UDC 517.956.3

ON PRACTICAL STABILITY OF DISCRETE INCLUSIONS WITH SPATIAL COMPONENTS

ПРО ПРАКТИЧНУ СТІЙКІСТЬ ДИСКРЕТНИХ ВКЛЮЧЕНЬ ІЗ ПРОСТОРОВИМИ КОМПОНЕНТАМИ

V. V. Pichkur, Ya. M. Linder

Taras Shevchenko Nat. Univ. of Kyiv
Volodymyrska str., 64, Kyiv, 01601, Ukraine
e-mail: vpichkur@gmail.com
yaroslav.linder@gmail.com

M. S. Tairova

Odessa I. I. Mechnikov Nat. Univ.
Dvoryanska str., 2, Odessa, 65082, Ukraine
masonmas@gmail.com

We analyze properties of the maximal set of initial conditions of the problem of weak practical stability of discrete inclusions with spatial components. We prove compactness and properties of the boundary and interior of the maximal set of practical stability. In the linear case, we obtain the Minkowski function and the inverse Minkowski function of the optimal set of initial conditions.

Проаналізовано властивості максимальної множини початкових умов задачі слабкої практичної стійкості дискретних включень із просторовою компонентою. Доведено компактність та властивості границі і внутрішності максимальної множини практичної стійкості. У лінійному випадку отримано функцію Мінковського та обернену функцію Мінковського оптимальної множини початкових умов.

1. Introduction. Theory of discrete systems is intensively developed over the past decades. The reason of such advancement is a broad range of their application. Discrete systems are used to describe behaviour of processes in technical, economic, chemical and many other fields. Discrete systems have significant advantages in comparison with continuous ones. They are simpler and can be used for solutions approximation to various classes of problems. Therefore, mathematical approaches to the solution analysis of discrete systems are intensively developed. Quantitative and qualitative research techniques for discrete systems, including conditions of stability and practical stability, conditions of invariance and robustness are discussed in [1 – 8]. Dynamic systems with impulse impact are intermediary type of systems between the discrete and continuous ones [9 – 11]. The focus of this work is on properties of maximal set of weak practical stability for discrete system with set-valued right-hand side and spatial components [2, 4, 12 – 14]. Set-valued systems with spatial components generalize control systems with set-valued observations [15].

We prove compactness and properties of boundary and interior of the maximal set of practical stability. In the case of linear dynamic components, we obtain the Minkowski function and the inverse Minkowski function of the optimal set of initial conditions.

Basic notations. As usual, we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by $\|\cdot\|$ the Euclidean norm in it, while $\langle\cdot,\cdot\rangle$ stands for the scalar product that generates the Euclidean norm in $\mathbb{R}^n$. The set of internal points and the boundary of the set A are written $\operatorname{int} A$ and ∂A respectively. Denote also by S the unit sphere with center at origin and $K_r(a)$ the closed ball with radius r and center at $a \in \mathbb{R}^n$. Let $c(A,\psi) = \sup_{a\in A}\langle a,\psi\rangle$ with $\psi \in \mathbb{R}^n$ be the support function of a set $A \subseteq \mathbb{R}^n$, $\operatorname{comp}\left(\mathbb{R}^n\right)$ stands for the set of all non-empty compact sets in $\mathbb{R}^n$, while $\operatorname{conv}\left(\mathbb{R}^n\right)$ is the set of all non-empty convex compact sets in $\mathbb{R}^n$, $A^\sigma = A + K_\sigma(0)$ is the σ-extension of the set $A \subseteq \mathbb{R}^n$, $\alpha(\cdot,\cdot)$ is the Hausdorff metric [16].

2. The maximal set of initial conditions. Let D be a bounded closed domain in $\mathbb{R}^m$. We consider a discrete inclusion together with a set-valued mapping $B_k\colon \mathbb{R}^m \to \operatorname{comp}\left(\mathbb{R}^n\right)$, $k = 0,1,\ldots,N$,

$$\begin{cases} x(k+1) \in f_k(x(k)), & k = 0,\ldots,N-1, \\ B_k(x(k)), & k = 0,\ldots,N. \end{cases} \tag{1}$$

Here $x \in D$, multifunctions $f_k\colon D \to \operatorname{comp}(D)$ are continuous, $x(k) = x(k,x_0)$ is a solution of the discrete inclusion in (1) satisfying the initial condition $x(0) = x_0,\ k = 0,\ldots,N$. Denote by $X(k,x_0)$ the corresponding attainability set at a discrete time $k = 0,\ldots,N$. Multifunctions B_k satisfy the Lipschitz condition on D. It means that there are numbers $L_k > 0$ such that

$$\alpha\left(B_k(x), B_k(y)\right) \le L_k\|x-y\|$$

for any $x,y \in D,\ k = 0,\ldots,N$. The first inclusion in (1) is called the dynamic component. The mapping B_k is called the spatial component of system (1) associated with the dynamic component.

Let $\Phi(k) \in \operatorname{comp}\left(\mathbb{R}^n\right)$ be a state constraint, $B_k(x(k,0)) \subseteq \Phi(k),\ k = 0,\ldots,N,\ I_0 \subseteq \Phi(0)$. We say that r-condition takes place if there exists $r > 0$ such that for each $x_0 \notin K_r(0)$ and for any solution $x(k,x_0)$ of discrete inclusion in (1) there exists $k \in \{0,\ldots,N\}$ such that

$$B_k(x(k,x_0)) \nsubseteq \Phi(k).$$

In particular, we can use a stronger condition $B_k(X(k,x_0)) \subseteq \mathbb{R}^n/\Phi(k)$.

Definition 1. *System (1) is called $\{I_0,\Phi(k),0,N\}$-weak stable if for any $x_0 \in I_0$ there exists a solution $x(k,x_0)$ of discrete inclusion (1) such that $B_k(x(k,x_0)) \subseteq \Phi(k),\ k = 0,\ldots,N$.*

Definition 2. *We say that $I_* \subseteq \Phi(0)$ is the maximum set of weak practical stability of system (1) with the state constraints $\Phi(k),\ k = 0,\ldots,N$, if system (1) is $\{I_*,\Phi(k),0,N\}$-weak stable and $I_0 \subseteq I_*$ for any set $I_0 \subseteq \Phi(0)$ such that weak $\{I_0,\Phi(k),0,N\}$-stability of the system (1) takes place.*

Consider the following theorems.

Theorem 1. *If a sequence of solutions of system (1) is regular, then it tends to a solution of the system (1).*

Proof. Suppose there exists a sequence $N_k^m = B_k(x_m(k))$ so that

$$\lim_{m\to\infty} \alpha\left(N_k^m, N_k\right) = 0, \quad k = 0, \ldots, N,$$

where N_k is a compact set, $x_m(k)$ is a solutions sequence of (1). The sequence $x_m(k)$ is bounded on D and there exists some subsequence which we can denote by $x_m(k)$, $\lim_{m\to\infty} x_m(k) = x(k)$. According to [17] $x(k)$ is a solution of the inclusion in (1), $k \in 0, \ldots, N$. Hence, from the Lipschitz condition it follows that

$$\alpha\left(B_k(x_m(k)), B_k(x(k))\right) \leq L_k \left\|x_m(k) - x(k)\right\| \to 0, \quad m \to \infty.$$

Therefore, $B_k(x(k)) = N_k$.

Theorem 2. *Suppose that* Π *is a set of solutions* $B_k(x(k))$ *of system (1) such that* $B_k(x(k)) \subseteq \subseteq \Phi(k)$, $k \in 0, \ldots, N$. *Then for any sequence from* Π *there exists a regular subsequence tending to some point in* Π.

Proof. Assume that $B_k(x_m(k))$ is a sequence such that

$$B_k(x_m(k)) \subseteq \Phi(k), \quad k = 0, \ldots, N, \quad m = 1, 2, \ldots.$$

We perform the following procedure for all $k \in 0, \ldots, N$ in a sequential order. Since $\mathrm{comp}\left(\Phi(k)\right)$ is compact space, from the sequence $B_k(x_m(k))$ one can select a regular subsequence (see [16]). Redefine $B_k(x_m(k))$ as a subsequence of the sequence $B_k(x_m(k))$ eliminating the members with numbers not belonging to the regular subsequence of the sequence $B_k(x_m(k))$. Therefore,

$$\lim_{m\to\infty} \alpha\left(B_k\left(x_m(k)\right), N_k\right) = 0, \quad N_k \in \mathrm{comp}(\Phi(k)), \quad k = 0, \ldots, N.$$

According to theorem 1 the equality $B_k(x(k)) = N_k$ holds true.

Theorem 3. *The maximum set of initial conditions* I_* *is compact.*

Proof. According to the r-condition $I_* \subseteq K_r(0)$. Thus, I_* is bounded. Let us prove that I_* is closed. Take a sequence $x_p \in I_*$, $\lim_{p\to\infty} x_p = x_0$. By definition of the set I_* for any point $x_p \in I_*$ there exists a solution $x_p(k, x_p)$ of the discrete inclusion in (1) such that

$$B_k(x_p(k, x_p)) \subseteq \Phi(k).$$

According to theorem 2 from a sequence of solutions (1) one can select a regular subsequence tending to some solution in (1), and $B_k(x(k, x_0)) \subseteq \Phi(k)$ for any $k = 0, \ldots, N$. Therefore, $x_0 \in I_*$.

Theorem 4. *Let* $x_0 \in \partial I_*$, $x(k, x_0)$ $k = 0, 1, \ldots, N$, *be a solution of the discrete inclusion in (1) and* $B_k(x(k, x_0)) \subseteq \Phi(k)$, *then there exists* $\bar{k} \in \{0, \ldots, N\}$ *such that*

$$\partial B_{\bar{k}}(x(k, x_0)) \cap \partial\Phi(\bar{k}) \neq \varnothing.$$

Proof. By contradiction. Assume that there exists a solution $x(k, x_0)$ of the discrete inclusion in (1) such that $B_k(x(k, x_0)) \subseteq \Phi(k)$ and

$$\partial B(x(k, x_0)) \cap \partial\Phi(k) = \varnothing, \quad k = 0, \ldots, N.$$

It means that there exists $\varepsilon > 0$ such that

$$(B(x(k, x_0)))^{\varepsilon} \subseteq \Phi(k)$$

for any $k \in 0, \ldots, N$. Taking into account the continuity property of the mapping $z \mapsto$ $\mapsto B_k(x(k, z)),\ z \in D$, there exists $\delta > 0$ such that for $z_0 \in K_\delta(x_0)$ the following statement holds true:

$$B_i(x(i, z_0)) \subseteq (B_i(x(i, x_0)))^{\varepsilon_0},$$

where $\varepsilon_0 = \min \varepsilon_i > 0,\ i = 0, 1, \ldots, N$. Thus, $B_i(x(i, z_0)) \subseteq \Phi(i),\ i = 0, 1, \ldots, N,\ z_0 \in K_\delta(x_0)$. Therefore $K_\delta(x_0) \subseteq I_*$. It means that $x_0 \in \operatorname{int} I_*$. This contradiction proves the theorem.

3. Practical stability of discrete systems with linear dynamic and spatial components. Consider a linear discrete inclusion

$$x(k+1) \in A(k)x(k) + U(k). \tag{2}$$

Here $A(k)$ is a nondegenerate matrix of dimension $(n \times n),\ U(k) \in \operatorname{conv}(\mathbb{R}^n),\ 0 \in int\, U(k)$, $k = 0, \ldots, N-1$. Consider further a set component $B_k \colon \mathbb{R}^m \to \operatorname{conv}\left(\mathbb{R}^n\right)$ of the linear form

$$B_k(x) = \Xi(k)x + V(k),$$

where $\Xi(k)$ is a matrix of dimension $(m \times n),\ k = 0, \ldots, N,\ V(k) \in \operatorname{conv}(\mathbb{R}^n)$. Suppose $\Phi(k) \in \operatorname{conv}(\mathbb{R}^n)$ is a set of state constraints, $k \in 0, \ldots, N$. The attainability set of system (2) is defined by [4]

$$X(k, x_0) = \Theta(k)x_0 + \Omega(k), \quad k = 1, \ldots, N,$$

where $\Theta(k) = \Theta(0, k-1),\ \Theta(i, k) = A_k \ldots A_{i+1}A_i$,

$$\Omega(k) = \sum_{i=1}^{k} \Theta(i, k-1)U(i-1), \quad k = 1, \ldots, N.$$

Hence,

$$B_k(X(k, x_0)) = \Xi(k)\Theta(k)x_0 + \Xi(k)\Omega(k) + V(k).$$

The following theorem takes place.

Theorem 5. *If* $\Phi(k) \in \operatorname{conv}\left(\mathbb{R}^n\right)$, *then* $I_* \in \operatorname{conv}\left(\mathbb{R}^n\right)$.

Proof. Suppose that $x_0,\ y_0$ are arbitrary points from the set I_*. We show that for $\lambda \in [0, 1]$ the following inclusion $\lambda x_0 + (1-\lambda)y_0 \in I_*$ holds true. According to the definition of the set I_* there exist $\omega_1(k) \in \Omega(k),\ \omega_2(k) \in \Omega(k)$ such that

$$\Xi(k)\Theta(k)x_0 + \Xi(k)\omega_1(k) + V(k) \subseteq \Phi(k),$$

$$\Xi(k)\Theta(k)y_0 + \Xi(k)\omega_2(k) + V(k) \subseteq \Phi(k), \quad k = 0, \ldots, N.$$

Since $\Omega(k),\ V(k)$ are convex, for $\lambda \in [0, 1]$,

$$\lambda(\Xi(k)\Theta(k)x_0 + \Xi(k)\omega_1(k) + V(k)) +$$
$$+ (1-\lambda)(\Xi(k)\Theta(k)y_0 + \Xi(k)\omega_2(k) + V(k)) \subseteq \Phi(k).$$

From $\Omega(k) \in \operatorname{conv}(\mathbb{R}^n)$ we have $\lambda\omega_1(k) + (1-\lambda)\omega_2(k) \in \Omega(k)$. Therefore, $\lambda x_0 + (1-\lambda) y_0 \in \in I_*$.

Theorem 5 is proved.

Denote by $\Sigma(U),\ \Sigma(\Omega)$ sets of selections of the maps $U,\ \Omega$ respectively. Suppose that for all $k = 0, 1, \ldots, N$ and $\psi \in S$

$$c(\Phi(k), \psi) - c\,(\Omega(k), \Xi^*(k)\psi) - c\big(V(k), \psi\big) > 0 \tag{3}$$

takes place.

Theorem 6. *The Minkowski function* $m_*(x_0) = \inf\{\lambda > 0 \colon x_0 \in \lambda I_*\}$ *of the set* I_* *is equal to*

$$\min_{w \in \Sigma(\Omega)} \max_{k \in 0,\ldots,N} \max_{\psi \in S} \frac{\langle x_0, \Theta^*(k)\Xi^*(k)\psi\rangle}{c(\Phi(k), \psi) - c(V(k), \psi) - \langle w(k), \Xi^*(k)\psi\rangle}$$

so that $I_* = \{x_0 \in \mathbb{R}^n \colon m_*(x_0) \leq 1\}$.

Proof. Consider $\dfrac{x_0}{\lambda} \in I_*$. There exists $w(\cdot) \in \Sigma(\Omega)$ such that for all $k = 0, 1, \ldots, N$ inclusion

$$\Xi(k)\Theta(k)\,\frac{x_0}{\lambda} + \Xi(k) w(k) + V(k) \subseteq \Phi(k)$$

holds. Therefore,

$$\frac{1}{\lambda}\,\langle \Xi(k)\Theta(k)x_0, \psi\rangle + \langle \Xi(k) w(k), \psi\rangle + c(V_k, \psi) \leq c(\Phi(k), \psi)$$

for all $k \in 0, 1, \ldots, N$ and $\psi \in S$. Taking into account equation (3), we obtain that there exists $w(\cdot) \in \Sigma(\Omega)$ such that

$$\lambda \geq \max_{k \in 0,\ldots,N} \max_{\psi \in S} \frac{\langle x_0, \Theta^*(k)\Xi^*(k)\psi\rangle}{c(\Phi(k), \psi) - c(V(k), \psi) - \langle w(k), \Xi^*(k)\psi\rangle}.$$

Hence,

$$\lambda \geq \min_{w \in \Sigma(\Omega)} \max_{k \in 0,\ldots,N} \max_{\psi \in S} \frac{\langle x_0, \Theta^*(k)\Xi^*(k)\psi\rangle}{c(\Phi(k), \psi) - c(V(k), \psi) - \langle w(k), \Xi^*(k)\psi\rangle}.$$

From the Minkowski function definition, it follows that the theorem is proven.

Corollary 1. *Suppose that for any solution* $x(k, x_0)$ *of inclusion (2) statement* $B_k(x(k, x_0)) \subseteq \subseteq \Phi(k)$ *takes place and there exists* $\bar{k} \in \{0, \ldots, N\}$ *such that* $\partial B_{\bar{k}}(x(\bar{k}, x_0)) \cap \partial\Phi(\bar{k}) \neq \varnothing$. *Then* $x_0 \in \partial I_*$.

Corollary 2. *The inverse Minkowski function of the set* I_*

$$d_*(x_0) = \sup\{\lambda > 0 \colon \lambda x_0 \in I_*\} =$$

$$= \max_{w \in \Sigma(\Omega)} \min_{k \in 0,\ldots,N} \min_{\psi \in P(k)} \frac{c(\Phi(k), \psi) - c(V(k), \psi) - \langle w(k), \Xi^*(k)\psi\rangle}{\langle x_0, \Theta^*(k)\Xi^*(k)\psi\rangle},$$

where $P(k) = \{\psi \in S \colon \langle x_0, \Theta^*(k)\Xi^*(k)\psi\rangle > 0\}$. *In this case,*

$$I_* = \cup_{e \in S}\big[0, d_*(e)\big]e.$$

4. Conclusion. In the paper practical stability of discrete inclusions with spatial components was analyzed. Compactness and other properties of boundary and interior of the maximal set of practical stability were proven. In the case of linear dynamic components, the Minkowski function and the inverse Minkowski function of the optimal set of initial conditions were obtained.

References

1. *Halanay A., Wexler D.* Qualitative theory of impulsive systems. – Moscow: Mir, 1971.
2. *Bashnyakov A. N., Pichkur V. V., Hitko I. V.* On maximal initial data set in problems of practical stability of discrete system // J. Automat. Inf. Sci. – 2011. – **43**, № 3. – P. 1 – 8.
3. *Michel A., Hou L., Liu D.* Stability of dynamical systems. – Boston: Birkhäuser, 2008.
4. *Pichkur V. V., Sasonkina M. S.* Maximum set of initial conditions for the problem of weak practical stability of a discrete inclusion // J. Math. Sci. – 2013. – **194**, № 4. – P. 414 – 425.
5. *Pichkur V. V., Sasonkina M. S.* Practical stabilization of discrete control systems // Int. J. Pure Appl. Math. – 2012. – **81**, № 6. – P. 877 – 884.
6. *Antsaklis P., Michel A.* Linear systems. – Boston: Birkhäuser, 2006.
7. *Ravi P. Agarwal.* Difference equations and inequalities. Theory, methods and applications. – Singapore: Nat. Univ. Singapore, 1968.
8. *Galor O.* Discrete dynamical systems. – Berlin: Springer, 2007.
9. *Linder Ya. N., Pichkur V. V.* Conditions of weak practical stability of differential inclusions with impulse impact // J. Automat. Inf. Sci. – 2011. – **43**, № 11. – P. 57 – 65.
10. *Kapustyan O. V., Perestyuk M. O., Romanyuk I. V.* Stability of global attractors of impulsive infinite-dimensional systems // Ukr. Math. J. – 2018. – **70**, № 1. – P. 30 – 41.
11. *Kichmarenko O. D., Karpycheva M. L.* General averaging scheme for discrete equations with variable delay // J. Math. Sci. – 2017. – **226**, № 3. – P. 270 – 284.
12. *Garashchenko F. G., Pichkur V. V.* On optimal sets of practical stability // J. Automat. Inf. Sci. – 1999. – **31**, № 4-5. – P. 1 – 12.
13. *Garashchenko F. G., Pichkur V. V.* Optimal sets of practical stability of differential inclusions // J. Automat. Inf. Sci. – 2003. – **35**, № 1-4. – P. 5 – 12.
14. *Garashchenko F. G., Pichkur V. V.* On properties of maximal set of external practical stability of discrete systems // J. Automat. Inf. Sci. – 2016. – **48**, № 3. – P. 46 – 53.
15. *Kurzhanski A., Vályi I.* Ellipsoidal calculus for estimation and control. – Basel: Birkhäuser, 1997. – 321 p.
16. *Polovinkin E. S., Balashov M. V.* Elements of convex and strongly convex analysis. – Moscow: Fizmatlit, 2004 (in Russian).
17. *Pichkur V. V., Sasonkina M. S.* On the continuous dependence of discrete inclusions on the initial conditions solutions // Tavr. Vesnik Informatiki i Matematiki. – 2011. – № 6. – P. 73 – 80 (in Russian).

Received 22.05.19,
after revision — 14.09.19