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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH DOUBLE SINGULARITY IN CONDITIONALLY STABLE CASE

АСИМПТОТИЧНА ПОВЕДІНКА РОЗВ'ЯЗКІВ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ПОДВІЙНОЮ СИНГУЛЯРНІСТЮ В УМОВНО СТІЙКОМУ ВИПАДКУ

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The double singular perturbation for boundary-value problem for nonlinear system of ordinary differential equations is considered. For a formal asymptotic solution, constructed by the method of boundary functions and generalized inverse matrices and projectors, we prove an asymptotic property of the formal series.

Розглянуто подвійно сингулярне збурення граничної задачі для нелінійної системи звичайних диференціальних рівнянь. Для формального асимптотичного розв'язку, який побудовано за методом граничних функцій та узагальнених обернених матриць і проекторів, доведено асимптотичну властивість формального ряду.

1. Introduction. In the paper we investigate the asymptotic behavior of the formal solution of the boundary-value problem (BVP)

$$\varepsilon \frac{dx}{dt} = Ax + \varepsilon F(t, x, \varepsilon, f(t, \varepsilon)) + \varphi(t), \quad t \in [a, b], \quad 0 < \varepsilon \ll 1, \tag{1}$$

$$lx(\cdot) = h, \quad h \in R^n, \tag{2}$$

where ε is a small positive parameter. The BVP (1), (2) will be considered under the following conditions:

(C_1) The $(n \times n)$-matrix A with constant elements has p eigenvalues with negative real part, and the remaining $(n-p)$ eigenvalues have positive real part, i.e., $\lambda_j \in \sigma(A)$, $\operatorname{Re}\lambda_j < 0$, $j = \overline{1, p}$, and $\operatorname{Re}\lambda_j > 0$, $j = \overline{p+1, n}$.

(C_2) The vector-function $\varphi(t)$ is an n-dimensional vector-function of the class $C^\infty([a, b])$.

(C_3) The function $F(t, x, \varepsilon, f(t, \varepsilon))$ is an n-dimensional vector-function, having arbitrary order continuous partial derivatives with respect to all arguments in the domain $G = [a, b] \times \times D_x \times [0, \overline{\varepsilon}] \times D_f$, where $D_x \in R^n$ is some neighborhood of the solution $x_0(t)$ of the degenerate system $Ax_0(t) + \varphi(t) = 0$, $D_f \in R^p$ is a bounded and closed domain, $0 < \overline{\varepsilon} \ll 1$. The function $f(t, \varepsilon)$ is smooth of arbitrary order with respect to all arguments in the domain $G_1 = [a, b] \times (0, \overline{\varepsilon}]$ and its values belongs to D_f.

(C_4) l is a linear n-dimensional bounded vector functional, $l = \operatorname{col}\left(l^1, \ldots, l^n\right)$, $l \in \in (C[a, b] \to R^n, R^n)$.

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We assume that the function f from (1) contains singular elements (for example, $f = f(\exp(-t/\varepsilon), \sin(t/\varepsilon)))$. Thus the system (1) has a double singularity. On the one hand, the small parameter ε appears before the derivative, and, on the other hand, it brings in a singularity of the function f.

The condition (C_1) shows that we consider system (1) in a conditionally stable case [13, 14].

We seek to determine existence and uniqueness of a n-dimensional asymptotic solution $x(t,\varepsilon)$ of the BVP (1), (2) such that $x(\cdot,\varepsilon) \in C^1([a,b])$, $x(t,\cdot) \in C((0,\varepsilon_0])$ and $\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t)$.

The construction of the asymptotic solution of problem (1), (2) is based on the boundary functions method (see, for example, [13, 14]). The initial research for a Cauchy problem with double singularity is carried out in [9] in the case $\operatorname{Re}\lambda_i < 0\ \forall i,\ \lambda_i \in \sigma(A)$. The BVP in this case are analyzed in the papers [7, 11].

Primary research of the problem (1), (2) with conditions $(C_1)-(C_4)$ is conducted in [5] and [6]. In the papers [5, 6] the formal asymptotic solutions of the BVP (1), (2) have been constructed in various cases. In [6] we considered the case which uses generalized inverse matrices and projectors [1, 3, 10].

2. Preliminary results and problem formulation. In the paper [6] a formall asymptotic solution of the BVP (1), (2) was obtained after introducing a second parameter μ and studyind the BVP with two parameters $\varepsilon \in [0,\overline{\varepsilon}]$ and $\mu \in (0,\overline{\varepsilon}]$, $0 < \overline{\varepsilon} \ll 1$,

$$\begin{gathered} \varepsilon\dot{z} = Az + \varepsilon F(t,z,\varepsilon,f(t,\mu)) + \varphi(t), \quad t \in [a,b], \\ lz(\cdot) = h, \quad h \in R^n. \end{gathered} \tag{3}$$

The solution of the BVP (3) be found in a unique formal expression of the form

$$z(t,\varepsilon,\mu) = \sum_{k=0}^{\infty} \left(z_k(t,\mu) + \Pi_k(\tau,\mu) + Q_k(\nu,\mu)\right)\varepsilon^k, \quad \tau = \frac{t-a}{\varepsilon}, \quad \nu = \frac{t-b}{\varepsilon}. \tag{4}$$

After the determination of $z_k(t,\mu)$, $\Pi_k(\tau,\mu)$, and $Q_k(\nu,\mu)$, a solution of (1), (2) takes the form

$$x(t,\varepsilon) = \sum_{k=0}^{\infty} \left(z_k(t,\varepsilon) + \Pi_k(\tau,\varepsilon) + Q_k(\nu,\mu)\right)\varepsilon^k.$$

Using the condition (C_1) it is easy to obtain functions $z_k(t,\varepsilon)$, which are elements of the regular series. Elements of the singular series $\Pi_k(\tau,\varepsilon)$, $\tau = \dfrac{t-a}{\varepsilon}$ and $Q_k(\nu,\mu)$, $\nu = \dfrac{t-b}{\varepsilon}$, were obtained by solving sequential linear differential systems. We will indicate some of the operations made in the article [6] and necessary for the present work.

We substitute series (4) into system (3) and we represent the function $F(t,z,\varepsilon,f(t,\mu))$ in the form

$$F\left(t, \sum_{k=0}^{\infty} \left(z_k(t,\mu) + \Pi_k(\tau,\mu) + Q_k(\nu,\mu)\right)\varepsilon^k, \varepsilon, f(t,\mu)\right) = \overline{F}(t,\varepsilon,\mu) + \Pi F(\tau,\varepsilon,\mu) + QF(\nu,\varepsilon,\mu),$$

where

$$
\begin{aligned}
\overline{F}(t,\varepsilon,\mu) &= F\left(t, \sum_{k=0}^{\infty} z_k(t,\mu)\varepsilon^k, \varepsilon, f(t,\mu)\right),\\
\Pi F(\tau,\varepsilon,\mu) &= F\left(a+\varepsilon\tau, \sum_{k=0}^{\infty} \left(z_k(a+\varepsilon\tau,\mu)+\Pi_k(\tau,\mu)\right)\varepsilon^k, \varepsilon, f(a+\varepsilon\tau,\mu)\right) -\\
&\quad - F\left(a+\varepsilon\tau, \sum_{k=0}^{\infty} z_k(a+\varepsilon\tau,\mu)\varepsilon^k, \varepsilon, f(a+\varepsilon\tau,\mu)\right), \qquad (5)\\
QF(\nu,\varepsilon,\mu) &= F\left(b+\varepsilon\nu, \sum_{k=0}^{\infty} \left(z_k(b+\varepsilon\nu,\mu)+Q_k(\nu,\mu)\right)\varepsilon^k, \varepsilon, f(b+\varepsilon\nu,\mu)\right) -\\
&\quad - F\left(b+\varepsilon\nu, \sum_{k=0}^{\infty} z_k(b+\varepsilon\nu,\mu)\varepsilon^k, \varepsilon, f(b+\varepsilon\nu,\mu)\right).
\end{aligned}
$$

We decompose the functions $\overline{F}(t,\varepsilon,\mu)$, $\Pi F(\tau,\varepsilon,\mu)$, $QF(\nu,\varepsilon,\mu)$ in Taylor series in a neighborhood of the points $(t, z_0(t), 0, f)$, $(a, z_0(a)+\Pi_0(\tau), 0, f)$, $(b, z_0(b)+Q_0(\nu), 0, f)$, respectively. We get

$$
\overline{F}(t,\varepsilon,\mu) = \sum_{k=0}^{\infty} \overline{F}_k(t,\mu)\varepsilon^k,
$$

where

$$
\overline{F}_k(t,\mu) = \begin{cases} F(t, z_0(t), 0, f(t,\mu)), & k=0,\\[2ex] F_z(t, z_0(t), 0, f(t,\mu))z_k(t,\mu)+\\ \quad +g_k\left(z_0(t),\ldots,z_{k-1}(t,\mu), f(t,\mu)\right), & k=1,2,\ldots. \end{cases} \qquad (6)
$$

In (6) the functions g_k contain derivative up to the $(k-1)$th order of the function $F(t,z,\varepsilon,f(t,\mu))$ with respect to z and ε, calculated in the point $(t, z_0(t), 0, f)$

$$
\Pi F(\tau,\varepsilon,\mu) = \sum_{k=0}^{\infty} \Pi F_k(\tau,\mu)\varepsilon^k,
$$

where

$$
\Pi F_k(\tau,\mu) = \begin{cases} F(a, z_0(a,\mu)+\Pi_0(\tau), 0, f(a,\mu)) -\\ \quad -F(a, z_0(a,\mu), 0, f(a,\mu)), & k=0,\\[2ex] F_z(a, z_0(a)+\Pi_0(\tau), 0, f(a,\mu))\Pi_k(\tau,\mu)+\\ \quad +G_k\left(\tau, \Pi_0(\tau),\ldots,\Pi_{k-1}(\tau,\mu), f(a,\mu)\right), & k=1,2,\ldots. \end{cases} \qquad (7)
$$

The function G_k contain derivatives up to the $(k-1)$th order of the function $F(t, z, \varepsilon, f(t, \mu))$ with respect to t, z and ε in the point $(a, z_0(a) + \Pi_0(\tau), 0, f(a, \mu))$, and the derivatives up to the $(k-1)$th order of the function $z_k(t, \mu)$ with respect to t in the point (a, μ),

$$QF(\nu, \varepsilon, \mu) = \sum_{k=0}^{\infty} QF_k(\nu, \mu)\varepsilon^k,$$

where

$$QF_k(\nu, \mu) = \begin{cases} F(b, z_0(b) + Q_0(\nu), 0, f(b, \mu)) - \\ \quad -F(b, z_0(b), 0, f(b, \mu)), & k = 0, \\ \\ F_z(b, z_0(b) + Q_0(\nu), 0, f(b, \mu))Q_k(\nu, \mu) + \\ \quad +R_k\left(\nu, Q_0(\nu), \ldots, Q_{k-1}(\nu, \mu), f(b, \mu)\right), & k = 1, 2, \ldots. \end{cases} \tag{8}$$

The functions R_k contain derivatives up to the $(k-1)$th order of the function $F(t, z, \varepsilon, f(t, \mu))$ with respect to t, z and ε in the point $(b, z_0(b) + Q_0(\nu), 0, f(b, \mu))$, and the derivatives up to the $(k-1)$th order of the function $z_k(t, \mu)$ with respect to t in the point (b, μ).

A similar approach is used for the BVP (1), (2), which does not contain the function $f(t, \varepsilon)$, in the articles [4, 8]. In these articles and in the article [6] Lemma 1 is essential. The problem connected with condition (C_1) for a differential equation in a Banach space is discussed, for example, in [2].

Lemma 1. *Let the matrix A satisfy the condition (C_1), P be a spectral projection on the left half plane of the matrix A, and functions $g(\tau) \in C(0, +\infty)$, $\overline{g}(\nu) \in C(-\infty, 0)$ satisfy the inequalities*

$$\|g(\tau)\| \le C^* \exp(-\alpha^* \tau), \quad C^* > 0,\ \alpha^* > 0, \quad \tau \ge 0,$$

$$\|\overline{g}(\nu)\| \le \overline{C}^* \exp(\overline{\alpha}^* \nu), \quad \overline{C}^* > 0, \quad \overline{\alpha}^* > 0, \quad \nu \le 0.$$

Then the systems $\dfrac{dx}{d\tau} = Ax + g(\tau)$, $\tau \in [0, +\infty)$ and $\dfrac{dy}{d\nu} = Ay + \overline{g}(\nu)$, $\nu \in (-\infty, 0]$, have particular solutions $(L_\tau g)(\tau)$ and $(L_\nu \overline{g})(\nu)$, respectively, in the forms

$$(L_\tau g)(\tau) = \int_0^{+\infty} K(\tau, s) g(s) ds \quad \textit{and} \quad (L_\nu \overline{g})(\nu) = \int_{-\infty}^{0} \overline{K}(\nu, s) \overline{g}(s)\, ds,$$

satisfying the inequalities

$$\|(L_\tau g)(\tau)\| \le C \exp(-\gamma \tau),\ \ \tau \ge 0; \quad \|(L_\nu \overline{g})(\nu)\| \le \overline{C} \exp(\overline{\gamma} \nu),\ \ \nu \le 0,$$

where $C, \overline{C}, \gamma, \overline{\gamma}$ are certain positive constants, and

$$K(\tau, s) = \begin{cases} X(\tau) P X^{-1}(s), & 0 \le s \le \tau < +\infty, \\ \\ -X(\tau)(I - P) X^{-1}(s), & 0 \le \tau \le s < +\infty, \end{cases}$$

$$\overline{K}(\nu, s) = \begin{cases} -X(\nu)(I - P) X^{-1}(s), & -\infty < \nu \le s \le 0, \\ \\ X(\nu) P X^{-1}(s), & -\infty < s \le \nu < 0. \end{cases}$$

Let the linear system $\dfrac{dx}{dt} = Ax$ have a fundamental solution matrix $X(t) = \exp(At)$, $X(0) = E_n$ and B be an $(n\times n)$ nonsingular constant matrix such that $B^{-1}AB = \operatorname{diag}(A_+, A_-)$, where A_+ is an $(p\times p)$-matrix having eigenvalues with negative real parts, $\operatorname{Re}\lambda_i < 0$, $i = \overline{1,p}$, and A_- is $((n-p)\times(n-p))$-matrix having eigenvalues with positive real parts, $\operatorname{Re}\lambda_i > 0$, $i = \overline{p+1,n}$.

The system $\dfrac{dx}{dt} = Ax$ has stable manifold S^+ in the form $S^+\colon \overline{\overline{x}} = H\overline{x}$, where $H = B_{21}B_{11}^{-1}$ is an $((n-p)\times p)$-matrix, and unstable manifold S^- in the form $S^-\colon \overline{x} = \overline{H}\overline{\overline{x}}$, where $\overline{H} = B_{12}B_{22}^{-1}$ is an $(p\times(n-p))$-matrix. The cells B_{ij}, $i,j = 1,2$, are elements of the block representation of the matrix $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}$.

Let $X_p(\tau) = X(\tau)\begin{pmatrix} E_p \\ H\end{pmatrix}$ be a $(n\times p)$-matrix, $X_{n-p}(\nu) = X(\nu)\begin{pmatrix} \overline{H} \\ E_{n-p}\end{pmatrix}$ be a $(n\times(n-p))$-matrix.

Let us introduce the following notations:

$D_1(\varepsilon) = lX_p(\cdot) = lX_p\left(\dfrac{(\cdot)-a}{\varepsilon}\right)$ is a $(n\times p)$-matrix;

$D_2(\varepsilon) = lX_{n-p}(\cdot) = lX_{n-p}\left(\dfrac{(\cdot)-b}{\varepsilon}\right)$ is a $(n\times(n-p))$-matrix;

$D(\varepsilon) = (D_1(\varepsilon)D_2(\varepsilon))$ is a $(n\times n)$-matrix.

Let $D(\varepsilon) = D_0 + O\left(\varepsilon^q\exp(-\alpha/\varepsilon)\right)$, D_0 be a $(n\times n)$-matrix, with constant elements. In this case $\operatorname{rang} D_0 = r < n$ and D_0^{-1} does not exist. But according to [1, 3, 10] we can use unite Mur – Penrous pseudoinverse matrix of D_0, which is written as D_0^+. More details in connection to pseudoinverse matrices can be found in the cited literature. Let P_{D_0} and $P_{D_0^*}$ be $(n\times n)$-matrices (orthogonal projections) projecting R^n onto $N(D_0) = \ker D_0$ and onto $N(D_0^*) = \ker D_0^*$, respectively, i.e., $P_{D_0}\colon R^n \to \ker D_0$, $P_{D_0^*}\colon R^n \to \ker D_0^*$, $D_0^* = D_0^T$, $P_{D_0}^2 = P_{D_0}$, $P_{D_0^*}^2 = P_{D_0^*}$. Having in mind that $\operatorname{rang} D_0 = r < n$, then $\operatorname{rang} P_{D_0} = \operatorname{rang} P_{D_0^*} = n - r = q$. There exists q linearly independent columns in the $(n\times n)$-matrix P_{D_0} and q linearly independent rows in the $(n\times n)$-matrix $P_{D_0^*}$. By $P_{D_{0q}}$ we denote the $(n\times q)$-matrix consisting of q arbitrary linearly independent columns of P_{D_0}, and by $P_{D_{0q}^*}$ we denote the $(q\times n)$-matrix consisting of q arbitrary linearly independent rows of $P_{D_0^*}$.

Theorem 1 [6]. *Suppose that the following conditions are satisfied:*

(H_1) (C_1) – (C_4);

(H_2) *the matrix* $D(\varepsilon)$ *has the representation* $D(\varepsilon) = D_0 + O\left(\varepsilon^q\exp(-\alpha/\varepsilon)\right)$, $q\in N$, $\alpha > 0$, *and* $\operatorname{rank} D_0 = r < n$;

(H_3) $P_{D_0^*}h_0 = 0$, *where* $h_0 = h + l\left(A^{-1}\varphi(\cdot)\right)$;

(H_4) *the nonlinear equation* $P_{D_0^*}h_1(\varepsilon,\mu,\xi_0) = 0$ *for all* $0 < \varepsilon \le \overline{\varepsilon}$, $0 < \mu \le \overline{\varepsilon}$ *has a unique bounded solution with respect to* $\xi_0 = \psi_0(\varepsilon,\mu) \in R^q$, *where*

$$h_1(\varepsilon,\mu,\xi_0) = -lz_1(\cdot,\mu) - l(L_\tau \Pi F_0)(\cdot,\mu,\xi_0) - l(L_\nu QF_0)(\cdot,\mu,\xi_0)$$

at $\Pi F_0(\tau,\mu) = F(a, z_0(a,\mu) + \Pi_0(\tau), 0, f(a,\mu)) - F(a, z_0(a,\mu), 0, f(a,\mu))$ *and* $QF_0(\nu,\mu) = F(b, z_0(b) + Q_0(\nu), 0, f(b,\mu)) - F(b, z_0(b), 0, f(b,\mu))$.

Then the principal part of the expansion (4) has the forms

$$z_0(t,\mu) = -A^{-1}\varphi(t),$$
$$\Pi_0(\tau) = X_p(\tau)[P_{D_{0q}}]_p\psi_0(\varepsilon,\mu) + X_p(\tau)[D_0^+ h_0]_p, \tag{9}$$
$$Q_0(\nu) = X_{n-p}(\nu)[P_{D_{0q}}]_{n-p}\psi_0(\varepsilon,\mu) + X_{n-p}(\nu)\left[D_0^+ h_0\right]_{n-p},$$

respectively.

We introduce the following notations:

$$h_k(\varepsilon,\mu,\xi_{k-1}) = D_1(\varepsilon,\mu)\xi_{k-1} + S_{k-1}(\varepsilon,\mu), \quad \xi_{k-1} \in R^q, \quad k \ge 2,$$

$$D_1(\varepsilon,\mu) = -l\left(F_z\left(a, z_0(a) + \Pi_0\left(\frac{(\cdot)-a}{\varepsilon}\right), 0, f(a,\mu)\right) X_p(\cdot)[D_{0q}]_p + \right.$$
$$\left. + F_z\left(b, z_0(b) + Q_0\left(\frac{(\cdot)-b}{\varepsilon}\right), 0, f(b,\mu)\right) X_{n-p}(\cdot)[D_{0q}]_{n-p}\right),$$

$$S_k(\varepsilon,\mu) = -l z_{k+1}(\cdot,\mu) - l\left(F_z\left(a, z_0(a) + \Pi_0(\cdot), 0, f(a,\mu)\right)\overline{\Phi}_k(\cdot,\varepsilon,\mu) + \right.$$
$$\left. + F_z(b, z_0(b) + Q_0(\cdot), 0, f(b,\mu))\overline{\overline{\Phi}}_k(\cdot,\varepsilon,\mu)\right) -$$
$$- l\left(G_k(\cdot, \Pi_0(\cdot), \ldots, \Pi_{k-1}(\cdot,\mu), f(a,\mu)) + \right.$$
$$\left. + R_k(\cdot, Q_0(\cdot), \ldots, Q_{k-1}(\cdot,\mu), f(b,\mu))\right), \quad k \ge 1, \tag{10}$$

$$\overline{\Phi}_k(\tau,\varepsilon,\mu) = X_p(\tau)\left[D_0^+\overline{h}_k\right]_p + (L_\tau\overline{\Pi F}_k)(\tau,\mu),$$
$$\overline{\overline{\Phi}}_k(\nu,\varepsilon,\mu) = X_{n-p}(\nu)\left[D_0^+\overline{h}_k\right]_{n-p} + (L_\nu\overline{QF}_k)(\nu,\mu),$$
$$\overline{h}_k = h_k(\varepsilon,\mu,\xi_{k-1}) = h_k(\varepsilon,\mu,\psi_{k-1}(\varepsilon,\mu)) = \overline{h}_k(\varepsilon,\mu),$$
$$\overline{D}_1(\varepsilon,\mu) = P_{D_{0q}} D_1(\varepsilon,\mu),$$
$$\overline{S}_k(\varepsilon,\mu) = -P_{D_{0q}^*} S_k(\varepsilon,\mu), \quad k \ge 1.$$

It should be noted that in (10) $D_1(\varepsilon,\mu)$ is an $(n \times q)$-matrix, $S_k(\varepsilon,\mu)$ is an n-vector, $\overline{D}_1(\varepsilon,\mu)$ is a $(q \times q)$-matrix and $\overline{S}_k(\varepsilon,\mu)$ is a q-vector. Besides, the functions G_k contain derivative up to the $(k-1)$th order of the function a $F(t,z,\varepsilon,f(t,\mu))$ with respect to t, z, and ε, in the point $(a, z_0(a)+\Pi_0(\tau), 0, f(a,\mu))$, and derivative up to the $(k-1)$th order of the function $z_k(t,\mu)$ with respect to t in the point (a,μ). The functions R_k contain derivatives up to the $(k-1)$th order of the function $F(t,z,\varepsilon,f(t,\mu))$ with respect to t, z, and ε, in the point $(b, z_0(b)+Q_0(\nu), 0, f(b,\mu))$, and derivatives up to the $(k-1)$th order of the function $z_k(t,\mu)$ with respect t in the point (b,μ).

Theorem 2 [6]. *Let the conditions* (H_1) – (H_4) *of the Theorem 1 and the condition*

(H_5) $\operatorname{rank} \overline{D}_1(\varepsilon,\mu) \neq 0$ $\forall 0 < \varepsilon \leq \overline{\varepsilon},\ 0 < \mu \leq \overline{\varepsilon}$

be fulfilled. Then the coefficients $z_k(t,\mu)$, $\Pi_k(\tau,\mu)$, *and* $Q_k(\nu,\mu)$ *for* $k \geq 1$ *of the series (4) have the forms*

$$z_k(t,\mu) = -\left(A^{-1}\right)^{k+1} \frac{d^k\varphi(t)}{dt^k} - \sum_{i=k}^{1} (A^{-1})^i \frac{d^{i-1}}{dt^{i-1}} \overline{F}_{k-i}(t,\mu),$$

$$\Pi_k(\tau,\mu) = -X_p(\tau)[P_{D_{0q}}]_p \overline{D}_1^{-1}(\varepsilon,\mu)\overline{S}_k(\varepsilon,\mu) + \overline{\Phi}_k(\tau,\varepsilon,\mu),$$

$$Q_k(\nu,\mu) = -X_{n-p}(\nu)[P_{D_{0q}}]_{n-p} \overline{D}_1^{-1}(\varepsilon,\mu)\overline{S}_k(\varepsilon,\mu) + \overline{\overline{\Phi}}_k(\nu,\varepsilon,\mu).$$

Theorem 3 [6]. *Let the conditions* (H_1) – (H_5) *of the Theorem 2 be fulfilled. Then*

1) the functions $z_k(t,\mu)$, $k \geq 0$, *are bounded, i. e.,* $\exists M_k > 0$: $\|z_k(t,\mu)\| \leq M_k$ $\forall t \in [a,b]$, $\mu \in (0,\overline{\varepsilon}]$, $k \geq 0$;

2) the boundary functions $\Pi_k(\tau,\mu)$ *and* $Q_k(\nu,\mu)$, $k \geq 0$, *decrease exponentially at* $\tau \to \infty$ *and* $\nu \to -\infty$ *respectively,* $\mu \in (0,\overline{\varepsilon}]$.

In the next section we will show that the obtained formal series (4) is asymptotic.

2. Main results. In BVP (3) we make a change of variables,

$$u(t,\varepsilon,\mu) = z(t,\varepsilon,\mu) - Z_n(t,\varepsilon,\mu), \tag{11}$$

where $Z_n(t,\varepsilon,\mu) = \sum_{k=0}^{n} \left[z_k(t,\mu) + \Pi_k(\tau,\mu) + Q_k(\nu,\mu)\right]\varepsilon^k$ is the nth partial sum of the series (4). Keeping in mind the expressions (9) in Theorem 1, the notations (10) and z_k, Π_k, Q_k, $k \geq 1$, in Theorem 2 we obtain that the new variable u satisfy the BVP

$$\varepsilon\dot{u} = Au + H_n(t,u,\varepsilon,\mu), \quad lu(\cdot,\varepsilon,\mu) = 0, \tag{12}$$

where

$$H_n(t,u,\varepsilon,\mu) = \varepsilon F(t, u + Z_n, \varepsilon, f(t,\mu)) + L(t,\tau,\nu,\varepsilon,\mu). \tag{13}$$

The expression $L(t,\tau,\nu,\varepsilon,\mu)$ have the form

$$L(t,\tau,\nu,\varepsilon,\mu) = \Delta_1 + \Delta_2 + \Delta_3, \tag{14}$$

where

$$\begin{aligned}
\Delta_1 &= A\sum_{k=1}^{n} z_k(t,\mu)\varepsilon^k - \sum_{k=0}^{n} \frac{d}{dt} z_k(t,\mu)\varepsilon^{k+1}, \\
\Delta_2 &= A\sum_{k=0}^{n} \Pi_k(\tau,\mu)\varepsilon^k - \sum_{k=0}^{n} \frac{d}{d\tau} \Pi_k(\tau,\mu)\varepsilon^k, \\
\Delta_3 &= A\sum_{k=0}^{n} Q_k(\nu,\mu)\varepsilon^k - \sum_{k=0}^{n} \frac{d}{d\nu} Q_k(\nu,\mu)\varepsilon^k.
\end{aligned} \tag{15}$$

For the function $H_n(t,u,\varepsilon,\mu)$ from (13) we will prove three lemmas.

Lemma 2. *Let* $u = 0$ *in (13). Then the function* $H_n(t,0,\varepsilon,\mu)$ *with* $C > 0$, $t \in [a,b]$, $\varepsilon \in (0,\varepsilon_1]$, $\mu \in (0,\varepsilon_1]$, $0 < \varepsilon_1 < \bar{\varepsilon}$, *satisfies the inequality* $\|H_n(t,0,\varepsilon,\mu)\| \le C\varepsilon^{n+1}$.

Proof. By means of equalities (5)–(8) for the representation of $H_n(t,0,\varepsilon,\mu)$ we obtain

$$\begin{aligned} H_n(t,0,\varepsilon,\mu) &= \varepsilon F(t,Z_n,\varepsilon,f) + L(t,\tau,\nu,\varepsilon,\mu) = \\ &= \varepsilon F\left(t, \sum_{k=0}^{n}[z_k + \Pi_k + Q_k]\,\varepsilon^k, \varepsilon, f\right) + L(t,\tau,\nu,\varepsilon,\mu) = \\ &= \Delta_0 + \Delta_1 + \Delta_2 + \Delta_3, \end{aligned} \tag{16}$$

where $\Delta_0 = \varepsilon\overline{F} + \varepsilon\Pi F + \varepsilon QF$, and Δ_i, $i = 1,2,3$ it, are given by (15). For a representation of the sums Δ_i we use (5) for Δ_0, Theorem 2 for Δ_1. For representation Δ_2 and Δ_3 we consider receiving the boundary functions Π_k, Q_k by linear BVPs, from which result of the boundary function in Theorem 2,

$$\frac{d\Pi_k(\tau)}{d\tau} = A\Pi_k(\tau) + \overline{\Pi F}_k(\tau,\mu), \quad \tau \in \left[0, \frac{b-a}{\varepsilon}\right], \quad \mu \in (0,\varepsilon],$$

$$\frac{dQ_k(\nu)}{d\nu} = AQ_k(\nu) + \overline{QF}_k(\nu,\mu), \quad \nu \in \left[\frac{a-b}{\varepsilon}, 0\right], \quad \mu \in (0,\varepsilon],$$

$$l\left(\Pi_k\left(\frac{(\cdot)-a}{\varepsilon},\mu\right) + Q_k\left(\frac{(\cdot)-b}{\varepsilon},\mu\right)\right) = \begin{cases} h - l(z_0(\cdot,\mu)), & k = 0, \\ -l(z_k(\cdot,\mu)), & k = 1,2,\ldots, \end{cases}$$

$$\overline{\Pi F}_k(\tau,\mu) = \begin{cases} 0, & k = 0, \\ \Pi F_{k-1}(\tau,\mu), & k = 1,2,\ldots, \end{cases}$$

$$\overline{QF}_k(\nu,\mu) = \begin{cases} 0, & k = 0, \\ QF_{k-1}(\tau,\mu), & k = 1,2,\ldots. \end{cases}$$

Thus we get

$$\Delta_0 = \varepsilon\overline{F} + \varepsilon\Pi F + \varepsilon QF = \sum_{k=0}^{n}\overline{F}_k\varepsilon^{k+1} + \sum_{k=0}^{n}\Pi F_k\varepsilon^{k+1} + \sum_{k=0}^{n}QF_k\varepsilon^{k+1},$$

$$\Delta_1 = A\sum_{k=1}^{n} z_k(t,\mu)\varepsilon^k - \sum_{k=0}^{n}\frac{d}{dt}\,z_k(t,\mu)\varepsilon^{k+1} = -\sum_{k=1}^{n}\overline{F}_{k-1}(t,\mu)\varepsilon^k - \frac{d}{dt}\,z_n\varepsilon^{n+1},$$

$$\Delta_2 = A\sum_{k=0}^{n}\Pi_k\varepsilon^k - \sum_{k=0}^{n}\frac{d}{d\tau}\,\Pi_k\varepsilon^k = -\sum_{k=1}^{n}\overline{\Pi F}_{k-1}(\tau,\mu)\varepsilon^k,$$

$$\Delta_3 = A\sum_{k=0}^{n}Q_k\varepsilon^k - \sum_{k=0}^{n}\frac{d}{d\nu}\,Q_k\varepsilon^k = \ldots = -\sum_{k=1}^{n}\overline{\Pi Q}_{k-1}(\nu,\mu)\varepsilon^k.$$

We substitute Δ_i, $i = 0, 1, 2, 3$, in (16) and obtain

$$H_n(t, 0, \varepsilon, \mu) = (-Az_n(t, \mu) + \Pi F_n(\tau, \mu) + QF_n(\nu, \mu))\, \varepsilon^{n+1}.$$

Let the following inequalities be fulfilled:

$$\|A\| \le C_0, \quad C_0 > 0, \quad \|z_n(t, \mu)\| \le M_n, \quad M_n > 0.$$

On the other hand Theorem 2 shows that by the equalities (14) and (15) we have

$$\|\Pi F_n(\tau, \mu)\| \le \overline{C}_n \exp(-\alpha_n \tau), \quad \overline{C}_n > 0, \quad \alpha_n > 0, \quad \tau \ge 0,$$

$$\|QF_n(\nu, \mu)\| \le \overline{\overline{C}}_n \exp(\overline{\alpha}_n \nu), \quad \overline{\overline{C}}_n > 0, \quad \overline{\alpha}_n > 0, \quad \nu \le 0.$$

Then

$$\begin{aligned} \|H_n(t, 0, \varepsilon, \mu)\| &= \| - Az_n(t, \mu) + \Pi F_n(\tau, \mu) + QF_n(\nu, \mu)\|\varepsilon^{n+1} \le \\ &\le [\|A\|\|z_n(t, \mu)\| + \|\Pi F_n(\tau, \mu)\| + \|QF_n(\nu, \mu)\|]\, \varepsilon^{n+1} \le \\ &\le \left[C_0 M_n + \overline{C}_n \exp(-\alpha_n \tau) + \overline{\overline{C}}_n \exp(\overline{\alpha}_n \nu)\right] \varepsilon^{n+1}. \end{aligned}$$

Keeping in mind that at $t \in [a, b]$, $\tau = \dfrac{t - a}{\varepsilon} \ge 0$, $\nu = \dfrac{t - b}{\varepsilon} \le 0$, for the upper bound, we finally obtain

$$\|H_n(t, 0, \varepsilon, \mu)\| \le \left[C_0 M_n + \overline{C}_n.1 + \overline{\overline{C}}_n.1\right] \varepsilon^{n+1} = C\varepsilon^{n+1},$$

where $C = C_0 M_n + \overline{C}_n + \overline{\overline{C}}_n$.

Let us also estimate the function $H_n(t, u, \varepsilon, \mu)$.

Lemma 3. *There exists a constant $C^* > 0$ such that $\|H_n(t, u, \varepsilon, \mu)\| \le C^* \varepsilon$ for $\|u\| \le 2R$, $R > 0$.*

Proof. From (13) we get

$$H_n(t, u, \varepsilon, \mu) = \varepsilon F(t, u + Z_n, \varepsilon, f(t, \mu)) + L(t, \tau, \nu, \varepsilon, \mu).$$

Then

$$\begin{aligned} H_n(t, u, \varepsilon, \mu) &= \varepsilon F(t, u + Z_n, \varepsilon, f(t, \mu)) - \varepsilon F(t, Z_n, \varepsilon, f(t, \mu)) + \\ &+ \varepsilon F(t, Z_n, \varepsilon, f(t, \mu)) + L(t, \tau, \nu, \varepsilon, \mu) = \\ &= \varepsilon\, [F(t, u + Z_n, \varepsilon, f(t, \mu)) - F(t, Z_n, \varepsilon, f(t, \mu))] + \\ &+ \varepsilon F(t, Z_n, \varepsilon, f(t, \mu)) + L(t, \tau, \nu, \varepsilon, \mu) = \\ &= \varepsilon \int\limits_0^1 F_x(t, Z_n + \theta u, \varepsilon, f(t, \mu)) d\theta u + H_n(t, 0, \varepsilon, \mu). \end{aligned}$$

According to the condition (C_3) and $x = Z_n + \theta u$ the function F_x is continuous with respect to all arguments in the domain G, i. e., there exists a constant $\widetilde{C} > 0$ such that $\|F_x\| \le \widetilde{C}$. Using the last lemma we obtain the estimate

$$\|H_n(t,u,\varepsilon,\mu)\| \le \varepsilon \int\limits_0^1 \|F_x(t, Z_n + \theta u, \varepsilon, f(t,\mu))\| d\theta \|u\| + \|H_n(t,0,\varepsilon,\mu)\| \le$$

$$\le \varepsilon\widetilde{C}\|u\| + C\varepsilon^{n+1} \le 2R\widetilde{C}\varepsilon + C\varepsilon^{n+1} \le C^*\varepsilon,$$

where $2R\widetilde{C} \le C^* < 2R\widetilde{C} + \widetilde{\widetilde{C}}$, $0 < \widetilde{\widetilde{C}} \ll 1$.

Lemma 4. *Let, in the some neighborhood of the degenerate solution $\|z_0\| < \delta$, we have $\|z\| \le \le \rho < \delta$ and $t \in [a,b]$, $\varepsilon \in [0,\varepsilon_1]$, $\mu \in (0,\varepsilon_1]$. Then there is a positive constant $\overline{K}_1$ such that in case of $\|\overline{u}\| \le \overline{\delta}$ and $\|\overline{\overline{u}}\| \le \overline{\delta}$, where $0 < \overline{\delta} < \delta$ and $\overline{\delta} + \rho < \delta$, the function $H_n(t,u,\varepsilon,\mu)$ satisfies the inequality*

$$\|\Delta H_n\| = \|H_n(t,\overline{u},\varepsilon,\mu) - H_n(t,\overline{\overline{u}},\varepsilon,\mu)\| \le \overline{K}_1\varepsilon\|\overline{u} - \overline{\overline{u}}\|.$$

Proof. From (13) we get

$$\Delta H_n = \varepsilon\left[F(t, \overline{u} + z_n, \varepsilon, f(t,\mu)) - F(t, \overline{\overline{u}} + z_n, \varepsilon, f(t,\mu))\right].$$

The estimate of the last difference is realized analogously to Lemma 3. Consequently,

$$\|\Delta H_n\| \le \overline{K}_1\varepsilon\|\overline{u} - \overline{\overline{u}}\|.$$

Let B be a $(n \times n)$-matrix satisfying the condition $B^{-1}AB = \operatorname{diag}(A_+\ A_-)$, where $\operatorname{Re}\lambda_i(A_+) < 0$, $j = \overline{1,p}$, $\operatorname{Re}\lambda_i(A_-) > 0$, $j = \overline{p+1,n}$.

In the system (12) we make a change of variables,

$$u(t,\varepsilon,\mu) = B\begin{pmatrix} \eta(t,\varepsilon,\mu) \\ \delta(t,\varepsilon,\mu) \end{pmatrix},$$

where $\eta(t,\varepsilon,\mu) = (\eta_1,\ldots,\eta_p)^T$, $\delta(t,\varepsilon,\mu) = (\delta_1,\ldots,\delta_{n-p})^T$. Then (8) take the form

$$\begin{aligned} \varepsilon\dot{\eta}(t,\varepsilon,\mu) &= A_+\eta + \left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_p, \\ \varepsilon\dot{\delta}(t,\varepsilon,\mu) &= A_-\delta + \left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_{n-p}, \end{aligned} \tag{17}$$

$$lB\begin{pmatrix} \eta(t,\varepsilon,\mu) \\ \delta(t,\varepsilon,\mu) \end{pmatrix} = 0, \tag{18}$$

where

$$\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu) = B^{-1}H_n(t,\eta,\delta,\varepsilon,\mu) =$$

$$= B^{-1}\varepsilon F\left(t, B\begin{pmatrix} \eta \\ \delta \end{pmatrix} + Z_n, \varepsilon, f(t,\mu)\right) + B^{-1}L(t,\tau,\nu,\varepsilon,\mu). \tag{19}$$

The notations $\left[\widetilde{H}_n\right]_p$ and $\left[\widetilde{H}_n\right]_{n-p}$ are used to denote the first p and the later $n-p$ elements of the vector function $\widetilde{H}_n$, respectively.

According to Lemmas 2 and 4 it is possible to assert that for the equality (19) we have the inequalities

$$\|\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\| \le \widetilde{C}_1\varepsilon^{n+1}, \quad \widetilde{C}_1 > 0,$$
$$\|\Delta\widetilde{H}_n\| = \|\widetilde{H}_n(t,\overline{\eta},\overline{\delta},\varepsilon,\mu) - \widetilde{H}_n(t,\overline{\overline{\eta}},\overline{\overline{\delta}},\varepsilon,\mu) \le \widetilde{\widetilde{C}}_1\varepsilon\left(\|\overline{\eta}-\overline{\overline{\eta}}\| + \|\overline{\delta}-\overline{\overline{\delta}}\|\right), \quad \widetilde{\widetilde{C}}_1 > 0. \tag{20}$$

The system of differential equations (17) transforms to a system of integral equations,

$$\eta(t,\varepsilon,\mu) = W(t,s,\varepsilon,\mu)\eta(a,\varepsilon,\mu) + \int\limits_a^t W(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_p ds,$$
$$\delta(t,\varepsilon,\mu) = \overline{W}(t,s,\varepsilon,\mu)\delta(b,\varepsilon,\mu) + \int\limits_t^b \overline{W}(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_{n-p} ds, \tag{21}$$

where $\eta(a,\varepsilon,\mu)$ and $\delta(b,\varepsilon,\mu)$ are arbitrary vectors depending on ε and μ. In (21) the fundamental matrices $W(t,s,\varepsilon,\mu)$ and $\overline{W}(t,s,\varepsilon,\mu)$ are solutions of the systems

$$\varepsilon\frac{d\eta}{dt} = A_+\eta, \quad W(s,s,\varepsilon,\mu) = E_p \quad \text{and} \quad \varepsilon\frac{d\delta}{dt} = A_-\delta, \quad \overline{W}(s,s,\varepsilon,\mu) = E_{n-p}$$

respectively. If $\sigma > 0$ and $K_0 > 0$, then the inequalities

$$\|W(t,s,\varepsilon,\mu)\| \le K_0\exp\left(-\sigma\frac{t-s}{\varepsilon}\right), \quad a \le s \le t \le b,$$
$$\|\overline{W}(t,s,\varepsilon,\mu)\| \le K_0\exp\left(-\sigma\frac{s-t}{\varepsilon}\right), \quad a \le t \le s \le b, \tag{22}$$

are fulfilled. In consequence we will consider the iterative process

$$\eta^0(a,\varepsilon,\mu) = 0, \quad \delta^0(b,\varepsilon,\mu) = 0,$$
$$\eta^i(t,\varepsilon,\mu) = W(t,s,\varepsilon,\mu)\eta(a,\varepsilon,\mu) + \int\limits_a^t W(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n\left(t,\eta^{i-1},\delta^{i-1},\varepsilon,\mu\right)\right]_p ds, \tag{23}$$
$$\delta^i(t,\varepsilon,\mu) = \overline{W}(t,s,\varepsilon,\mu)\delta(b,\varepsilon,\mu) + \int\limits_t^b \overline{W}(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n\left(t,\eta^{i-1},\delta^{i-1},\varepsilon,\mu\right)\right]_{n-p} ds.$$

Our further aim is to show that the integral system (21) has a uniquel and continuous solution. Therefore we introduce a Banach space M [12] which consists of the all continuous n-dimensional functions

$$y(t,\varepsilon,\mu) = (\eta_1(t,\varepsilon,\mu),\ldots,\eta_p(t,\varepsilon,\mu),\delta_1(t,\varepsilon,\mu),\ldots,\delta_{n-p}(t,\varepsilon,\mu))^T$$

in the domain $G_2 = [a,b] \times (0,\overline{\varepsilon}] \times (0,\overline{\varepsilon}] = \{(t,\varepsilon,\mu) | t \in [a,b], \varepsilon \in (0,\overline{\varepsilon}], \mu \in (0,\overline{\varepsilon}]\}$ with the norm

$$\|y(t,\varepsilon,\mu)\| = \sum_{i=1}^{p} \max_{G_2} |\eta_i(t,\varepsilon,\mu)| + \sum_{i=1}^{n-p} \max_{G_2} |\delta_i(t,\varepsilon,\mu)|$$

and the distance between the elements

$$\|y_2(t,\varepsilon,\mu) - y_1(t,\varepsilon,\mu) = \sum_{i=1}^{p} \max_{G_2} \left|\eta_i^2(t,\varepsilon,\mu) - \eta_i^1(t,\varepsilon,\mu)\right| + \sum_{i=1}^{n-p} \max_{G_2} \left|\delta_i^2(t,\varepsilon,\mu) - \delta_i^1(t,\varepsilon,\mu)\right|.$$

The right-hand side of (21) we consider as an action of the operator $L(\cdot)$ on the vector function

$$y(\cdot,\varepsilon,\mu) = \begin{pmatrix} \eta(\cdot,\varepsilon,\mu) \\ \delta(\cdot,\varepsilon,\mu) \end{pmatrix},$$

$$L(y) = \begin{bmatrix} L_1(y) \\ L_2(y) \end{bmatrix} = \begin{bmatrix} W(t,s,\varepsilon,\mu)\eta(a,\varepsilon,\mu) + \int\limits_a^t W(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_p ds \\ \overline{W}(t,s,\varepsilon,\mu)\delta(b,\varepsilon,\mu) + \int\limits_t^b \overline{W}(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_{n-p} ds \end{bmatrix}. \qquad (24)$$

We will prove that the operator $L(y)$ is a contractive operator.

Lemma 5. *Let the conditions be fulfilled:*

1) $\widetilde{\widetilde{C}}_1 < \sigma$ *and* $C^* < \sigma$,

2) $0 < K_0 < \frac{1}{4}$,

3) $0 < R < \dfrac{2K_0C^*}{\sigma(1-2K_0)}$,

4) $0 < \varepsilon \le \overline{\varepsilon} < \dfrac{\sigma}{2K_0C^*}(1-2K_0)R$,

5) $\|\eta(a,\varepsilon,\mu)\| \le R$, $\|\delta(b,\varepsilon,\mu)\| \le R$.

Then the operator $L(y)$ is a contraction.

Proof. Step 1. Primary we show that the operator $L(y)$ maps the space M into itself,

$$\|L(y)\| \le \max_{G_2} \left\| W(t,s,\varepsilon,\mu)\eta(a,\varepsilon,\mu) + \int\limits_a^t W(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_p ds \right\| +$$

$$+ \max_{G_2} \left\| \overline{W}(t,s,\varepsilon,\mu)\delta(b,\varepsilon,\mu) + \int\limits_t^b \overline{W}(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_{n-p} ds \right\| \le$$

$$\le \overline{\Delta}_1 + \overline{\Delta}_2, \qquad (25)$$

where

$$\overline{\Delta}_1 = \max_{G_2} \left[\|W(t,s,\varepsilon,\mu)\| \, \|\eta(a,\varepsilon,\mu)\| + \|\overline{W}(t,s,\varepsilon,\mu)\| \, \|\delta(b,\varepsilon,\mu)\| \right],$$

$$\overline{\Delta}_2 = \max_{G_2} \left\{ \int_a^t \|W(t,s,\varepsilon,\mu)\| \frac{1}{\varepsilon} \, \|[\widetilde{H}_n(s,\eta,\delta,\varepsilon,\mu)]_p\| ds \right\} +$$

$$+ \max_{G_2} \left\{ \int_t^b \|\overline{W}(t,s,\varepsilon,\mu)\| \frac{1}{\varepsilon} \, \|[\widetilde{H}_n(s,\eta,\delta,\varepsilon,\mu)]_{n-p}\| ds \right\}.$$

Keeping in mind (22), the condition 4, 5 of the present lemma and Lemma 3 for estimates of $\overline{\Delta}_1$ and $\overline{\Delta}_2$ we have

$$\overline{\Delta}_1 \le \max_{G_2} \left[K_0 \exp\left(-\sigma \frac{t-s}{\varepsilon}\right) R + K_0 \exp(-\sigma \frac{s-t}{\varepsilon}) R \right] \le K_0 R (1+1) = 2K_0 R,$$

$$\overline{\Delta}_2 \le \max_{G_2} \int_a^t K_0 \exp\left(-\sigma \frac{t-s}{\varepsilon}\right) \frac{1}{\varepsilon} C^* \varepsilon ds + \max_{G_2} \int_t^b K_0 \exp\left(-\sigma \, \frac{s-t}{\varepsilon}\right) \frac{1}{\varepsilon} C^* \varepsilon \, ds =$$

$$= K_0 C^* \max_{G_2} \left[\int_a^t \exp\left(-\sigma \frac{t-s}{\varepsilon}\right) ds + \int_t^b \exp\left(-\sigma \, \frac{s-t}{\varepsilon}\right) ds \right] =$$

$$= K_0 C^* \max_{G_2} \frac{\varepsilon}{\sigma} \left[1 - \exp\left(-\sigma \, \frac{t-a}{\varepsilon}\right) - \exp\left(-\sigma \, \frac{b-t}{\varepsilon}\right) + 1 \right] \le$$

$$\le K_0 C^* \frac{\overline{\varepsilon}}{\sigma} \, 2 \left[1 - \exp\left(-\sigma \frac{b-a}{\overline{\varepsilon}}\right) \right] \le 2K_0 C^* \, \frac{\overline{\varepsilon}}{\sigma}.$$

We substitute the estimates for $\overline{\Delta}_1$, $\overline{\Delta}_2$ in (25) and get

$$\|L(y)\| \le \overline{\Delta}_1 + \overline{\Delta}_2 \le 2K_0 R + 2K_0 C^* \, \frac{\overline{\varepsilon}}{\sigma}.$$

Keeping in mind the condition 4, we get $\|L(y)\| \le R$, that is, the operator $L(y)$ maps the space M into itself.

Step 2. We will estimate the difference $L(y_2) - L(y_1)$ (see (24)). According to (22), second inequality in (20) and condition 4 we obtain

$$\|L(y_2)-L(y_1)\| = \left\|\begin{bmatrix} L_1(y_2)-L_1(y_1) \\ L_2(y_2)-L_2(y_1)\end{bmatrix}\right\| =$$

$$= \left\|\begin{bmatrix} \int\limits_a^t W(t,s,\varepsilon,\mu)\,\frac{1}{\varepsilon}\left[\widetilde{H}_n\left(\eta^2,\delta^2,s,\varepsilon,\mu\right)-\widetilde{H}_n\left(\eta^1,\delta^1,s,\varepsilon,\mu\right)\right]_p ds \\ \int\limits_t^b \overline{W}\left(t,s,\varepsilon,\mu\right)\frac{1}{\varepsilon}\left[\widetilde{H}_n\left(\eta^2,\delta^2,s,\varepsilon,\mu\right)-\widetilde{H}_n\left(\eta^1,\delta^1,s,\varepsilon,\mu\right)\right]_{n-p} ds\end{bmatrix}\right\| \le$$

$$\le \int\limits_a^t K_0\exp\left(-\sigma\,\frac{t-s}{\varepsilon}\right)\frac{1}{\varepsilon}\,\widetilde{\widetilde{C}}_1\varepsilon\left(\left\|\eta^2-\eta^1\right\|+\left\|\delta^2-\delta^1\right\|\right)ds+$$

$$+\int\limits_t^b K_0\exp\left(-\sigma\,\frac{s-t}{\varepsilon}\right)\,\frac{1}{\varepsilon}\,\widetilde{\widetilde{C}}_1\varepsilon\left(\left\|\eta^2-\eta^1\right\|+\left\|\delta^2-\delta^1\right\|\right)ds \le$$

$$\le \frac{2K_0\widetilde{\widetilde{C}}_1}{\sigma}\,\varepsilon\left(1-\exp\left(-\sigma\,\frac{b-a}{\varepsilon}\right)\right)\|y_2-y_1\| \le \frac{2K_0\widetilde{\widetilde{C}}_1}{\sigma}\,\varepsilon\|y_2-y_1\| \le$$

$$\le \frac{2K_0\widetilde{\widetilde{C}}_1}{\sigma}\overline{\varepsilon}\|y_2-y_1\| < \frac{2K_0\widetilde{\widetilde{C}}_1}{\sigma}\,\frac{\sigma}{2K_0C^*}(1-2K_0)R\|y_2-y_1\| =$$

$$= \underbrace{\frac{\widetilde{\widetilde{C}}_1}{C^*}\,(1-2K_0)R}_{\Theta}\,\|y_2-y_1\| = \Theta\|y_2-y_1\|.$$

The conditions 1–3 of the lemma show that $0<\Theta<1$. Consequently, the operator $L(y)$ is a contraction.

We introduce the following notations:

$$\overline{D}(\varepsilon) = l\left(\begin{bmatrix} B_{11}W(\cdot,a,\varepsilon) & B_{12}\overline{W}(\cdot,b,\varepsilon) \\ B_{21}W(\cdot,a,\varepsilon) & B_{22}\overline{W}(\cdot,b,\varepsilon)\end{bmatrix}\right) - (n\times n)\text{-matrix}, \tag{26}$$

$$\overline{g}(\varepsilon,\mu) = -l\left(B\begin{bmatrix} \int\limits_a^{(\cdot)} W(\cdot,s,\varepsilon)\,\frac{1}{\varepsilon}\left[\widetilde{H}_n(s,\eta,\delta,\varepsilon,\mu)\right]_p ds \\ \int\limits_{(\cdot)}^b \overline{W}(\cdot,s,\varepsilon)\frac{1}{\varepsilon}\left[\widetilde{H}_n(s,\eta,\delta,\varepsilon,\mu)\right]_{n-p} ds\end{bmatrix}\right) - (n\times 1)\text{-vector}. \tag{27}$$

Let $\overline{D}(\varepsilon)=\overline{D}_0+O\left(\varepsilon^s\exp\left(-\dfrac{\alpha}{\varepsilon}\right)\right)$, where $\overline{D}_0$ is a constant $(n\times n)$-matrix.

Theorem 4. *Let the conditions* (H_1) – (H_5) *(see Theorems 1 and 2), the conditions of Lemma 5, and the conditions* $\operatorname{rank}\overline{D}_0 = n,\ K_3 b_1\|B\| \le 2$, *where* $\|\overline{D}_0^{-1}\| \le K_3,\ K_3 > 0,\ \|l(\psi)\| \le b_1\|\psi\|,$ $b_1 > 0$, *be satisfied. Then there exist constants* $\varepsilon^* > 0,\ \widetilde{C}^* > 0$ *such that the problem (1), (2) has a unique solution* $x(t,\varepsilon)$ *and it satisfies the inequality*

$$\|x(t,\varepsilon) - X_n(t,\varepsilon)\| \le \widetilde{C}^*\varepsilon^{n+1} \tag{28}$$

for $t \in [a,b]$ *and* $\varepsilon \in (0,\varepsilon^*]$.

Proof. To prove that (1), (2) has the only solution satisfying (28) means to prove that the boundary problem (3) has a unique solution satisfying

$$||z(t,\varepsilon,\mu) - Z_n(t,\varepsilon,\mu)|| \le \widetilde{C}^*\varepsilon^{n+1}$$

for $t \in [a,b]$, $\varepsilon \in (0,\varepsilon^*]$ and $\mu \in (0,\varepsilon^*]$.

Therefore (3) we make the following replacement (11) and obtain the boundary problem (12). To prove the theorem it is sufficient to show that (12) has a unique solution such that $\|u(t,\varepsilon,\mu)\| \le \widetilde{C}^*\varepsilon^{n+1}$. For system (12) we make a change of the variable

$$u(t,\varepsilon,\mu) = B\begin{pmatrix} \eta(t,\varepsilon,\mu) \\ \delta(t,\varepsilon,\mu) \end{pmatrix},$$

where $\eta(t,\varepsilon,\mu) = (\eta_1,\ldots,\eta_p)^T$, $\delta(t,\varepsilon,\mu) = (\delta_1,\ldots,\delta_{n-p})^T$, and consider the equivalent integral equation (21).

Lemma 5 shows that system (21) has only a solution which does not go out of the area

$$\Omega = \{(t,\eta,\delta,\varepsilon,\mu) | t \in [a,b],\ \|\eta\| \le R, \|\delta\| \le R, \varepsilon \in (0,\overline{\varepsilon}], \mu \in (0,\overline{\varepsilon}]\}$$

and depends on arbitrary constant vectors $\eta(a,\varepsilon,\mu)$ and $\delta(b,\varepsilon,\mu)$. The determination of the vectors $\eta(a,\varepsilon,\mu)$ and $\delta(b,\varepsilon,\mu)$ is performed using the algebraic system (18),

$$lu(\cdot,\varepsilon,\mu) = lB\begin{pmatrix} \eta(\cdot,\varepsilon,\mu) \\ \delta(\cdot,\varepsilon,\mu) \end{pmatrix} = 0.$$

We substitute η and δ of (21) in the last equation, and according to the notations of (26), (27) we find that $\eta(a,\varepsilon,\mu)$ and $\delta(b,\varepsilon,\mu)$ satisfy the algebraic system

$$\overline{D}(\varepsilon)\begin{bmatrix} \eta(a,\varepsilon,\mu) \\ \\ \delta(b,\varepsilon,\mu) \end{bmatrix} = \overline{g}(\varepsilon,\mu).$$

After dropping exponentially small elements in $\overline{D}(\varepsilon)$, we get the system

$$\overline{D}_0\begin{bmatrix} \eta(a,\varepsilon,\mu) \\ \\ \delta(b,\varepsilon,\mu) \end{bmatrix} = \overline{g}(\varepsilon,\mu). \tag{29}$$

Under the condition of the theorem, the system (29) has a solution

$$\begin{bmatrix} \eta(a,\varepsilon,\mu) \\ \\ \delta(b,\varepsilon,\mu) \end{bmatrix} = \overline{D}_0^{-1}\overline{g}(\varepsilon,\mu). \tag{30}$$

An estimate of (30), according to the conditions of the theorem and the proof of a Lemma 5, has the form

$$\left\|\begin{bmatrix} \eta(a,\varepsilon,\mu) \\ \delta(b,\varepsilon,\mu) \end{bmatrix}\right\| \le \left\|\overline{D}_0^{-1}\right\| \|\overline{g}(\varepsilon,\mu)\| \le K_3\|\overline{g}(\varepsilon,\mu)\| = K_3 b_1 \|B\|\overline{\Delta}_2 \le 2R.$$

In the latter inequality we used the estimate for $\overline{\Delta}_2$ (see step 1 of the Lemma 5) and the condition $K_3 b_1\|B\| \le 2$. Thus for integral equations (21) we find the representation

$$\begin{aligned} \eta(t,\varepsilon,\mu) &= W(t,s,\varepsilon,\mu)\left[\overline{D}_0^{-1}\overline{g}(\varepsilon,\mu)\right]_p + \int_a^t W(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_p ds, \\ \delta(t,\varepsilon,\mu) &= \overline{W}(t,s,\varepsilon,\mu)\left[\overline{D}_0^{-1}\overline{g}(\varepsilon,\mu)\right]_{n-p} + \int_t^b \overline{W}(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\left[\widetilde{H}_n(t,\eta,\delta,\varepsilon,\mu)\right]_{n-p} ds. \end{aligned} \tag{31}$$

For the integral equations (31) we apply the iteration process (23). For the first approximation of (23), keeping in mind Lemma 3 and that $u = (\eta_1,\ldots,\eta_p,\delta_1,\ldots,\delta_{n-p})$ we find

$$\begin{aligned} \|u_1 - u_0\| &\le \sum_{i=1}^{p}\max\left|\eta_i^1 - \eta_i^0\right| + \sum_{i=1}^{n-p}\max\left|\delta_i^1 - \delta_i^0\right| = \\ &= \max_{G_2}\left\|\eta^1 - \eta^0\right\| + \max_{G_2}\left\|\delta^1 - \delta^0\right| \le \\ &\le \max_{G_2}\left[\|W(t,a,\varepsilon,\mu)\|\|\eta(a,\varepsilon,\mu)\| + \|\overline{W}(t,b,\varepsilon,\mu)\|\|\delta(b,\varepsilon,\mu)\|\right] + \\ &\quad + \max_{G_2}\left\{\int_a^t \left\|W(t,a,\varepsilon,\mu)\frac{1}{\varepsilon}\right\|\left\|\left[\widetilde{H}_n(s,0,0,\varepsilon,\mu)\right]_p\right\| ds + \right. \\ &\quad \left. + \int_t^b \|\overline{W}(t,b,\varepsilon,\mu)\|\frac{1}{\varepsilon}\left\|\left[\widetilde{H}_n(s,0,0,\varepsilon,\mu)\right]_{n-p}\right\| ds\right\} = \overline{\Delta}_1 + \overline{\Delta}_2^0, \end{aligned}$$

where

$$\overline{\Delta}_2^0 = \max_{G_2}\left\{\int_a^t \left\|W(t,s,\varepsilon,\mu)\frac{1}{\varepsilon}\right\|\left\|\left[\widetilde{H}_n(s,0,0,\varepsilon,\mu)\right]_p\right\| ds + \right.$$

$$+\int\limits_t^b \left\|\overline{W}(t,s,\varepsilon,\mu)\right\| \frac{1}{\varepsilon} \left\|\left[\widetilde{H}_n(s,0,0,\varepsilon,\mu)\right]_{n-p}\right\| ds \Bigg\}.$$

It is possible to prove that $\overline{\Delta}_1 \leq K^1\varepsilon^{n+1}$, $K^1 > 0$, $\overline{\Delta}_2^0 \leq K^2\varepsilon^{n+1}$, $K^2 > 0$. Consequently,

$$\|u_1 - u_0\| \leq K\varepsilon^{n+1} \quad \text{or} \quad \|u_1 - u_0\| \leq \frac{\alpha}{2}, \quad \alpha = 2K\varepsilon^{n+1}.$$

With the help of the second inequalities in (20) we obtain

$$\left\|u^j - u^{j-1}\right\| \leq \frac{1}{2} \left\|u^{j-1} - u^{j-2}\right\|.$$

Therefore we have

$$\|u_1 - u_0\| \leq \frac{\alpha}{2},\ \|u_2 - u_1\| \leq \frac{\alpha}{2^2},\ \ldots,\ \|u_j - u_{j-1}\| \leq \frac{\alpha}{2^j}.$$

Then, in the domain G_2,

$$\|u_j(t,\varepsilon,\varepsilon\mu)\| \leq \sum_{k=1}^{j} \|u_k(t,\varepsilon,\mu) - u_{k-1}(t,\varepsilon,\mu)\| \leq$$

$$\leq \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{j-1}} + \ldots\right) \frac{\alpha}{2} \leq K\varepsilon^{n+1}.$$

Let $\lim_{j\to\infty} u_j(t,\varepsilon,\varepsilon\mu) = u(t,\varepsilon,\varepsilon\mu)$. Then (22) becomes an identify. Then there exist constants $C^* > 0$ and $\varepsilon^* > 0$, $\varepsilon^* \leq \bar{\varepsilon}$ such that, in the region,

$$\Omega_1 = \{(t,\eta,\delta,\varepsilon,\mu) | t \in [a,b], \|\eta\| \leq R, \|\delta\| \leq R, \varepsilon \in (0,\varepsilon^*], \mu \in (0,\varepsilon^*]\}$$

we have

$$\|u(t,\varepsilon,\mu)\| = \left\|\begin{bmatrix} \eta(a,\varepsilon,\mu) \\ \delta(b,\varepsilon,\mu) \end{bmatrix}\right\| \leq \widetilde{C}^*\varepsilon^{n+1}.$$

Theorem 4 shows that the obtained formal series (4) is asymptotic and $\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t)$.

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