P_+(0)P_-(0) - P_-(0)^2 + P_-(0) =$$
$$= -2P_+(0)P_-(0) + 2P_+(0)P_-(0) + P_+(0) + P_-(0) - I = D.$$

From the obtained equation it follows that $DDD = D$, which means that D is generalized-invertible and $D = D^-$.

Thus, Lemma 1 is proved.

Remark 4. The main results of [18] are obtained under the assumption, that projectors $P_+(0)$ and $P_-(0)$ commute. From Lemma 1 and the previous theorem we obtain results from [18].

Remark 5. Note that a similar result is considered in Noetherian case in [23]. Since the Noetherian operator is generalized-invertible, then this result also satisfies the conditions of the proved theorem.

Nonlinear differential equations in Banach space with an unbounded operator in the linear part. In the Banach space $\mathbf{B}$ let us consider the differential equation

$$\frac{dx(t,\varepsilon)}{dt} = A(t)x(t,\varepsilon) + \varepsilon Z(x(t,\varepsilon), t, \varepsilon) + f(t). \tag{17}$$

We look for a bounded solution $x(t,\varepsilon)$ of equation (17), which turns into one of the solutions $x(t,0) = x_0(t,c)$ of the generating equation at $\varepsilon = 0$.

To find the necessary condition, we assume that the operator function $Z(x,t,\varepsilon)$ satisfies the requirement:

$$Z(\cdot,\cdot,\cdot) \in C[\|x - x_0\| \le q] \times BC(\mathbb{R}, \mathbf{B}) \times C[0, \varepsilon_0],$$

where q is a positive constant (continuity around the generating solution).

We show that this problem can be solved by using the operator equation

$$F(c) = \int\limits_{-\infty}^{+\infty} H(t)Z(x_0(t,c), t, 0)dt = 0. \tag{18}$$

We will call it the *equation for generated elements*.

Theorem 3 (necessary condition)**.** *Suppose that the homogeneous equation (6) is an exponential dichotomous on semi-axes* $\mathbb{R}_0^+$ *and* $\mathbb{R}_0^-$ *with projector-valued operator functions* $P_+(t)$ *and* $P_-(t)$ *respectively, and nonlinear equation (17) as a bounded solution* $x(\cdot,\varepsilon)$, *which turns into one of the solutions of the generated equation (6) with the element* $c = c^0 : x(t,0) = x_0\left(t, c^0\right)$ *for* $\varepsilon = 0$. *Then this element must satisfy the equation for generating elements (18).*

Proof. If equation (17) has a bounded solutions $x(t,\varepsilon)$, then, according to the proven theorem 2, the solvability condition must be satisfied

$$\int\limits_{-\infty}^{+\infty} H(t)\big\{f(t)+\varepsilon Z(x(t,\varepsilon),t,\varepsilon)\big\}dt = 0. \tag{19}$$

By using condition (9), we obtain that condition (19) is equivalent to

$$\varepsilon\int\limits_{-\infty}^{+\infty} H(t)Z(x(t,\varepsilon),t,\varepsilon)dt = 0.$$

After simplification by epsilon $\varepsilon \neq 0$ we obtain

$$\int\limits_{-\infty}^{+\infty} H(t)Z(x(t,\varepsilon),t,\varepsilon)dt = 0.$$

For $\varepsilon \to 0,\ x(t,\varepsilon) \to x_0(t,c^0)$. Finally (by using the continuity of the operator function $Z(x,t,\varepsilon)$), we obtain

$$F\left(c^0\right) = \int\limits_{-\infty}^{+\infty} H(t)Z(x_0\left(t,c^0\right),t,0)dt = 0,$$

that proves the theorem.

To obtain a sufficient condition for the existence of bounded solutions of equation (17) additionally, we assume that operator function $Z(x,t,\varepsilon)$ is strongly differentiable in the neighborhood of the generated solution

$$Z(\cdot,t,\varepsilon) \in C^1[\|x-x_0\| \leq q].$$

Let us show that this problem can be solved with the help of the operator

$$B_0 = \int\limits_{-\infty}^{+\infty} H(t)A_1(t)T(t,0)P_+(0)P_{N(D)}dt \colon \mathbf{B} \to \mathbf{B},$$

where $A_1(t) = Z^1(v,t,\varepsilon)\big|_{v=x_0;\varepsilon=0}$ (derivative in the sense of Frechet).

Theorem 4 (sufficient condition)**.** *Suppose that homogeneous equation (6) admits exponential dichotomy on semi-axes $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ with projector-valued operator functions $P_+(t)$ and $P_-(t)$ respectively, and the equation (7) has bounded solutions. Suppose that the following conditions for the operator B_0 are satisfied:*

(i) the operator B_0 is generalized-invertible;

(ii) $P_{N(B_0^*)}P_{N(D^*)}P_-(0) = 0$.

Then for an arbitrary element $c = c^0 \in \mathbf{B}$, that satisfies the equation for generating elements (18), there is at least one weak bounded solution of equation (17).

This solution can be found by an iterative process

$$\overline{y}_{k+1}(t,\varepsilon) = \varepsilon G\left[Z\left(x_0\left(\tau,c^0\right)+y_k,\tau,\varepsilon\right)\right](t,0),$$

$$c_k = -B_0^- \int\limits_{-\infty}^{+\infty} H(\tau) \left\{ A_1(\tau) \overline{y}_k(\tau, \varepsilon) + \mathcal{R}(y_k(\tau, \varepsilon), \tau, \varepsilon) \right\} d\tau,$$

$$y_{k+1}(t, \varepsilon) = T(t, 0) P_+(0) P_{N(D)} c_k + \overline{y}_{k+1}(t, \varepsilon),$$

$$x_k(t, \varepsilon) = x_0 \left(t, c^0\right) + y_k(t, \varepsilon), \quad k = 0, 1, 2, \ldots, \quad y_0(t, \varepsilon) = 0,$$

$$x(t, \varepsilon) = \lim_{k \to \infty} x_k(t, \varepsilon).$$

Proof. In equation (17) we perform the following substitution of variables $x(t, \varepsilon) = = x_0 \left(t, c^0\right) + y(t, \varepsilon)$, where the element c^0 satisfies (18). Thus, we derive the equation in respect to y:

$$\frac{dy(t)}{dt} = A(t) y(t) + \varepsilon Z \left(x_0 \left(t, c^0\right) + y(t, \varepsilon), t, \varepsilon\right). \tag{20}$$

It is necessary to find a bounded solution

$$y(t, \varepsilon) \colon y(\cdot, \varepsilon) \in BC^1(\mathbb{R}, \mathbf{B}), y(t, \cdot) \in C[0, \varepsilon_0], y(t, 0) = 0.$$

It is obvious that the solvability of equation (20) is equivalent to the solvability of equation (17). This condition for y looks as follows:

$$\int\limits_{-\infty}^{+\infty} H(t) Z(x_0 \left(t, c^0\right) + y(t, \varepsilon), t, \varepsilon) dt = 0. \tag{21}$$

If condition (21) holds, the set of bounded solutions of equation (20) will look like this:

$$y(t, \varepsilon) = T(t, 0) P_+(0) P_{N(D)} c + \overline{y}(t, \varepsilon),$$

where

$$\overline{y}(t, \varepsilon) = \varepsilon G[Z(x_0 + y, \tau, \varepsilon)](t, 0).$$

Since the operator $Z(x, t, \varepsilon)$ is differentiable by Frechet in the neighborhood of the generating solution, then the idea holds for it

$$Z \left(x_0 \left(t, c^0\right) + y(t, \varepsilon), t, \varepsilon\right) = Z \left(x_0 \left(t, c^0\right), t, 0\right) + A_1(t) y(t, \varepsilon) + \mathcal{R}(y(t, \varepsilon), t, \varepsilon),$$

where $A_1(t) = Z^{(1)}(v, t, \varepsilon)\big|_{v = x_0, \varepsilon = 0}$, and for members $\mathcal{R}(y, t, \varepsilon)$ of higher-order for y the equations hold

$$\mathcal{R}(0, t, 0) = 0, \qquad \mathcal{R}_x^{(1)}(0, t, 0) = 0.$$

The condition (21) can then be rewritten as follows:

$$\int\limits_{-\infty}^{+\infty} H(t) \left\{ Z \left(x_0 \left(t, c^0\right), t, 0\right) + A_1(t) \left\{ T(t, 0) P_+(0) P_{N(D)} c + \overline{y}(t, \varepsilon) \right\} \right\} dt +$$

$$+\int\limits_{-\infty}^{+\infty} H(t)\mathcal{R}(y(t,\varepsilon),t,\varepsilon)dt = 0. \tag{22}$$

By using the notation, we rewrite the condition (22) as an operator equation with respect to c:

$$B_0 c = -\int\limits_{-\infty}^{+\infty} H(t)\left\{A_1(t)\overline{y}(t,\varepsilon) + \mathcal{R}(y(t,\varepsilon),t,\varepsilon)\right\}dt. \tag{23}$$

A necessary and sufficient condition for the solvability of the operator equation (23) will be the condition

$$P_{N(B_0^*)}\int\limits_{-\infty}^{+\infty} H(t)\left\{A_1(t)\overline{y}(t,\varepsilon) + \mathcal{R}(y(t,\varepsilon),t,\varepsilon)\right\}dt = 0,$$

which holds by assuming (ii) of Theorem 4:

$$P_{N\left(B_0^*\right)}H(t) = P_{N\left(B_0^*\right)}P_{N(D^*)}P_-(0)T(0,t) = 0.$$

Then the element c can be selected in the form

$$c = -B_0^-\int\limits_{-\infty}^{+\infty} H(t)\left\{A_1(t)\overline{y}(t,\varepsilon) + \mathcal{R}(y(t,\varepsilon),t,\varepsilon)\right\}dt.$$

So we obtain this operator system

$$\begin{aligned} &y(t,\varepsilon) = T(t,0)P_+(0)P_{N(D)}c + \overline{y}(t,\varepsilon),\\ &c = -B_0^-\int\limits_{-\infty}^{+\infty} H(t)\left\{A_1(t)\overline{y}(t,\varepsilon) + \mathcal{R}(y(t,\varepsilon),t,\varepsilon)\right\}dt,\\ &\overline{y}(t,\varepsilon) = \varepsilon G\left[Z(x_0+y,\tau,\varepsilon)\right](t,0). \end{aligned} \tag{24}$$

Here we need to introduce the vector $u = (y, c, \overline{y})^T \in \mathbf{B}\times\mathbf{B}\times\mathbf{B}$, which belongs to the Cartesian product $\mathbf{B}^3$ (T means a transposition operation). Considering the auxiliary operator

$$L_1 g = -B_0^-\int\limits_{-\infty}^{+\infty} H(t)A_1(t)g(t)dt.$$

Operator system (24) can be rewritten as follows:

$$u = \begin{bmatrix} 0 & T(t,0)P_+(0)P_{N(D)} & I\\ 0 & 0 & L_1\\ 0 & 0 & 0 \end{bmatrix} u + \begin{pmatrix} 0\\ -B_0^-\int_{-\infty}^{+\infty} H(t)\mathcal{R}(y,t,\varepsilon)dt\\ \varepsilon G[Z(x_0+y,\tau,\varepsilon)](t,0) \end{pmatrix}.$$

This operator system is equivalent to this one:

$$\begin{bmatrix} I & -T(t,0)P_+(0)P_{N(D)} & -I \\ 0 & I & -L_1 \\ 0 & 0 & I \end{bmatrix} u = \begin{pmatrix} 0 \\ -B_0^- \displaystyle\int_{-\infty}^{+\infty} H(t)\mathcal{R}(y,t,\varepsilon)dt \\ \varepsilon G[Z(x_0+y,\tau,\varepsilon)](t,0) \end{pmatrix}. \tag{25}$$

Let us enter the notation

$$M := \begin{bmatrix} I & -T(t,0)P_+(0)P_{N(D)} & -I \\ 0 & I & -L_1 \\ 0 & 0 & I \end{bmatrix}, \quad g = \begin{pmatrix} 0 \\ -B_0^- \displaystyle\int_{-\infty}^{+\infty} H(t)\mathcal{R}(y,t,\varepsilon)dt \\ \varepsilon G[Z(x_0+y,\tau,\varepsilon)](t,0) \end{pmatrix}.$$

The operator M has a bounded inverse M^{-1}:

$$M^{-1} = \begin{bmatrix} I & T(t,0)P_+(0)P_{N(D)} & T(t,0)P_+(0)P_{N(D)}L_1 + I \\ 0 & I & L_1 \\ 0 & 0 & I \end{bmatrix}.$$

We can check that so defined operator satisfies the equality

$$MM^{-1} = M^{-1}M = I$$

is checked by direct substitution.

We prove that this equality determines a bounded operator.

We need to prove, that there is a constant $c_1 > 0$, that for all $u \in \mathbf{B}^3$ the inequality $\left\|M^{-1}u\right\|_{\mathbf{B}^3} \le c_1\|u\|_{\mathbf{B}^3}$ holds. This inequality is equivalent [43] to the following one (with the constant $c_2 > 0$): for arbitrary $y, c, \overline{y} \in \mathbf{B}$

$$\left|\left|\left|M^{-1}(y,c,\overline{y})^T\right|\right|\right|_{\mathbf{B}^3} \le c_2\left(|||y|||_{\mathbf{B}} + |||c|||_{\mathbf{B}} + |||\overline{y}|||_{\mathbf{B}}\right),$$

$$M^{-1}(y,c,\overline{y})^T \begin{pmatrix} y + T(t,0)P_+(0)P_{N(D)}c + T(t,0)P_+(0)P_{N(D)}L_1\overline{y} + \overline{y} \\ c + L_1\overline{y}\overline{y} \end{pmatrix}.$$

Let's now prove the boundedness of the norm of each component of the vector in the Banach space $\mathbf{B}$:

$$\left|\left|\left|y + T(\cdot,0)P_+(0)P_{N(D)}c + T(\cdot,0)P_+(0)P_{N(D)}L_1\overline{y} + \overline{y}\right|\right|\right|_{\mathbf{B}} \le$$
$$\le |||y|||_{\mathbf{B}} + \left|\left|\left|T(\cdot,0)P_+(0)P_{N(D)}\right|\right|\right|_{\mathbf{B}} |||c|||_{\mathbf{B}} + \left|\left|\left|T(\cdot,0)P_+(0)P_{N(D)}L_1\overline{y}\right|\right|\right|_{\mathbf{B}} +$$
$$+ |||\overline{y}|||_{\mathbf{B}} \le |||y|||_{\mathbf{B}} + c_1\,|||c|||_{\mathbf{B}} + c_2\,|||\overline{y}|||_{\mathbf{B}};$$

Similarly,

$$|||c + L_1\overline{y}|||_{\mathbf{B}} \le |||c|||_{\mathbf{B}} + |||L_1|||_{\mathbf{B}}\,|||\overline{y}|||_{\mathbf{B}} \le |||c|||_{\mathbf{B}} + c_3\,|||\overline{y}|||_{\mathbf{B}}.$$

So,

$$\left|\left|\left|L^{-1}(y,c,\overline{y})^T\right|\right|\right|_{\mathbf{B}^3} \leq |||y|||_{\mathbf{B}} + (c_1+1)\,|||c|||_{\mathbf{B}} + (1+c_2+c_3)\,|||\overline{y}|||_{\mathbf{B}} \leq$$
$$\leq c_4\left(|||y|||_{\mathbf{B}} + |||c|||_{\mathbf{B}} + |||\overline{y}|||_{\mathbf{B}}\right),$$

where $c_4 = \max\{1, 1+c_1, 1+c_2+c_3\}$. From this, we obtain the boundedness of the operator L^{-1}.

Then this operator system (24) we write in the form

$$u = M^{-1}g = M^{-1}S(\varepsilon)u,$$

where the operator $S(\varepsilon)$ is generally nonlinear. Varying the parameter ε and by using boundedness of the operator M^{-1}, we can derive that the operator $M^{-1}S(\varepsilon)$ is nonexpanding. Then it follows from the principle of contracting mappings [43] that the operator system (24) has the only fixed point, which defines the bounded solution of (17).

The relationship between the necessary and sufficient conditions. To establish a connection between the necessary and sufficient conditions of the solvability of the equation (17) let us first define the auxiliary statement.

Corollary 1. *Assume that the operator $F(c)$ has a derivative in the sense of Frechet $F^{(1)}(c)$ for each element c^0 of the Banach space $\mathbf{B}$, which satisfies the equation for the generated elements (18). If the operator B_0 has a bounded inverse, then the equation (18) has the only bounded for each c^0 solution on the whole axis.*

Proof.

$$F^{(1)}(c)[h] = \int\limits_{-\infty}^{+\infty} H(t)\; Z^{(1)}(v,t,\varepsilon)\Big|_{v=x_0,\varepsilon=0} \left[x_0^{(1)}(t,s,c)[h]\right] dt.$$

This representation follows from the theorem of the superposition of differential mappings in Banach space [45]. Let us find the derivative of the solution $x_0^{(1)}(t,c)$ for c. Since $x_0(t,c) = T(t,0)P_+(0)P_{N(D)}c + (G[f])(t,0)$, then [45]

$$x_0^{(1)}(t,c)[h] = \frac{\partial x_0(t,c+\alpha h)}{\partial\alpha}\bigg|_{\alpha=0} =$$
$$= \frac{\partial}{\partial\alpha}\left(T(t,0)P_+(0)P_{N(D)}c + \alpha T(t,0)P_+(0)P_{N(D)}h + (G[f])(t,0)\right)\bigg|_{\alpha=0} =$$
$$= \frac{\partial}{\partial\alpha}\left(T(t,0)P_+(0)P_{N(D)}c\right)\bigg|_{\alpha=0} + \frac{\partial}{\partial\alpha}\left(\alpha T(t,0)P_+(0)P_{N(D)}h\right)\bigg|_{\alpha=0} +$$
$$+ \frac{\partial}{\partial\alpha}(G[f])(t,0)\bigg|_{\alpha=0} =$$
$$= T(t,0)P_+(0)P_{N(D)}h,$$

and

$$Z^{(1)}(v,t,\varepsilon)\Big|_{v=x_0,\varepsilon=0} = A_1(t).$$

In the end, we will get

$$F^{(1)}(c)[h] = \int\limits_{-\infty}^{+\infty} H(t)A_1(t)T(t,0)P_+(0)P_{N(D)}dt[h] = B_0[h].$$

Because $F^{(1)}(c) = B_0$ is invertible, the equation (18) has only one solution, and so the equation (17) has the only one bounded on the whole axis solution.

Remark 6. Since the operator $F^{(1)}(c)$ is invertible, then for the operator B_0 conditions (i) and (ii) of Theorem 4 hold. Then the equation (17) will have a unique bounded solution for each $c^0 \in \mathbf{B}$. Thus, the condition of invertibility of $F^{(1)}(c)$ connects necessary and sufficient conditions. In the finite-dimensional case, the condition of invertibility of $F^{(1)}(c)$ is equivalent to the condition of the simplicity of the root c^0 of the solution for generating constants [15].

Generalized bounded solutions of linear evolutionary equations in locally convex functions. The analysis of differential equations in locally convex and Frechet spaces is an actual task even today. Such tasks are widely used by many areas in mathematics, in particular, mathematical physics.

In this part of the paper, the generalization of e-dichotomy for evolutionary equations in locally convex spaces with unbounded operator function is considered. Necessary and sufficient conditions for the existence of the generalized bounded solutions for differential equations of the first order in Frechet space with an unbounded operator are stated.

Statement of the problem. In the complete locally convex space E we consider inhomogeneous differential equation with unbounded operator function $A(t)$:

$$\frac{dx(t)}{dt} = A(t)x + f(t), \quad t \in J, \tag{26}$$

where for each $t \in J$, $A(t)$ is a linear closed operator function with independent from time t, dense domain $D(A(t)) = D \subset E$, the vector function $f(t)$ is bounded and independent on J.

Assume [46], that there is bounded operator function $U(t)$, $t \in J$, with domain $D(U(t)) = D$ such that for each $x \in D$ it satisfies the equation

$$\frac{d}{dt}U(t)x = A(t)U(t)x, \quad U(0) = I.$$

Since the set D is dense on E, then $U(t)$ can be extended by continuity to the whole space E. The operator function $U(t)$ is traditionally called *the evolutionary operator*, which corresponds to a homogeneous equation

$$\frac{dx(t)}{dt} = A(t)x(t), \quad t \in J. \tag{27}$$

For simplicity, we additionally assume that for each t, there is an evolutionary operator $U(t)$ that has bounded inverse $U^{-1}(t)\colon E \to E$ (so-called parabolic case).

Definition 4. *If $f\colon J \to E$ is continuous, bounded and $U(t)$ is an evolutionary operator, then the vector function $u(t)$ is called a generalized (weak) solution of the equation (26), if $u(t)$ is continuous and the following equality holds for an arbitrary $t \in J$:*

$$u(t) = U(t)U^{-1}(\tau)u(\tau) + \int\limits_{\tau}^{t} U(t)U^{-1}(s)f(s)ds, \quad t \geq \tau. \tag{28}$$

Definition 5. *We say that the equation (27) admits an exponential dichotomy on* $J,$ *if there is a projector* $P \in \mathcal{L}(E)$ *such that for an arbitrary semi-norm* $q \in \operatorname{Spec} E$ *there exist semi-norm* $p \in \operatorname{Spec} E,$ *constants* $K \geq 1,\ \alpha > 0$ *such that the following inequalities are satisfied:*

$$q(U(t)PU^{-1}(s)\xi) \leq Ke^{-\alpha(t-s)}p(\xi), \quad t \geq s,$$

$$q(U(t)(I-P)U^{-1}(s)\xi) \leq Ke^{-\alpha(s-t)}p(\xi), \quad s \geq t,$$

where $\operatorname{Spec} E$ *is the set of all semi-norms, that are defined on* $E.$

This definition, introduced in the papers [38, 47], helps to analyze the question of the existence of generalized solutions by analogy with how it was done in Banach space.

Theorem 5. *Let the homogeneous equation (27) be exponentially-dichotomous on axes* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P_+ *and* P_- *respectively. Then:*

(i) equation (26) has generalized bounded solutions if and only if operator equation

$$P_+\xi_1 - (I - P_-)\xi_2 = g, \tag{29}$$

has solutions; here

$$g = \int\limits_{-\infty}^{0} P_- U^{-1}(\tau) f(\tau) d\tau + \int\limits_{0}^{+\infty} (I - P_+) U^{-1}(\tau) f(\tau) d\tau;$$

(ii) if the solvability conditions of the equation (29) are satisfied, equation (26) has bounded on the whole axis solutions, which are defined as follows:

$$x(t,\xi_1,\xi_2) = \begin{cases} U(t)P_+\xi_1 - \displaystyle\int_t^{+\infty} U(t)P_+U^{-1}(\tau)f(\tau)d\tau + \\ \qquad + \displaystyle\int_0^t U(t)(I-P_+)U^{-1}(\tau)f(\tau)d\tau, \quad t \geq 0, \\ U(t)(I-P_-)\xi_2 + \displaystyle\int_{-\infty}^t U(t)P_-U^{-1}(\tau)f(\tau)d\tau - \\ \qquad - \displaystyle\int_t^0 U(t)(I-P_-)U^{-1}(\tau)f(\tau)d\tau, \quad t \leq 0. \end{cases} \tag{30}$$

Proof. The solutions of equation (26), bounded on the semi-axes, have the form (30). We can prove that for example (30) define bounded solutions for a nonnegative real axis $t \geq 0.$

From the definition of evolutionary operator for the equation (26) we have

$$\frac{d(U(t)P_+\xi_1)}{dt} = A(t)U(t)P_+\xi_1.$$

From the fact that equation (27) is e-dichotomous, it follows that for an arbitrary semi-norm $q \in \operatorname{Spec} E$ there exists a semi-norm $p \in \operatorname{Spec} E$ such that

$$q(U(t)P_+\xi_1) \leq K_1 e^{-\alpha t} p(\xi_1), \quad t \geq 0.$$

Thus, $U(t)P_+\xi_1$ defines the set of bounded solutions of the homogeneous equation (27):

$$\frac{d\left(-\displaystyle\int_t^{+\infty} U(t)P_+U^{-1}(\tau)f(\tau)d\tau + \int_0^t U(t)(I-P_+)U^{-1}(\tau)f(\tau)d\tau\right)}{dt} =$$

$$= U(t)P_+U^{-1}(t)f(t) + A(t)\int_0^t U(t)P_+U^{-1}(\tau)f(\tau)d\tau +$$

$$+ U(t)(I - P_-)U^{-1}(t)f(t) - A(t)\int_t^{+\infty} U(t)P_+U^{-1}(\tau)f(\tau)d\tau =$$

$$= f(t) + A(t)\left\{-\int_t^{+\infty} U(t)P_+U^{-1}(\tau)f(\tau)d\tau + \int_0^t U(t)(I - P_+)U^{-1}(\tau)f(\tau)d\tau\right\}.$$

We prove the boundedness of one of the integrals

$$q\left(\int_t^{+\infty} U(t)P_+U^{-1}(\tau)f(\tau)d\tau\right) \le \int_t^{+\infty} q(U(t)P_+U^{-1}(\tau)f(\tau))d\tau \le$$

$$\le \int_t^{+\infty} K_1 e^{-\alpha(t-\tau)}p(f(\tau))d\tau \le \sup_{\tau\in\mathbb{R}} p(f(\tau))\frac{K_1}{\alpha}.$$

The boundedness of the remaining integrals is proved in the same way. In order the equation (30) to define generalized bounded solutions on the whole axis, it is necessary and sufficient that

$$x(0+, \xi_1, \xi_2) = x(0-, \xi_1, \xi_2).$$

This condition is equivalent to the solvability of the operator equation

$$P_+\xi_1 - (I - P_-)\xi_2 = \int_{-\infty}^0 U(t)P_-U^{-1}(\tau)f(\tau)d\tau + \int_0^{+\infty} U(t)(I - P_+)U^{-1}(\tau)f(\tau)d\tau. \tag{31}$$

Under the condition of solvability of this equation, the generalized bounded solutions of equation (26) are derived by substitution in (30) its solutions.

Thus, Theorem 5 is proved.

Let us say a few words about the theorem. Let us consider an auxiliary operator

$$S := (P_+, P_- - I)$$

and vector $\xi = (\xi_1, \xi_2)^T$. Then the equation (29) can be written in the form

$$S\xi = g.$$

Since the introduced operator S is not necessarily normally solvable (for an arbitrary sequence $g_n \in R(S)$ such that $g_n \to g,\ n \to \infty,$ we have that there exist a sequence

$$z_n = \left(z_n^1, z_n^2\right)^T\colon g_n = Sz_n = P_+z_n^1 + (P_- - I)z_n^2.$$

But from this representation and the convergence of g_n to the element g in the space F it doesn't follow that separately each of the sequences $P_+z_n^1$ and $(P_- - I)z_n^2$ are convergent. Therefore, it can not be guaranteed that $g \in R(S))$ and in the general case a condition $P_{(S^*)}g = 0$ does not guarantee the solvability of equation (29). Therefore, equation (29) may not be normally solvable [48]. Here, the representation (30) gives generalized (weak) bounded solutions only on semi-axes. To obtain a complete result on the entire axis, the additional conditions are required on operator $P_+ - I + P_-$, which will allow obtaining the solvability of equation (29). Therefore, we will consider the same equation, but in the space of Frechet F. His geometry makes it possible to introduce the notion of strong generalized-invertible operator [49] and thus clarify the obtained theorem and formulate the completed result.

Remark 7. Recall that since Frechet space is decomposed into direct algebraic sum of subspaces, it also decomposes into a topological direct sum of these subspaces [50]. This fact makes the whole study of the equation (29) simpler in Frechet space than in general topological spaces, and helps to complete the obtained theorem.

Remark 8. Depending on the which topology of the solutions in zero points the gluing is made

$$x\,(0+, \xi_1, \xi_2) = x(0-, \xi_1, \xi_2), \tag{32}$$

we obtain the different types of solutions of the equivalent operator equation (29). Everything depends on the space in which we define these solutions. If evolutionary operator $U(t)$ is defined on the whole space F (for example, when the corresponding operator function $A(t)$ is strongly continuous $|||A||| = \sup_{t\in\mathbb{R}} \|A(t)\| < \infty$), then the solution of the operator equation (31), which is equivalent to the condition (32) is contained in the initial space F and depending on whether the right side of the set of values of the operator S (operator S is normally-solvable $(R(S) = \overline{R(S)})$) we have either a general solution or a quasi-solution of the operator equation (31). If the operator S is not normally-solvable, then when the right side of the operator equation belongs to the closure of image of the operator S and do not belong to its set of values $(g \in \overline{R(S)}/R(S))$ we obtain generalized solutions. In the general case of unbounded operator coefficients, the evolution operator $U(t)$ can be determined not on the whole space F but the condition of the solvability of operator equation (31) does not guarantee that $g \in R(S)$. Then for the operator equation (31) the following case is possible, when $g \in \overline{R(S)}$, but $g \notin R(S)$. Moreover, under these conditions it can happen that $\xi_1, \xi_2 \in \overline{F}/F$, where the space $\overline{F}$ is obtained by the complement the output space according to a certain topology. Depending on the way the topology is determined, we will have different types of solutions of operator equation (31) and accordingly to the equation (29) and the original equation. Note also that it is possible to obtain the situation when $\xi_1, \xi_2 \in D(A(t)) = D$ and $g \in R(S)$. So we obtain the classical solutions. If $\xi_1, \xi_2 \notin D(A(t)) = D$ and $g \in R(S)$ then we obtain the classical solutions [51]; if $\xi_1, \xi_2 \in F,\ g \in \overline{R(S)}/R(S)$ or $\xi_1, \xi_2 \in D(A(t))$, $g \in \overline{R(S)}/R(S)$ we get generalized solutions. If the right-hand side of $g \notin \overline{R(S)},\ g \in F/\overline{R(S)}$ we get generalized quasi-solutions. If the operator S is normally-solvable, then generalized quasi-solutions $(g \in F/R(S))$ coincide with quasi-solutions [52].

Theorem 6. *Let homogeneous equation (27) admits an exponentially dichotomy on semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors P_+ and P_- respectively, and the operator*

$$D = P_+ - I + P_- : F \to F$$

is strong (X, Y) generalized-invertible.

Then:

(i) in order for there to exist generalized bounded on the whole axis solutions (26), it is necessary and sufficient for the vector function $f \in BC(\mathbb{R}, F)$ *to satisfy the condition*

$$\int\limits_{-\infty}^{+\infty} H(t)f(t)dt = 0, \tag{33}$$

where $H(t) = \left(I - \overline{D}D^{-}_{X,Y}\right) P_{-}U^{-1}(t),\ \overline{D}$ *is an extension of operator* D *on the complement of the space* $\overline{F}$*;*

(ii) if condition (33) is satisfied, then generalized solutions of the equation (26) will have the form

$$x_0(t, c) = U(t)P_{+}P_{N(\overline{D})}c + \overline{(G[f])}(t) \quad \forall c \in \overline{F}, \tag{34}$$

where

$$\overline{(G[f])}(t) = \begin{cases} \displaystyle\int_0^t U(t)U^{-1}(\tau)P_{+}f(\tau)d\tau - \int_t^{+\infty} U(t)U^{-1}(\tau)(I - P_{+})f(\tau)d\tau + \\ \displaystyle\qquad + U(t)P_{+}D^{-}_{X,Y}\left[\int_0^{\infty} U^{-1}(\tau)(I - P_{+})f(\tau)d\tau + \right. \\ \displaystyle\qquad \left. + \int_{-\infty}^0 U^{-1}(\tau)P_{-}f(\tau)d\tau\right], \quad t \geq 0, \\ \displaystyle\int_{-\infty}^t U(t)U^{-1}(\tau)P_{-}f(\tau)d\tau - \int_t^0 U(t)U^{-1}(\tau)(I - P_{-})f(\tau)d\tau + \\ \displaystyle\qquad + U(t)(I - P_{-})D^{-}_{X,Y}\left[\int_0^{\infty} U^{-1}(\tau)(I - P_{+})f(\tau)d\tau + \right. \\ \displaystyle\qquad \left. + \int_{-\infty}^0 U^{-1}(\tau)P_{-}f(\tau)d\tau\right], \quad t \leq 0, \end{cases}$$

is generalized Green's operator, extended on $\overline{F},\ D^{-}_{X,Y}$ *is strong* (X, Y) *generalized-invertible to the operator* D.

Taking into account the introduced definitions, the ideas of proofs are the same as before.

Corollary 2. *Let the homogeneous equation (27) in a complete locally convex space* E *is exponentially-dichotomous on the whole axis with projector* P*. Then for an arbitrary and bounded on the whole axis and continuous vector function* f*, there is a unique bounded solution of equation (26). This solution looks like this*

$$x(t) = \int\limits_{-\infty}^{+\infty} G(t, \tau)f(\tau)d\tau, \tag{35}$$

where

$$G(t,\tau)=\begin{cases}U(t)PU^{-1}(\tau), & t\geq\tau,\\ -U(t)(I-P)U^{-1}(\tau), & t<\tau.\end{cases}$$

Proof. There are subspaces $E_1 = PE$ and $E_2 = (I - P)E$ that are connected to projectors P and $I - P$ respectively. The subspace E_1 consists of those initial values of solutions of the equation (27), which remain bounded when $t \to \infty$, and subspace E_2 from the initial values of solutions, which remain bounded when $t \to -\infty$. Since these subspace intersects only at zero, then equations (27) have no trivial bounded solutions. Therefore, equation (26) has a unique solution, which is defined by (35).

Remark 9. In the case when E is a Banach space and the operator function $A(t)$ for each t is continuous and bounded operator ($A(t) \in \mathcal{L}(E)$), we obtain the known results [8].

Remark 10. In the case when E is quasi-complete barreled space and the operator $A(t) = A$ is regular ($A\colon E \to E$), we obtain the results of [46]. Here the condition of dichotomy is equivalent to the fact that the spectrum of the operator A does not interact with the imaginary axis. Thus, in this case for spectral projectors $P_1 = P$ and $P_2 = I-P$ the following representation holds [46]:

$$P_k=\frac{1}{2\pi}\int\limits_{\Gamma_k}R_\lambda d\lambda,$$

and, accordingly, the Green's function $G(\cdot)$ can be written as

$$G(t)=\begin{cases}e^{tA}P_1=\dfrac{1}{2\pi}\displaystyle\int_{\Gamma_1}e^{\lambda t}R_\lambda d\lambda, & t>0,\\ -e^{tA}P_2=-\dfrac{1}{2\pi}\displaystyle\int_{\Gamma_2}e^{\lambda t}R_\lambda d\lambda, & t<0.\end{cases}$$

If A is a sectorial operator, we obtain the results of bounded solutions, which are given in [17].

Corollary 3. *Let the homogeneous equation (27) is exponentially dichotomous on semi-axes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors P_+ and P_- respectively and the operator*

$$D = P_+ - I + P_- : F \to F,$$

has generalized-invertible.

Then:

(i) in order for equations (26) bounded on the whole axis to exist, it is necessary and sufficient, that the vector function $f \in BC(\mathbb{R}, F)$ satisfies the condition

$$\int\limits_{-\infty}^{+\infty}H(t)f(t)dt=0, \tag{36}$$

where $H(t) = (I - DD^-)\,P_-U^{-1}(t)$;

(ii) if condition (36) holds, the bounded solutions of the equation (26) will look as follows:

$$x_0(t,c)=U(t)P_+P_{N(D)}c+(G[f])(t),\quad c\in F, \tag{37}$$

where

$$(G[f])(t) = \begin{cases} \int_0^t U(t)U^{-1}(\tau)P_+f(\tau)d\tau - \int_t^{+\infty} U(t)U^{-1}(\tau)(I-P_+)f(\tau)d\tau + \\ \quad +U(t)P_+D^-\left[\int_0^\infty U^{-1}(\tau)(I-P_+)f(\tau)d\tau + \right. \\ \quad \left. + \int_{-\infty}^0 U^{-1}(\tau)P_-f(\tau)d\tau\right], \quad t \geq 0, \\ \int_{-\infty}^t U(t)U^{-1}(\tau)P_-f(\tau)d\tau - \int_t^0 U(t)U^{-1}(\tau)(I-P_-)f(\tau)d\tau + \\ \quad +U(t)(I-P_-)D^-\left[\int_0^\infty U^{-1}(\tau)(I-P_+)f(\tau)d\tau + \right. \\ \quad \left. + \int_{-\infty}^0 U^{-1}(\tau)P_-f(\tau)d\tau\right], \quad t \leq 0, \end{cases}$$

is generalized Green operator.

Example 1. Consider the equation (26) as a countable system with a diagonal operator

$$A(t) = \operatorname{diag}\left\{\underbrace{\tanh t, \ldots, \tanh t}_{k}, -\tanh t, -\tanh t, \ldots\right\}$$

in spaces $l^2_{\text{loc}}(\mathbf{C})$ $\Big($with a system of semi-norms $\left\|(x_1, x_2, \ldots, x_n, \ldots)\right\|^2_{n, l^2_{\text{loc}}} = \sum_{i=1}^n |x_i|^2, \ n \in$ $\in \mathbb{N}\Big)$ or Kothe space with different weight vectors (see definition, for example, in [46]). Then equation (26) is exponentially dichotomous on semi-axes with projectors

$$P_+ = \operatorname{diag}\left\{\underbrace{0, \ldots, 0}_{k}, 1, 1, \ldots\right\}, \qquad P_- = \operatorname{diag}\left\{\underbrace{1, \ldots, 1}_{k}, 0, 0, \ldots\right\},$$

respectively.

The evolutionary operator has the form

$$U(t) = \operatorname{diag}\left\{\underbrace{\frac{e^t + e^{-t}}{2}, \ldots, \frac{e^t + e^{-t}}{2}}_{k}, \frac{2}{e^t + e^{-t}}, \frac{2}{e^t + e^{-t}}, \ldots\right\}.$$

Thus, according to the theorem we get that

$$D = P_+ - I + P_- = 0, \qquad P_{N(D)} = P_{N(D^*)} = I,$$

and since $\dim R\left[P_{N(D^*)}P_-\right] = k$, then operator $P_{N(D^*)}P_-$ is finite-dimensional, $H(t) =$ $= \operatorname{diag}\{H_k(t), 0\}$, where $H_k(t) = \operatorname{diag}\left\{2/\left(e^t + e^{-t}\right), \ldots, 2/\left(e^t + e^{-t}\right)\right\}$ is $(k \times k)$-dimensional matrix. Necessary and sufficient condition of the existence of generalized bounded solutions in Frechet space will take the following form:

$$\int\limits_{-\infty}^{+\infty} H_k(t)f(t)dt = 0 \quad \Leftrightarrow \quad \begin{cases} \int_{-\infty}^{+\infty} \dfrac{f_1(t)}{e^t + e^{-t}}\, dt = 0, \\ \dots \\ \int_{-\infty}^{+\infty} \dfrac{f_k(t)}{e^t + e^{-t}}\, dt = 0. \end{cases}$$

In contrast to the case of Banach spaces, the topology of this Frechet space is determined by a set of semi-norms $\|(x_1, x_2, \dots, x_n, \dots)\|^2_{n, l^2_{\text{loc}}} \sum_{i=1}^{n} |x_i|^2$, which are not norms, and therefore such functions do not form a Banach space.

Bounded and almost periodic solutions. In this section, the results of previous researches are applied for finding bounded and almost periodic solutions of operator-differential equations.

Definition 6 [8]**.** *A continuous vector function $f(t)$ with values in Banach space E, defined for $t \in \mathbb{R}$, is almost periodic if for any $\varepsilon > 0$ there exist such $L > 0$, that each interval of length not less than L contains a point τ, for which*

$$\|f(t) - f(t+\tau)\| < \varepsilon.$$

Consider equation (26) in Banach space with a constant (in general case with unbounded) operator and almost periodic free term $f(t)$. Let us define $\Delta(f)$ "the set that is not almost periodic" [53]. Recall that the point $\lambda \in J$ is called the point "almost-periodicity" function f, if there is such a neighborhood of this point that the convolution $f * \varphi$ is an almost periodic function for any $\hat{\varphi} \in C_0^\infty$ with the support, which belongs to this neighborhood ($\hat{\varphi}$ is Fourier transformation for φ). The complement to this set is called "set of almost periodicity". Let also c_0 be the space of sequences $(x_n)_{n \in \mathbb{N}}$, the elements of which are convergent to zero.

The following statement is fair.

Statement 2. *If the set $\Delta(f)$ is sparse and fulfil one of the following conditions:*

(a) the space F does not contain the c_0 as a subspace;

(b) vector function f is weakly compact;

then, under the condition of existence of (29), any bounded solution of (26) in the form (30) is almost periodic.

This statement is a consequence of the Theorem 4.6 and Theorem 5 from [53]. Note that if there exist a wider class of almost periodic problems, namely, an equation with an asymptotic almost periodic functions.

Definition 7 [54]**.** *The continuous function $x(t)$ is called asymptotically almost periodic if it can be given as*

$$x(t) = f(t) + r(t), \quad t \in \mathbb{R},$$

where f is almost periodic, r is a continuous function that satisfies the condition

$$\lim_{t \to \infty} r(t) = 0.$$

This definition can be demonstrated on such examples.

Examples. In order to emphasize once again the difference in the study of the questions concerning the existence of bounded solutions in the Frechet and Banach spaces, we present the following example. Consider the equation

$$\dot{x}_m(t) + thtx_m(t) = e^{-\frac{t}{m}}, \quad m \in \mathbb{N}, \tag{38}$$

in Frechet space, the topology of which is generated by the system of semi-norms:

$$|x|_n = \sup_{t\in[-n;\infty)} |x(t)|, \quad n \geq 0.$$

Note that in contrast to Banach space of continuous and bounded on the entire axis functions $BC(\mathbb{R},\mathbb{R})$ the right-hand side of this equation does not belong to the space $BC(\mathbb{R},\mathbb{R})$, but belongs to Frechet space defined above. So,

$$\sup_{t\in\mathbb{R}} \left| e^{-\frac{t}{m}} \right| = +\infty,$$

but

$$\left| e^{-\frac{t}{m}} \right|_n = e^{\frac{n}{m}} < +\infty$$

for each n. This means that the given function is unbounded in Banach space; however, it is bounded in Frechet space. Frechet space of asymptotically almost periodic functions with the defined above topology is denoted as $AAP(\mathbb{R},\mathbb{R})$ [54], which means the function $e^{-\frac{t}{m}} \in$ $\in AAP(\mathbb{R},\mathbb{R})$. Then equation (38) has an asymptotically almost periodic bounded on the whole axis solution in the following form:

$$x_m(t) = \frac{2c_m}{e^t + e^{-t}} + \frac{e^{\left(1-\frac{1}{m}\right)t}}{\left(1-\dfrac{1}{m}\right)(e^t + e^{-t})} - \frac{e^{-\left(1+\frac{1}{m}\right)t}}{\left(1+\dfrac{1}{m}\right)(e^t + e^{-t})}.$$

Another example of the existence of a bounded solution in Frechet space could be such a differential equation in the space of generalized functions $D'(\mathbb{R})$:

$$x'(t) + \tanh t\, x(t) = \delta(t), \tag{39}$$

where δ is delta-function of Dirac. Equation (39) can be written in the form

$$\big(x', \varphi\big) + (\tanh t\, x, \varphi) = (\delta, \varphi) \quad \forall \varphi \in D(\mathbb{R})$$

or, by definition,

$$\big(x, -\varphi' + \tanh t\varphi\big) = (\delta, \varphi). \tag{40}$$

According to the theorem of the existence of bounded solutions, we obtain that the equation (39) can solve if and only if the right-hand side of (39) is orthogonal to bounded solutions of a homogeneous conjugate equation

$$-\varphi'(t) + \tanh t\, \varphi(t) = 0.$$

It is easy to see that the only bounded solution of the homogeneous equation is zero solution. For it, we get that

$$(\delta, \varphi) = (\delta, 0) = 0,$$

and so the solvability condition for equation (39) in the generalized sense (40) always holds. The generalized bounded solution can be given as

$$x(t) = \frac{\theta(t)}{\cosh t},$$

where $\theta(t)$ a Heaviside step function. In fact, $x(t)$ is a bounded solution, because

$$\big(x, -\varphi' + \tanh t\varphi\big) = \int\limits_{\mathbb{R}} x(\tau)\left(-\varphi'(\tau) + \tanh\tau\varphi(\tau)\right) d\tau = -\int\limits_0^{+\infty} \frac{\varphi'(\tau)}{\cosh\tau}\, d\tau + \int\limits_0^{+\infty} \frac{\sinh\tau}{\cosh^2\,\tau}\,\varphi(\tau) d\tau.$$

Integrating the first integral by parts gives us the following:

$$-\frac{\varphi(t)}{\cosh t}\bigg|_0^{+\infty} - \int\limits_0^{+\infty} \frac{\sinh\tau\varphi(\tau)}{\cosh^2\,\tau}\, d\tau + \int\limits_0^{+\infty} \frac{\sinh\tau}{\cosh^2\tau}\,\varphi(\tau) d\tau = \varphi(0) = (\delta, \varphi).$$

It is known [55] that $x(t) = \dfrac{\theta(t)}{\cosh t}$ is called the fundamental solution of the equation (39). By using this representation, we can see that for an arbitrary bounded on the entire axis generalized function $f(t) \in D'(\mathbb{R})$ bounded on the whole axis solution $x(t)$ of the equation

$$x'(t) + \tanh t x(t) = f(t)$$

can be found in this form

$$x(t) = \frac{\theta(t)}{\cosh t} * f(t),$$

where $*$ means the convolution operation [55].

If we consider the Banach space $\mathbf{B}$ with the topology generated by the set of semi-norms

$$\mathbf{B} \ni x \to |\varphi(x)|, \quad \varphi \in \mathbf{B}^*,$$

then we obtain a locally convex space, denoted by $\mathbf{B}_w$, which is valid for the following definition.

Definition 8 [54]**.** *The mapping* $x\colon \mathbb{R} \to \mathbf{B}$ *is called weakly almost periodic, if for an arbitrary* $\varphi \in \mathbf{B}^*$, *the mapping* $\varphi(x(t)) \in AP(\mathbb{R}, \mathbb{C})$. *The space of weak, almost periodic vector functions is denoted as* $WAP(\mathbb{R}, \mathbf{B})$.

Let us consider the equation $\dot{x}(t) = f(t)$ in Banach space $\mathbf{B}$ with the right-side $f(t) \in$ $\in AP(\mathbb{R}, \mathbf{B})$. As known from [56], in a finite-dimensional space, the boundedness of the operator $\int_0^t f(s) ds,\ t \in \mathbb{R}$, from an almost periodic function guarantees that the solution $x(t)$ will be almost periodic. In the Banach space, there is no such result, but the following fact is well known.

Theorem 7 [54]**.** *If $f(t) \in AP(\mathbb{R}, \mathbf{B})$ and*

$$x(t) = \int\limits_0^t f(s)ds, \quad t \in \mathbb{R},$$

then the boundedness x on $\mathbb{R}$ guarantees that $x \in WAP(\mathbb{R}, \mathbf{B})$. For $x \in AP(\mathbb{R}, \mathbf{B})$, is necessary and sufficient that $\mathcal{R}_x = \{x(t); t \in \mathbb{R}\}$ be relatively compact in $\mathbf{B}$.

From this theorem, the above statement and theorem 6 the next result follows.

Corollary 4. *Let homogeneous equation (27) be an exponentially dichotomous on semi-axes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors P_+ and P_- respectively, and the operator*

$$D = P_+ - I + P_- : F \to F$$

has generalized-inverse, where F is Banach space with a strong topology.

Then:

(i) in order to exist weak almost periodic solutions of equation (26), it is necessary and sufficient that the vector function $f \in AP(\mathbb{R}, F)$ satisfies the condition

$$\int\limits_{-\infty}^{+\infty} H(t)f(t)dt = 0, \tag{41}$$

where $H(t) = (I - DD^-)\, P_- U^{-1}(t)$;

(ii) if the condition (41) is satisfied, the weak almost periodic solutions of equation (26) will look as follows:

$$x_0(t, c) = U(t)P_+P_{N(D)}c + (G[f])(t), \quad c \in F, \tag{42}$$

where

$$(G[f])(t) = \begin{cases} \displaystyle\int_0^t U(t)U^{-1}(\tau)P_+f(\tau)d\tau - \int_t^{+\infty} U(t)U^{-1}(\tau)(I - P_+)f(\tau)d\tau + \\ \quad \displaystyle +U(t)P_+D^-\left[\int_0^{\infty} U^{-1}(\tau)(I - P_+)f(\tau)d\tau + \right. \\ \quad \displaystyle \left. + \int_{-\infty}^0 U^{-1}(\tau)P_-f(\tau)d\tau\right], \quad t \geq 0, \\ \displaystyle\int_{-\infty}^t U(t)U^{-1}(\tau)P_-f(\tau)d\tau - \int_t^0 U(t)U^{-1}(\tau)(I - P_-)f(\tau)d\tau + \\ \quad \displaystyle +U(t)(I - P_-)D^-\left[\int_0^{\infty} U^{-1}(\tau)(I - P_+)f(\tau)d\tau + \right. \\ \quad \displaystyle \left. + \int_{-\infty}^0 U^{-1}(\tau)P_-f(\tau)d\tau\right], \quad t \leq 0, \end{cases}$$

is generalized Green operator.

Proof. The validity of the solvability condition (41) follows from the previous theorem. The fact that $f \in AP(\mathbb{R}, F)$ guarantees that if the condition (41) holds the expression (42) defines the set of bounded on the entire axis solutions (26). Then, from the previous theorem [54, p. 144], follows that each solution is weakly almost periodic in Frechet space F_w with a weak topology $(x_0(t,c) \in WAP(\mathbb{R}, F_w))$.

Remark 11. Note that from the previous theorem (Theorem 5.3 [54, p. 140]) follows that the solution $x_0(t,c)$ belongs to $AP(\mathbb{R}, F)$ if and only if it has a relatively compact set of values in the space F.

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