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OPTIMAL MORSE – SMALE FLOWS WITH SINGULARITIES ON THE BOUNDARY OF SURFACE

ОПТИМАЛЬНІ ПОТОКИ МОРСА – СМЕЙЛА З ОСОБЛИВОСТЯМИ НА МЕЖІ ПОВЕРХНІ

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We consider optimal flows on noncompact surfaces with boundary, which have a minimum number of fixed points and all of them lie on the boundary of the surface. It is proved that the flow will be optimal if it has a single sink and a single source. The structure of optimal flows on simply connected region, Möbius strip, torus with a hole and Klein bottle with a hole are described.

Розглядаються оптимальні потоки на компактних поверхнях з межею, у яких мінімальне число нерухомих точок і всі вони лежать на межі поверхні. Доведено, що потік буде оптимальним, якщо він має єдиний стік і єдиний витік. Описано структуру всіх оптимальних потоків на однозв'язній області, листі Мьобіуса, торі з діркою та пляшці Клейна з діркою.

Introduction. Let M be a smooth surface with a boundary. We investigate the structure of flows on M with fixed points on the boundary.

Two vector fields X_1 and X_2, on closed surfaces M_1 and M_2, correspondingly, are called *topologically equivalent*, if there exists a homeomorphism $h\colon M_1 \to M_2$, that maps trajectories of the vector field X_1 into trajectories of the vector field X_2, preserving the orientation on the trajectories. It is also said that these vector fields have an identical structure.

A vector field X on a manifold M is called structurally stable, if there exists a neighbourhood U of the field X in a set of all vector fields on the manifold M such that an arbitrary $Y \in U$ is topologically equivalent to the field X.

We consider Morse – Smale flows, on connected surfaces with connected boundary, without closed trajectories and such that their singularities lie on the boundary of the surface and there is a minimum number of these singularities.

Definition. *A vector field X on a smooth manifold M with a boundary ∂M is called a Morse – Smale field, if it satisfies the following conditions:*

(1) the set $\Omega(X)$ of nonwandering points X consists of a finite number of singularities and closed orbits and all of them are hyperbolic;

(2) stable and unstable manifolds of two elements from $\Omega(X)$ cross transversally in $\operatorname{Int} M$, *and if there is a point in the nontransversal crossing on the boundary of these elements, then at least one of these elements is a singularity.*

A flow that is induced by a Morse – Smale vector field will be called a *Morse – Smale flow*.

The objective of this paper is to describe the structure of Morse – Smale flows on connected surfaces with a connected boundary without closed trajectories and such that their singularities lie on the boundary of the surface and there is a minimum number of these singularities.

On closed surfaces structurally stable vector fields are Morse – Smale fields [1, 2]. For larger dimension manifolds, except for Morse – Smale vector fields, there exist other structurally stable vector fields. For manifolds with boundary, an analogue of Morse – Smale fields was described in works [3 – 5].

On closed surfaces Morse – Smale flows form an open everywhere-dense set on the set of all flows. In addition, these flows are structurally stable on manifolds of an arbitrary dimension. Among flows for which the set of nonwandering points consists of a finite number of trajectories, structurally stable are only Morse – Smale flows. Analogous results are also true for manifolds with boundary. In particular, these results are proved in [3 – 5].

Structure classification of Morse – Smale vector fields on closed surfaces is obtained in the works of Peixoto [6], Sharko [7], for three-dimension manifolds in Umanskyi [8], Prishlyak [9] for fields with some restrictions.

In [10], a complete topological invariant was constructed for Morse – Smale flows with singularities on the boundary of a 2-dimensional disc.

2. Optimal flows. A Morse – Smale flow on a surface with boundary is called *simple*, if it doesn't have closed trajectories and all singularities lie on the boundary. A simple Morse – Smale flow on a surface with boundary is called *optimal*, if it has the least number of singularities among all simple Morse – Smale flows on this surface.

Theorem 1. *A simple Morse – Smale flow on a surface with boundary is optimal if and only if it has a single sink and a single source.*

Proof. To prove the theorem we should show: (1) a simple Morse – Smale flow on a surface with boundary has at least one sink and one source; (2) for a given surface with boundary there exist a simple Morse – Smale flow that has only one sink and one source; (3) if a flow has more than one sink or source, then it is not optimal.

Firstly, according to the definition of a Morse – Smale flow, every trajectory starts in a singularity. So far as the number of singularities is finite and not more than 2 trajectories come out of every of them, only a finite number of trajectories start in a saddle point. Thus, the rest of trajectories (of infinite number) start in sources. Analogously, it can be proved that at least one sink exists.

Secondly, it is known that every compact surface with one boundary component is diffeomorphic to a disc with glued ribbons. Therefore, it is enough to build a vector field with one sink and one source on the surface (differential of a diffeomorphism mapping the field into a field with the same properties on an arbitrary surface). Separatrices of the desired vector fields are shown on Fig. 1. Other trajectories start in sources and end in sinks.

Thirdly, consider doubling DM of M surface, that is received by gluing two M copies by trivial (identical) diffeomorphism of the boundary. As a result, we have a vector field on doubling with the same particularities. According to Poincare – Hopf theorem, Euler characteristic of surface DM is equal to the number of sources plus the number of sinks minus the number of saddle points. The formula shows that an increase of the number of sources or sinks makes the number of saddle points on the specified surface increase and, thus, makes the general number of singularities increase. It means that a flow with one source and one sink is optimal. The theorem is proved.

In what follows, for saddle singularities we will use the following names: (1) a-saddle for a singularity where two trajectories come into and one trajectory comes out (it is a sink when restricting a flow to the boundary), (2) b-saddle for a singularity where 2 trajectories come out and one trajectory comes in (it is a source when restricting the flow to the boundary) (see Fig. 2).

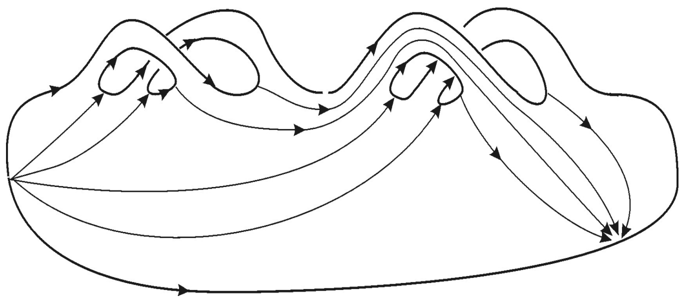

Fig. 1

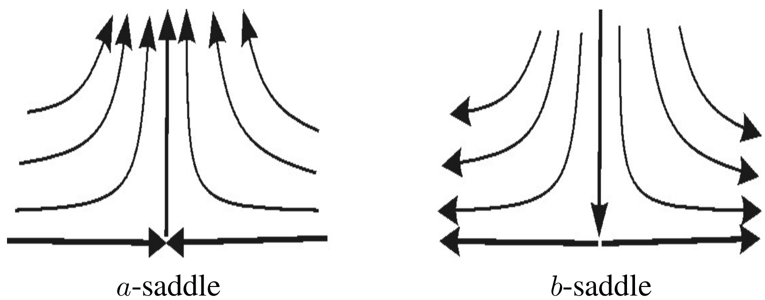

Fig. 2

3. Optimal flows on a closed simply connected region. Every optimal flow on a closed simply connected region has two singularities, a sink and a source. All these flows have the same structure as the flow illustrated on Fig. 3. It can be easily seen that if one fixes the middle point of every trajectory, except for the singularities, then we map homeomorphically the set of these middle points (a simple curve) to the same set of a standard flow. Subsequently, we continue the obtained homeomorphism to each trajectory.

4. Optimal flows on Möbius strip. Since the Klein bottle, which has Euler characteristics equal to zero, is Möbius strip doubling, the number of saddle points in an optimal flow on the Möbius strip is equal to two (the number of sources and sinks). Consider a trajectory α that starts in a source and ends in a saddle point, and does not lie on the Möbius strip boundary. Let us show that α does not partition the Möbius strip. Suppose, by contradiction, that α partitions the Möbius strip into two parts. Since the Möbius strip genus is equal to 1 and to sum of genuses of these parts, then one of the part has genus 0, in other words, it is a simply connected region. Denote it by D. While restricting the flow to the Möbius strip boundary, the considered saddle point and the source are sources of the restricted flow. Thus, the trajectory that comes out of a saddle must end in a sink. But it is impossible, because these points belong to different spaces, to D and to the relative complement of D.

Split the Möbius strip by a trajectory α. Then, the other saddle and the sink lie on opposite sides of the received quadrangle. The trajectory that comes out of this saddle and ends in a sink partitions the quadrangle into two quadrangles. Since these quadrangles have no singularities and separatrices inside, the structure of the flow is determined by its behaviour on the boundary of these quadrangles. According to unambiguity of the conducted constructions, the following theorem follows.

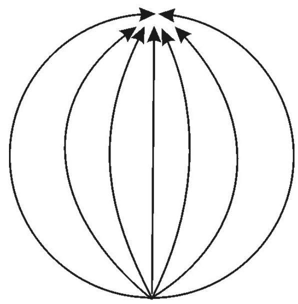

Fig. 3

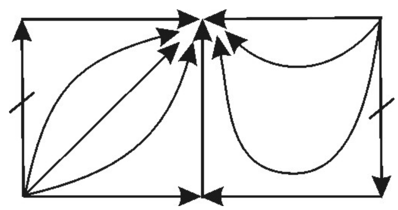

Fig. 4

Theorem 2. *All optimal flows on the Möbius strip have the same structure that is shown on Fig. 4.*

5. Optimal flows on torus with a hole. Calculating Euler characteristic as in Theorem 1, one can show that an optimal flow on torus with a hole, as on the Klein bottle with a hole, contains 6 singularities: a source, a sink, two a-saddles and two b-saddles. The sink and the source partition boundary of the surface into two arcs, so that two cases are possible:

(1) all saddle points lie on one arc;

(2) two saddle points lie on every arc.

Consider first case. We number singularities, bypassing boundary of the surface, so that the source has the number 1, and the sink the number 6. Then the point is a-saddle with the number 2, the b-saddle with the number 3, the a-saddle with the number 4 and the b-saddle with the number 5. The trajectory that comes out of the point 2 ends in the sink 6, and the trajectory, that comes into 3, starts in source 1. We cut the surface by these two trajectories. We get a simply connected space $U.$ To make this sure, we cut the obtained surface by two more trajectories that go in and out of other two saddles. Applying the Euler characteristics formula for a torus, we get three simply connected spaces. Gluing them together backwards by two trajectories, we get one simply connected space. From now on, we will consider U as an octagon with vertices $ABCDEFGH$ and indexes 13263162 correspondingly. By construction, the points with numbers 4 and 5 lie on DE side. The trajectory that enters the b-saddle with number 5 must start in a source. These vertices are vertices A or F of the octagon. If the trajectory starts at the vertex $F,$ then it splits the octagon into two parts, in one of which the point with number 4, lies and both sinks D and G lie in another one. Thus, the trajectory that comes out of vertex 4 cuts the given trajectory that enters vertex 5. And yet this is impossible for flows (all trajectories do not intersect). Thus, the trajectory that enters vertex 5 starts at vertex $A.$ Then, the trajectory that comes out of vertex 4 ends at vertex G (see Fig. 5).

Consider the second case. Cutting the surface with two trajectories, one of which comes out of the a-saddle, and another enters the b-saddle (for adjacent saddles), we obtain an octagon

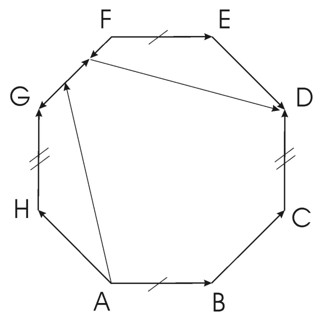

Fig. 5

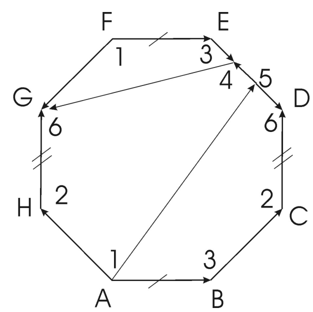

Fig. 6

$ABCDEFGH$ with vertices of the same types again. However, as opposed to the first case, two other saddle points will be located on the FG side. By considerations, analogous to the previous case, we conclude that the flow will have the structure represented in Fig. 6.

Theorem 3. *There are two optimal flows on a torus with a hole, which have a different structure. Every optimal flow has a structure represented in Fig. 5 or Fig. 6.*

Proof. As it was shown above, for every optimal flow, when cutting surface with two trajectories, an octagon $ABCDEFGH$ gets created, which we will consider as an equilateral octagon. In order to obtain the initial surface, one has to glue, by a homeomorphism, the side AD with the side FE in the octagon, and the side CD with the side $HG,$ respecting the order of the vertices. Notice, that the octagon symmetries reflect the flow into a flow with the same structure.

We show that there are no other optimal flows than the two described above. Since the movement direction side AB is the same as the one along the side $FE,$ and the direction along the side CD is the same as along the $HG,$ taking into consideration the octagon symmetries and the possibility to rename the vertices, we have only one option for the movement direction along these sides. For example, let it be the directions AB and $HG.$ Considering symmetries relatively to the line that comes through the centers of the sides AH and $DE,$ the movement direction along the side AH is set unambiguously. Let it be the direction from the vertex A to the vertex $H.$ Then A and F are sources, B and E are saddles, D and G are sinks, C and H are saddles. A pair of other saddle points is located on one of the sides that is not glued together. Because we cut by saddles that come out of the second and the third singularities, the sides AH and BC can not be these sides. Other two possible variants were considered above and specify two structures on the torus with a hole. The fact that these structures are different follows from the fact that their boundary limitations specify different flow structures on the boundary.

The theorem 3 is proved.

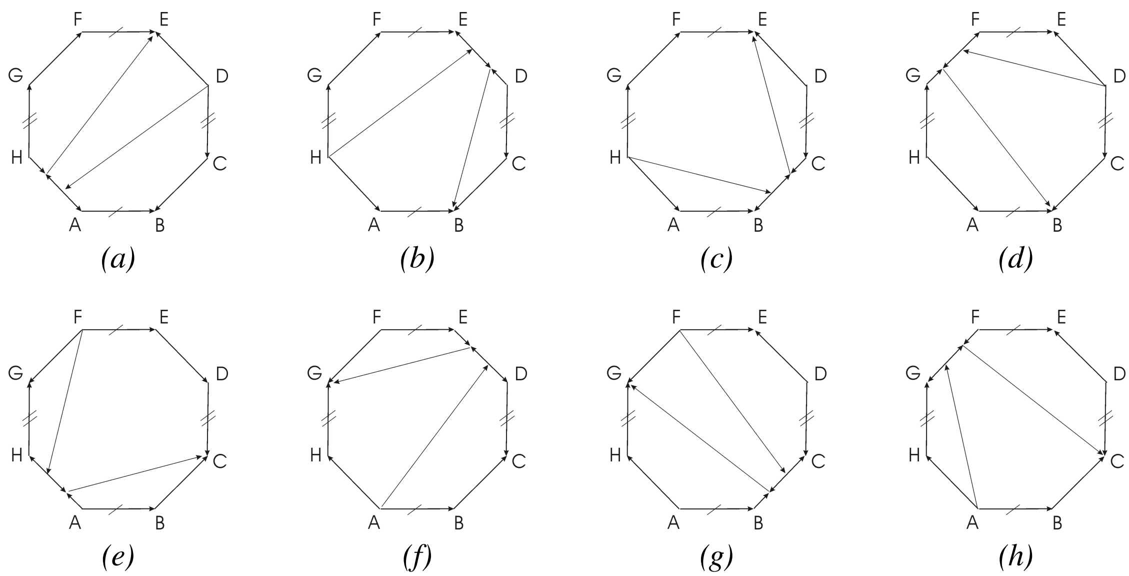

Fig. 7

6. Optimal flows on Klein bottle with a hole. Notice that the Klein bottle, like the torus, has Euler characteristic equal to zero. Then, with the same consideration as for the torus with a hole, after cutting the Klein bottle with a hole by two trajectories, we get octagon an $ABCDEFGH$ that we will consider as an equilateral octagon. Herewith, a Klein bottle with a hole can be obtained in two ways: (1) by gluing the side AB with the AC, the side EF with GH; (2) by gluing the side AB with FE, and the side CD with GH respecting the vertices order. The first case is not possible for an optimal Morse – Smale flow, because if there is one saddle (it has to be at least in one vertex) among the vertices $A,\ B,\ C,\ D$ ($E,\ F,\ G,\ H$ analogously), then the rest of the vertices are saddles, thus, the trajectory AB (EF) connects saddles, which is impossible for a Morse – Smale flow. Consider case two. For the side AH two orientations are possible. As in the case for the torus with a hole, a pair of saddles, that is left, is located on one of the sides, that do not glue together. Thus, altogether 8 cases of these locations are possible. They are shown in Fig. 7.

Herewith, we can cut an octagon by one of the trajectories and glue it by appropriate sides and repeat this procedure again, obtaining octagons that set the same structure as the previous one does. Then we get that these pairs of octagons give the same structure for the flow: (a) and (d), (b) and (h), (c) and (e), (f) and (g).

Theorem 4. *There are four optimal flows on a Klein bottle with a hole, which have different structures. Every optimal flow has one of these four structures.*

Proof. It follows from the previous discussion that every optimal flow on a Klein bottle with a hole has the structure, represented in Fig. 7 (a), (b), (c) or (f). We show that these four structures are different. Since cutting by separatrices in case (c) we get a hexagon and in other cases this does not occur, structure (c) is different from all others. Consider the quadrangles that are formed when cutting by separatrices in cases (a), (b) and (f). In case (b) the source and the sink are located in opposite vertices of the quadrangle, and, in other cases, in the adjacent ones. Therefore, structure (b) differs from others. In cases (a) and (f) a pentagon gets glued to a quadrangle by the edge adjacent to the sink that coincides with the quadrangle sink. Nevertheless, in every case this is done by different sides. Therefore, structures (a) and (f) are also different.

The Theorem 4 is proved.

It follows from the proof that if one changes the movement direction on all the trajectories of the flow, then structures (b) and (c) turn into themselves, and yet structures (a) and (f) turn into one another.

7. Conclusion. In the paper, flows on a surface with boundary were considered, with all singularities being hyperbolic and lying on the boundary of the surface. We prove a criterion for a flow to be optimal (it has a minimal number of singularities). The results obtained in the paper completely solve the problem of structural classification of optimal flows with hyperbolical singularities on the boundary of the surface of a torus with a hole, a Möbius strip, a Klein bottle with a hole. The authors expect that it is possible to receive similar results for other surfaces with boundary.

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