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THE DISCRETE NONLINEAR SCHRÖDINGER TYPE HIERARCHY, ITS FINITE DIMENSIONAL REDUCTION ANALYSIS AND NUMERICAL INTEGRABILITY SCHEME

ДИСКРЕТНА НЕЛІНІЙНА ІЄРАРХІЯ ТИПУ ШРЬОДІНГЕРА, АНАЛІЗ ЇЇ СКІНЧЕННОВИМІРНОЇ РЕДУКЦІЇ ТА ЧИСЕЛЬНА СХЕМА ІНТЕГРУВАННЯ

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We investigate discretizations of the integrable nonlinear Schrödinger dynamical system, well known as the Ablowitz – Ladik equation, the related symplectic structures and its finite dimensional invariant reductions. An effective scheme of invariant reducing the corresponding infinite system of ordinary differential equations to an equivalent finite system of ordinary differential equations with respect to the evolution parameter is developed. A finite set of recurrent algebraic regular relations, allowing to generate solutions of the discrete nonlinear Schrödinger dynamical system, is constructed, the related functional spaces of solutions is discussed. Finally, the Fourier transform approach to studying the solution set of the discrete nonlinear Schrödinger dynamical system and its functional-analytical aspects is analyzed.

Досліджуються дискретизації інтегровної нелінійної динамічної системи Шрьодінгера, відомої як рівняння Абловіца – Ладіка, відповідні симплектичні структури та її скінченновимірні інваріантні редукції. Побудовано ефективний алгоритм інваріантної редукції відповідної нескінченної системи звичайних диференціальних рівнянь до еквівалентної скінченної системи звичайних диференціальних рівнянь відносно параметра еволюції. Побудовано скінченну множину рекурентних алгебраїчних регулярних співвідношень, що дозволило побудувати розв'язки дискретної нелінійної динамічної системи Шрьодінгера, та розглянуто відповідні функціональні простори розв'язків. Проведено аналіз підходу перетворення Фур'є до вивчення множини розв'язків дискретної нелінійної динамічної системи Шрьодінгера та її функціонально-аналітичних аспектів.

1. Introduction. As is well known, soliton equations with constitute an wide class of integrable dynamical systems, which possess a lot very interesting mathematical properties, and describe diverse important physical phenomena. In particular, they are usually used to describe interactions between different solitary waves having physical applications. For example, the nonlinear Schrödinger equation describes the soliton propagation in a medium with both resonant and nonresonant nonlinearities [7, 46, 48] and the nonlinear interaction of high-frequency electrostatic wave with ion acoustic waves in plasma [18]. Due to the important role played by the soliton equations in many fields of physics such as hydrodynamics, solid-state physics, plasma physics, etc. they have received much attention in the literature. Especially there are of great interest their completely integrable descrete approximations of soliton equations, which have a great deal of applications in numerical analysis, computing simulations and related investigations. Our work is devoted to a thorough investigation of the three-point discretization

$$\begin{pmatrix} du_n/dt \\ d\bar{u}_n/dt \end{pmatrix} = K_n[u,\bar{u}] := \begin{pmatrix} i(u_{n+1} - 2u_n + u_{n-1}) - i\bar{u}_n u_n (u_{n+1} + u_{n-1}) \\ -i(2\bar{u}_n - u_{n+1} - u_{n-1}) + i\bar{u}_n u_n(\bar{u}_{n+1} + \bar{u}_{n-1}) \end{pmatrix} \tag{1.1}$$

for $n \in \mathbb{Z}$ of the integrable nonlinear Schrödinger dynamical system

$$\tilde{K}[\psi,\psi^*] := \begin{cases} \dfrac{d}{dt}\psi = i\psi_{xx} - 2i\alpha\psi\psi\psi^*, \\ \dfrac{d}{dt}\psi^* = -i\psi^*_{xx} + 2i\alpha\psi^*\psi\psi^* \end{cases} \tag{1.2}$$

on a functional manifold $\tilde{M} \subset L_2(\mathbb{R};\mathbb{C}^2)$, studying their related symplectic structures and finite-dimensional invariant reductions. The set of equations (1.1) is well known as the Ablowitz – Ladik (AL-DNLS) equation, whose Lax type integrability was first proven by Ablowitz and Ladik [1] and having many diverse applications [5, 21, 24, 30, 33, 35, 60] in physical and bilological sciences. Defining (1.1) as a smooth completely integrable Hamiltonian dynamical system on a discrete functional manifold $M_2 \subset l_2(\mathbb{Z};\mathbb{C}^2)$ with respect to the evolution parameter $t \in \mathbb{R}$, we developed an effective scheme of invariant reducing the corresponding infinite system of ordinary differential equations to a suitably determined finite system of ordinary differential equations, being a completely integrable finite-dimensional canonical Hamiltonian flow. The dynamical system (1.1) appears also to be a bi-Hamiltonian flow on the discrete functional manifold M_2 with respect to special noncanonical Poisson brackets (see, e.g., [9, 11, 22]). A constructed finite set of recurrent algebraic regular relations, allowing to generate solutions of the discrete nonlinear Schrödinger dynamical system, is analyzed in detail, the related functional spaces of solutions are also discussed. Based on the symplectic gradient-holonomic approach, devised before in [11, 54, 55] for the smooth nonlinear dynmamical systems on functional manifolds, we also investigate the differential-geometric and symplectic stuctures of the related hidden symmetries, responsible for the complete integrability of the Ablowitz – Ladik dynamical system (1.1). Finally, there is analyzed the Fourier transform approach to constructing the solution set of the discrete nonlinear Schrödinger dynamical system and its functional-analytical aspects.

2. Discrete dynamical systems integrability and reduction analysis. ***2.1. Preliminary notions and definitions.*** We consider an infinite-dimensional discrete manifold $M_m \subset l^2(\mathbb{Z};\mathbb{C}^m)$ for some integer $m \in \mathbb{Z}_+$ and a general nonlinear dynamical system of the form

$$\frac{dw}{dt} = K[w], \tag{2.1}$$

where $w \in M_m$ and $K\colon M_m \to T(M_m)$ is a Fréchet smooth nonlinear local mapping of M_m into its tangent space $T(M_m)$ and $t \in \mathbb{R}$ is the evolution parameter. As an example of the dynamical system (2.1) at $m = 2$ on a discrete manifold $M_2 \subset l_2(\mathbb{Z};\mathbb{C}^2)$, we will analyze the well-known [1, 44] AL-DNLS integrable system (1.1)

$$\begin{pmatrix} du_n/dt \\ d\bar{u}_n/dt \end{pmatrix} = K_n[u,\bar{u}] := \begin{pmatrix} i(u_{n+1} - 2u_n + u_{n-1}) - i\bar{u}_n u_n(u_{n+1} + u_{n-1}) \\ -i(2\bar{u}_n - u_{n+1} - u_{n-1}) + i\bar{u}_n u_n(\bar{u}_{n+1} + \bar{u}_{n-1}) \end{pmatrix}, \tag{2.2}$$

where the overbar denotes the complex conjugate on a functional manifold M_2 with $w = (u,v)^\intercal \in M_2$, and which, as was before mentioned, has many interesting applications [20, 21, 35] in a wide range of modern physics and biology problems.

To analyze the integrability properties of the differential-difference dynamical system (2.1), we shall develop a gradient-holonomic scheme related to those devised in [11, 32, 43, 54] for nonlinear dynamical systems defined on spatially one-dimensional functional manifolds and extended in [51] to include discrete manifolds.

Denote by $(\cdot,\cdot)$ the standard bilinear form (or pairing) on the space $T^*(M_m) \times T(M_m)$ naturally induced by the inner product in the Hilbert space $l^2(\mathbb{Z};\mathbb{C}^m)$. We define $\mathcal{D}(M_m)$ to be the space of smooth functionals on M_m, so for any $\gamma \in \mathcal{D}(M_m)$ one can define the gradient $\operatorname{grad}\gamma[w] \in T^*(M_m)$ as

$$\operatorname{grad}\gamma[u,\bar{u}] := \gamma^{\prime,*}[w]\cdot 1, \tag{2.3}$$

where the prime denotes the Fréchet derivative and "$*$" represents the conjugation with respect to the standard bracket on $T(M_m) \times T^*(M_m)$.

Definition 2.1. *A linear smooth operator $\vartheta\colon T^*(M_m) \to T(M_m)$ is called Poissonian on the manifold M_m, if the bilinear bracket*

$$\{\cdot,\cdot\}_\vartheta := (\operatorname{grad}(\cdot), \vartheta \operatorname{grad}(\cdot))$$

satisfies [3, 6, 9, 25, 54] the Jacobi identity on the space $\mathcal{D}(M_m)$ of all smooth functionals on M_m.

This means, in particular, that the bracket (2.3) satisfies the standard Jacobi identity on $\mathcal{D}(M_m)$.

Definition 2.2. *A linear smooth operator $\vartheta\colon T^*(M_m) \to T(M_m)$ is called Nötherian [9, 25, 54] with respect to the nonlinear dynamical system (2.1) if*

$$L_K\vartheta = \vartheta' K - \vartheta K^{\prime,*} - K'\vartheta = 0 \tag{2.4}$$

holds identically on the manifold M_m, where L_K is the Lie-derivative along the vector field $K\colon M_m \to T(M_m)$.

If the mapping $\vartheta\colon T^*(M_m) \to T(M_m)$ is invertible with inverse mapping $\vartheta^{-1} := \Omega : T(M_m) \to T^*(M_m)$, it is called *symplectic*. It then follows easily from (2.4) that

$$L_K\Omega = \Omega' K + \Omega K' + K^{\prime,*}\Omega = 0 \tag{2.5}$$

hold identically on M_m. Having now assumed that the manifold $M_m \subset l^2(\mathbb{Z};\mathbb{C}^2)$ is endowed with a smooth Poissonian structure $\vartheta\colon T^*(M_m) \to T(M_m)$, one can define the Hamiltonian system

$$\frac{dw}{dt} := -\vartheta \operatorname{grad} H[w], \tag{2.6}$$

corresponding to a Hamiltonian function $H \in \mathcal{D}(M_m)$. It follows directly from the definition (2.6) that the dynamical system

$$\frac{dw}{dt} = K[w] := -\vartheta \operatorname{grad} H[w]$$

satisfies the Nötherian conditions (2.4). We are studying the integrability [6, 9, 11, 50] of the discrete dynamical system (2.1). Accordingly we need to construct invariants with respect to it functions, called conservation laws, which are mutually commuting with respect to the Poisson bracket (2.3). The following Lax criterion [11, 39, 54] proves to be very useful.

Lemma 2.1. *Any smooth solution $\varphi \in T^*(M_m)$ to the Lax equation*

$$L_K\varphi = \frac{d\varphi}{dt} + K'^{,*}\varphi = 0, \tag{2.7}$$

satisfying the symmetry condition

$$\varphi' = \varphi'^{,*},$$

with respect to bracket $(\cdot,\cdot)$, is related to the conservation law

$$\gamma := \int_0^1 d\lambda(\varphi[w\lambda], w). \tag{2.8}$$

Proof. The expression (2.8) follows easily from the well-known Volterra homology equalities

$$\gamma = \int_0^1 \frac{d\gamma[w\lambda]}{d\lambda}\, d\lambda = \int_0^1 d\lambda \left(1, \gamma'[w\lambda]\cdot w\right) = \int_0^1 d\lambda \left(\gamma'^{,*}[w\lambda]\cdot 1, w\right) = \int_1^0 d\lambda(\operatorname{grad}\gamma[w\lambda], w)$$

and

$$(\operatorname{grad}\gamma[w])' = (\operatorname{grad}\gamma[w])'^{,*},$$

holding identically on M_m. Whence, one finds that there exists a function $\gamma \in \mathcal{D}(M_m)$ such that

$$L_K\gamma = 0, \quad \operatorname{grad}\gamma[w] = \varphi[w]$$

for any $w \in M_m$.

Lemma 2.1 is proved.

This result of Lax lemma is a direct consequence of the following generalized Nöther type result.

Lemma 2.2. *Let a smooth element $\psi \in T^*(M_m)$ satisfy the Nöther condition*

$$L_K\psi = \frac{d\psi}{dt} + K'^{,*}\psi = \operatorname{grad}\mathcal{L}_\psi \tag{2.9}$$

for some smooth functional $\mathcal{L}_\psi \in \mathcal{D}(M_m)$. Then the Hamiltonian representation

$$K = -\vartheta \operatorname{grad} H_\vartheta$$

holds, where

$$\vartheta := \psi' - \psi'^{,*}$$

and the Hamiltonian function is

$$H_\vartheta = (\psi, K) - \mathcal{L}_\psi.$$

It is easy to see that Lemma 2.1 follows from Lemma 2.2, if the conditions $\psi' = \psi'^{,*}$ and $\mathcal{L}_\psi = 0$ are imposed on (2.9).

Assume now that equation (2.9) allows an additional (nonsymmetric) smooth solution $\phi \in \in T^*(M_m)$:

$$L_K\phi = \frac{d\phi}{dt} + K'^{,*}\phi = \operatorname{grad}\mathcal{L}_\phi. \tag{2.10}$$

This means that our system (2.1) is bi-Hamiltonian:

$$-\vartheta \operatorname{grad} H_\vartheta = K = -\eta \operatorname{grad} H_\eta,$$

where, by definition,

$$\eta := \phi' - \phi'^{*}, \quad H_\eta = (\phi, K) - \mathcal{L}_\phi.$$

Definition 2.3. *One says that two Poissonian structures $\vartheta, \eta\colon T^*(M_m) \to T(M_m)$ on M_m are compatible [9, 25, 42, 54], if for any $\lambda, \mu \in \mathbb{R}$ the linear combination $\lambda\vartheta + \mu\eta\colon T^*(M_m) \to \to T(M_m)$ will be also Poissonian on M_m.*

It is easy to see that this condition is satisfied if, for instance, there exist an inverse $\vartheta^{-1}\colon T(M_m) \to T^*(M_m)$ and the composite map $\eta(\vartheta^{-1}\eta)\colon T^*(M_m) \to T(M_m)$ is also Poissonian on M_m.

Concerning the complete integrability of the infinite-dimensional dynamical system (2.1) on the discrete manifold M_m it is, in general, necessary, but not sufficient [11, 50, 54], to prove the existence of an infinite hierarchy of mutually commuting conservation laws with respect to the Poissonian structure (2.3).

Since in the case of Lax integrability of (2.1) there exist compatible Poissonian structures and related hierarchies of conservation laws, we shall focus our analysis by devising an integrability algorithm under the *a priori* assumption that the nonlinear dynamical system (2.1) on the manifold M_m is Lax integrable. This means that it possesses a Lax representation in the following general form:

$$\Delta f_n := f_{n+1} = l_n[w;\lambda] f_n, \tag{2.11}$$

where $f := \{f_n \in \mathbb{C}^r\colon n \in \mathbb{Z}\} \subset l^2(\mathbb{Z};\mathbb{C}^r)$ for some integer $r \in \mathbb{Z}_+$ and the matrices $l_n[w;\lambda] \in \in \operatorname{End}\mathbb{C}^r$, $n \in \mathbb{Z}$, in (2.11) are local matrix-valued functionals on M_m, depending on the "spectral" parameter $\lambda \in \mathbb{C}$, invariant with respect to our dynamical system (2.1).

As the Lax representation (2.11) is "local" with respect to the discrete variable $n \in \mathbb{Z}$, we shall assume for convenience that our manifold $M_m := M_m^{(N)} \subset l^\infty(\mathbb{Z}/N\mathbb{Z};\mathbb{C}^m)$ is periodic with respect to the discrete index $n \in \mathbb{Z}_N$, that is for any $n \in \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ and $\lambda \in \mathbb{C}$

$$l_n[w;\lambda] = l_{n+N}[w;\lambda] \tag{2.12}$$

for some integer $N \in \mathbb{Z}_+$. In this case the smooth functionals on $M_m^{(N)}$ can be represented as

$$\gamma := \sum_{n \in \mathbb{Z}_N} \gamma_n[w]$$

for some local Fréchet smooth densities $\gamma_n \colon M_m^{(N)} \to \mathbb{C}$, $n \in \mathbb{Z}_N$.

2.2. The gradient-holonomic scheme. Consider the representation (2.11) and define its fundamental solution $F_{m,n}(\lambda) \in \mathrm{Aut}\,(\mathbb{C}^r)$, $m, n \in \mathbb{Z}_N$, satisfying the equation

$$F_{m+1,n}(\lambda) = l_m[w;\lambda]F_{m,n}(\lambda)$$

and the condition

$$F_{m,n}(\lambda)|_{m=n} = \mathbf{1}$$

for all $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}_N$. Then the matrix function

$$S_n(\lambda) := F_{n+N,n}(\lambda) \tag{2.13}$$

is called the *monodromy* matrix for the linear equation (2.12) and satisfies for all $n \in \mathbb{Z}_N$ the Novikov – Lax relationship

$$S_{n+1}(\lambda)l_n = l_n S_n(\lambda). \tag{2.14}$$

It easy to compute that $S_n(\lambda) := \prod_{k=0}^{N-1} l_{n+k}[u;\lambda]$ owing to the periodicity condition (2.12). Construct now the generating functional

$$\bar{\gamma}(\lambda) := \mathrm{tr}\, S_n(\lambda), \tag{2.15}$$

where tr is the standard trace map, having the asymptotic expansion

$$\bar{\gamma}(\lambda) \sim \sum_{j \in \mathbb{Z}_+} \bar{\gamma}_j \lambda^{j_0 - j} \tag{2.16}$$

as $\lambda \to \infty$ for some fixed $j_0 \in \mathbb{Z}_+$. Then, owing to the obvious condition

$$D_n \bar{\gamma}(\lambda) = 0$$

for all $n \in \mathbb{Z}_N$, where we have introduced the "discrete" derivative

$$D_n := \Delta - 1,$$

we find that all functionals $\bar{\gamma}_j \in \mathcal{D}(M_m^{(N)})$, $j \in \mathbb{Z}_+$, are independent of the discrete index $n \in \mathbb{Z}_N$ and are simultaneously conservation laws for the dynamical system (2.1).

We now make an additional natural assumption, namely that the gradient vector

$$\bar{\varphi}(\lambda) := \mathrm{grad}\, \bar{\gamma}(\lambda)[w] = \mathrm{tr}\, l_n^{\prime ,*} \left(S_n(\lambda) l_n^{-1} \right), \tag{2.17}$$

solving the Lax determining equation (2.7), satisfies, owing to (2.14), for all $\lambda \in \mathbb{C}$,

$$z(\lambda)\vartheta\,\bar{\varphi}(\lambda) = \eta\bar{\varphi}(\lambda), \tag{2.18}$$

where $z\colon \mathbb{C} \to \mathbb{C}$ is a meromorphic function, and ϑ and $\eta\colon T^*(M_m^{(N)}) \to T(M_m^{(N)})$ are compatible Poissonian operators on the manifold $M_m^{(N)}$ that are Nötherian with respect to the dynamical system (2.1). Then it follows at once that the generating functional $\overline{\gamma}(\lambda) \in \mathcal{D}(M_m^{(N)})$ satisfies the commutation relationships

$$\{\bar{\gamma}(\lambda), \bar{\gamma}(\mu)\}_\vartheta = 0 = \{\bar{\gamma}(\lambda), \bar{\gamma}(\mu)\}_\eta \tag{2.19}$$

for all $\lambda, \mu \in \mathbb{C}$. Consequently, if we define on $M_{(N)}$ a generating dynamical system

$$\frac{dw}{d\tau} := -\vartheta \operatorname{grad} \bar{\gamma}(\lambda)[w]$$

as $\lambda \to \infty$, it follows from (2.19) that the hierarchy of functionals defined by the coefficients in (2.16) comprise its conservation laws.

With the importance of invariants and Poissonian structures related to the linear spectral problem (2.11) firmly in mind, we now describe its main Lie-algebraic properties and connections with the whole hierarchy of integrable differential-difference dynamical systems on the manifold M_m. More precisely, we sketch the Lie-algebraic aspects [22, 49, 57, 58] of the differential-difference dynamical systems associated with the Lax linear difference spectral problem (2.11). In this process we shall assume that $l_n := l_n[w;\lambda] \in G_n := GL^2(\mathbb{C}) \otimes \mathbb{C}(\lambda, \lambda^{-1})$ for $n \in \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ as $\lambda \to \infty$. To describe the related Lax integrable dynamical systems, we first define first the matrix product-group $G^N := \otimes_{j=1}^N G_j$ and its action $G^N \times M_G^{(N)} \to M_G^{(N)}$ on the phase space $M_G^{(N)} := \{l_n \in G_n \colon n \in \mathbb{Z}_N\}$, given as

$$\{g_n \in G_n \colon n \in \mathbb{Z}_N\} \times \{l_n \in G_n \colon n \in \mathbb{Z}_N\} = \{g_n l_n g_{n+1}^{-1} \in G_n \colon n \in \mathbb{Z}_N\}.$$

A functional $\gamma \in \mathcal{D}(M_G^{(N)})$ is invariant for this action iff the following discrete relationship:

$$\operatorname{grad} \gamma(l_n) l_n = l_{n+1} \operatorname{grad} \gamma(l_{n+1}) \tag{2.20}$$

holds for all $n \in \mathbb{Z}_N$.

We assume further that the matrix group G^N is identified with its tangent spaces $T_l(G^N)$, $l \in G^N$, which is locally isomorphic to the Lie algebra $\mathcal{G}^{(N)}$, where $\mathcal{G}^{(N)}$ is the corresponding Lie algebra of the Lie group G^N, which is isomorphic to the tangent space $T_e(G^N)$ at the group unity $e \in G^N$. With any element $l \in G^N$ there are associated, respectively, the left $\eta_l\colon \mathcal{G}^{(N)} \to T_l(G^N)$ and right $\rho_l\colon \mathcal{G}^{(N)} \to T_l(G^N)$ differentials of the left and right translations on the Lie group G^N, and their adjoint mappings $\rho_l^*\colon T_l^*(G^N) \to \mathcal{G}^{(N),*}$ and $\eta_l^*\colon T_l^*(G^N) \to \mathcal{G}^{(N),*}$, where

$$\begin{aligned} (\rho_l^* \operatorname{grad} \gamma(l), X) &= (\operatorname{grad}\gamma(l), Xl) = (l \operatorname{grad} \gamma(l), X) := \operatorname{Tr}(l \operatorname{grad} \gamma(l) X), \\ (\eta_l^* \operatorname{grad} \gamma(l), X) &= (\operatorname{grad} \gamma(l), lX) = (\operatorname{grad} \gamma(l) l, X) := \operatorname{Tr}(\operatorname{grad} \gamma(l) l X) \end{aligned} \tag{2.21}$$

for any $X \in \mathcal{G}^{(N)}$ and smooth functional $\gamma \in \mathcal{D}(G^N)$. Here $\mathrm{Tr}\colon G^N \to \mathbb{C}$ is the trace operation on the group G^N defined as

$$\mathrm{Tr}\, A := \operatorname*{res}_{\lambda=\infty} \sum_{j \in \mathbb{Z}_N} \mathrm{Sp}\, A_j[u, \bar{u}; \lambda]$$

for any $A \in G^N$. By virtue of (2.20) and (2.21), we can define the set

$$\{\Phi_n = \mathrm{grad}\, \gamma(l_n) l_n \in \mathcal{G}_n^* := T_e^*(G), \;\; n \in \mathbb{Z}_N\}$$

belonging to the space $\mathcal{G}^{(N),*} \simeq T_e^*(G^N)$ and satisfying the following invariance property:

$$\Phi_{n+1} = Ad_{l_n}^* \Phi_n(\lambda) = l_n^{-1} \Phi_n(\lambda) l_n \tag{2.22}$$

for any $n \in \mathbb{Z}_N$. The relationship (2.22) allows to define a function $\varphi\colon G^N \to \mathbb{C}$ invariant with respect to the adjoint action

$$G_n \times G_n \ni (g, S_n(\lambda)) \to ad_g S_n(\lambda) = g S_n(\lambda) g^{-1} \in G_n$$

for any $n \in \mathbb{Z}_N$ and such that

$$\gamma(l) = \varphi[S_N(\lambda)], \quad \Phi_N = \mathrm{grad}\, \varphi[S_N(\lambda)] S_N(\lambda), \tag{2.23}$$

where, by definition, the expression

$$S_N(\lambda) = \prod_{j=1}^{N} l_j[u, \bar{u}; \lambda] \tag{2.24}$$

coincides exactly with the proper monodromy matrix for the linear spectral problem (2.11). Owing to (2.22), the matrices $\Phi_n = \mathrm{grad}\, \varphi[S_n(\lambda)] S_n(\lambda) \in \mathcal{G}_n^*$, $n \in \mathbb{Z}_N$, can be reconstructed from (2.24). Therefore, we have [22, 58] the following Poissonian flow on the matrices $S_n(\lambda) \in \in G_n$, $n \in \mathbb{Z}_N$:

$$\frac{dS_n(\lambda)}{dt} = [\mathcal{R}(\mathrm{grad}\, \varphi[S_n(\lambda)] S_n(\lambda)), S_n(\lambda)] \tag{2.25}$$

with respect to the invariant Casimir function $\varphi \in I(\mathcal{G}_n^*)$ and the quadratic Poissonian structure

$$\{\gamma_1, \gamma_2\} := (l, [\mathrm{grad}\, \gamma_1(l), \mathcal{R}(l\, \mathrm{grad}\, \gamma_2(l))] + [\mathcal{R}(l\, \mathrm{grad}\, \gamma_1(l)), \mathrm{grad}\, \gamma_2(l)]) \tag{2.26}$$

for any functionals $\gamma_1, \gamma_2 \in \mathcal{D}(G^N)$, which is constructed by means of a skew-symmetric $\mathcal{R}$-structure $\mathcal{R}\colon \mathcal{G}^{(N)*} \to \mathcal{G}^{(N)}$. In particular, the equality

$$[\mathrm{grad}\, \varphi(S_n), S_n] = 0$$

holds for all $n \in \mathbb{Z}_N$.

Taking into account (2.23), one can rewrite (2.25) as

$$\frac{dS_n}{dt} = [\mathcal{R}(\operatorname{grad}\gamma(l_n)l_n), S_n],$$

for all $n \in \mathbb{Z}_N$. This together with (2.22) makes it possible to retrieve [34, 57] the related evolution of elements $l_n \in G_n$, $n \in \mathbb{Z}_N$:

$$\begin{gathered}\frac{dl_n}{dt} = p_{n+1}(l)l_n - l_n p_n(l),\\ p_n(l) := \mathcal{R}(\operatorname{grad}\gamma(l_n)l_n)\end{gathered} \tag{2.27}$$

from the relationships

$$\begin{gathered}S_n(\lambda) = \psi_n(l)S_N(\lambda)\psi_n^{-1}(l),\\ \psi_n(l) = \prod_{j=1}^{n} l_j[u, v; \lambda].\end{gathered}$$

The solution $f \in l^\infty(\mathbb{Z}, \mathbb{C}^2)$ to the linear spectral problem (2.11) satisfies the associated temporal evolution equation

$$\frac{df_n}{dt} = p_n(l)f_n \tag{2.28}$$

for any $n \in \mathbb{Z}$. It is easy to check that the compatibility condition for the linear equations (2.11) and (2.28) is equivalent to the discrete Lax representation (2.27), which upon reduction on the group manifold M_G, gives rise to the corresponding nonlinear Lax integrable dynamical system on the discrete manifold $M_m^{(N)}$. Hence, all Casimir invariant functions, when reduced on the manifold M_G, are in involution [23, 57, 58] with respect to the Poisson bracket (2.26).

Since the existence of an infinite hierarchy of mutually commuting conservation laws is a characteristic of the Lax integrability of the nonlinear dynamical system (2.1), this property can be effectively implemented into the scheme of our analysis. Namely, we have the following result.

Proposition 2.1. *The determiming Lax equation (2.7) allows the following asymptotic* (*as* $\lambda \to \infty$) *periodic solution* $\varphi(\lambda) \in T^*(M_m^{(N)})$:

$$\varphi_n(\lambda) \sim a_n(\lambda)\exp[\omega(t;\lambda)]\prod_{j=0}^{n}\sigma_j(\lambda), \tag{2.29}$$

where for all $n \in \mathbb{Z}$

$$\begin{gathered}a_n(\lambda) := (1, a_{(1),n}[w;\lambda], a_{(2),n}[w;\lambda], \ldots, a_{(m-1),n}[w;\lambda])^\tau,\\ a_{(k),n}(\lambda) \sim \sum_{s\in\mathbb{Z}_+} a_{(k),n}^{(s)}[w]\lambda^{-s+\tilde{a}}, \qquad \sigma_j(\lambda) \sim \sum_{s\in\mathbb{Z}_+} a_j^{(s)}[w]\lambda^{-s+\tilde{\sigma}},\end{gathered} \tag{2.30}$$

$1 \leq k \leq m-1$ *and* $\omega(t;\cdot)\colon \mathbb{C} \to \mathbb{C}$, $t \in \mathbb{R}$, *is a dispersion function. Moreover, the functional* $\gamma(\lambda) := \sum_{n\in\mathbb{Z}_N} \ln(\lambda^{-\tilde{\sigma}}\sigma_n[w;\lambda]) \in \mathcal{D}(M_m^{(N)})$ *is a generating function of conservation laws for the dynamical system (2.1).*

Proof. Lemma 2.1 and relationship (2.17) imply that the functional (2.15) is a conservation law for our dynamical system (2.1). Whence, expression (2.13) and equation (2.11) lead to the solution representation (2.29) for the Lax equation (2.7). Now, making use of the periodicity of the manifold $M_m^{(N)}$, it follows from the period translation of (2.29) that the functional

$$\gamma(\lambda) := \sum_{n\in\mathbb{Z}_N} \ln(\lambda^{-\tilde{\sigma}}\sigma_n[w;\lambda]) \sim \sum\nolimits_{j\in\mathbb{Z}_+} \gamma_j \lambda^{-j} \tag{2.31}$$

generates an infinite hierarchy of conservation laws to (2.1).

Proposition 2.1 is proved.

Thus, if we start the Lax integrability analysis of a given nonlinear dynamical system (2.21), it is necessary, as the first step, to study the asymptotic solutions (2.29) to the corresponding Lax equation (2.7). These solutions are then used to construct a related hierarchy of conservation laws in the functional form (2.31), taking into account expansions (2.30).

Remark 2.1. It is easy to observe that, owing to the arbitrariness of the period $N \in \mathbb{Z}_+$ of the manifold $M_m^{(N)}$, all of the finite-sum expressions obtained above can be generalized to the corresponding infinite-dimensional manifold $M_m \subset l^2(\mathbb{Z};\mathbb{C}^m)$, if the associated infinite series are convergent.

Since our dynamical system (2.1) induces a bi-Hamiltonian flow on the manifold $M_{(N)}$ under the above circumstances, the next step is to analyze the related compatible Poissonian or symplectic structures, satisfying, respectively, either equality (2.4) or equality (2.5). Before doing this, we shall need the following useful result.

Lemma 2.3. *All functionals* $\gamma_j \in \mathcal{D}(M_m^{(N)})$ *in the expansion (2.31) are mutually with respect to both Poissonian structures* $\vartheta,\eta\colon T^*(M_m^{(N)}) \to T(M_m^{(N)})$ *satisfying the gradient relationship (2.32).*

Proof. It follows from the representations (2.29) and (2.17) that the following asymptotic (as $\lambda \to \infty$) relationship holds:

$$\ln\bar{\gamma}(\lambda) \simeq \gamma(\lambda). \tag{2.32}$$

Since the generating function $\bar{\gamma}(\lambda) \in \mathcal{D}(M_m^{(N)})$ satisfies the commutation relationships (2.19), the same also holds, owing to (2.32), for the generating function $\gamma(\lambda) \in \mathcal{D}(M_m^{(N)})$.

Lemma 2.3 is proved.

We proceed now with an analytical approach to construction of the Poissonian structures $\vartheta,\eta\colon T^*(M_m^{(N)}) \to T(M_m^{(N)})$ for the dynamical system (2.1). Note that these Poissonian structures are also Nötherian for the whole hierarchy of dynamical systems

$$\frac{dw}{dt_j} := -\vartheta\,\mathrm{grad}\,\gamma_j[w], \tag{2.33}$$

where $t_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$, are the corresponding evolution parameters, and which, owing to (2.19), commute with each other on the manifold $M_m^{(N)}$. Therefore, it possible to apply Lemma 2.2 to any one of the dynamical systems (2.33) if the related vector fields commuting with (2.1) are assumed known.

To solve equation (2.9) for an element $\varphi \in T^*(M_m^{(N)})$ one can, in the case of a polynomial dynamical system (2.1), make use of the well-known asymptotic small parameter method [43, 54]. When applying this approach, it is necessary to take into account the following expansions at zero — element $(u, \bar{u})^\intercal = 0 \in M_m^{(N)}$ with respect to the small parameter $\mu \to 0$:

$$w := \mu w^{(1)}, \quad \varphi\left[w^{(1)}\right] = \varphi^{(0)} + \mu\varphi^{(1)}[w^{(1)}] + \mu^2\varphi^{(2)}[w^{(1)}] + \ldots,$$

$$\frac{d}{dt} = \frac{d}{dt_0} + \mu\,\frac{d}{dt_1} + \mu^2\,\frac{d}{dt_2} + \ldots,$$

$$K\left[w^{(1)}\right] = \mu K^{(1)}\left[w^{(1)}\right] + \mu^{(2)}K^{(2)}\left[w^{(1)}\right] + \ldots,$$

$$K'\left[w^{(1)}\right] = K'_0 + \mu K'_1\left[w^{(1)}\right] + \mu^2 K'_2\left[w^{(1)}\right] + \ldots,$$

$$\operatorname{grad}\mathcal{L}\left[w^{(1)}\right] = \operatorname{grad}\mathcal{L}^{(0)} + \mu\operatorname{grad}\mathcal{L}^{(1)}\left[w^{(1)}\right] + \mu^2\operatorname{grad}\mathcal{L}^{(2)}[w^{(1)}] + \ldots.$$

After solving the corresponding set of linear nonuniform functional equations

$$\frac{d\varphi^{(0)}}{dt_0} + K_0'^{*}\varphi^{(0)} = \operatorname{grad}\mathcal{L}^{(0)},$$

$$\frac{d\varphi^{(1)}}{dt_0} + K_0'^{*}\varphi^{(1)} = \operatorname{grad}\mathcal{L}^{(1)} - K_0'^{*}\varphi^{(0)},$$

$$\frac{d\varphi^{(2)}}{dt_0} + K_0'^{*}\varphi^{(2)} = \operatorname{grad}\mathcal{L}^{(2)} - K_1'^{*}\varphi^{(1)} - K_2'^{*}\varphi^{(0)}$$

and so on, using Fourier transforms applied to the suitable N-periodic functions, one can obtain the related Poissonian structure in the series form

$$\vartheta^{-1} = \varphi^{(0),\prime} - \varphi^{(0),\prime *} + \mu(\varphi^{(1),\prime} - \varphi^{(1),\prime *}) + \ldots$$

and finally set $\mu = 1$.

Another direct way of obtaining a Poissonian operator $\vartheta\colon T^*(M_m^{(N)}) \to T(M_m^{(N)})$ for (2.1) is the following: first reduce the Nötherian equation (2.4) to the set of linear nonuniform equations

$$\frac{d}{dt_0}(\vartheta_0\varphi^{(0)}) = K'_0(\vartheta_0\varphi^{(0)}),$$

$$\frac{d}{dt_0}(\vartheta_1\varphi^{(0)}) = K'_0(\vartheta_1\varphi^{(0)}) + \vartheta_0 K_1^{\prime,*}\varphi^{(0)} + K'_1\vartheta_0\varphi^{(0)},$$

$$\frac{d}{dt_0}(\vartheta_2\varphi^{(0)}) = K'_0(\vartheta_2\varphi^{(0)}) - \varphi^{(0)\prime}K^1 + \vartheta_0 K_2^{\prime,*}\varphi^{(0)} +$$

$$+ \vartheta_1 K_1^{\prime,*}\varphi^{(0)} + \vartheta_2 K_0^{\prime,*}\varphi^{(0)} + K'_1\vartheta_1\varphi^{(0)} + K'_2\vartheta_0\varphi^{(0)},$$

and then solve using the above small parameter asymptotics. The analytical expressions for actions $\vartheta_j\colon \varphi^{(0)} \to \vartheta_j\varphi^{(0)}$, $j \in \mathbb{Z}_+$ can now be used to retrieve them in operator form from the expansion

$$\vartheta = \vartheta_0 + \mu\vartheta_1 + \mu^2\vartheta_2 + \ldots,$$

by setting $\mu = 1$ at the end of the calculations. Similarly one can also construct the second Poissonian operator $\eta\colon T^*(M_m^{(N)}) \to T(M_m^{(N)})$ for the nonlinear dynamical system (2.1).

Now the next result follows directly from all of the above analysis.

Proposition 2.2. *Let a nonlinear dynamical system (2.1) on a discrete manifold $M_m^{(N)}$ admit both a nontrivial symmetric solution $\varphi \in T^*(M_m^{(N)})$ to the Lax equation (2.7) in the asymptotic as form (2.29) as $\lambda \to \infty$, generating an infinite hierarchy of nontrivial functionally independent conservation laws (2.31), and compatible nonsymmetric solutions ψ and $\phi \in T^*(M_m^{(N)})$ to the Nöther equations (2.9) and (2.10), respectively. Then this dynamical system is a Lax integrable bi-Hamiltonian flow on $M_m^{(N)}$ with respect to two compatible Poissonian structures*

$$\vartheta, \eta\colon T^*\left(M_m^{(N)}\right) \to T\left(M_m^{(N)}\right),$$

whose adjoint Lax representation

$$\frac{d\Lambda}{dt} = \left[\Lambda, K^{\prime,*}\right], \tag{2.34}$$

where $\Lambda := \vartheta^{-1}\eta$, is the so-called recursion operator. This operator can be transformed, in virtue of the gradient relationship (2.18), to the standard discrete Lax form

$$\frac{dl_n}{dt} = [p_n(l), l_n] + (D_n p_n(l))l_n$$

for some matrix $p_n(l) \in \operatorname{End}\mathbb{C}^r$ describing the temporal evolution

$$\frac{df_n}{dt} = p_n(l)f_n$$

related to (2.11), for $f \in l^\infty(\mathbb{Z};\mathbb{C}^r)$.

Remark 2.2. Inasmuch as all Hamiltonian flows (2.32) commute with each other and the dynamical system (2.1), and since they possess the same Poissonian and compatible (ϑ, η)-pair, the analytical algorithm described above can also be applied to any other flow commuting with (2.1).

Solutions to the discrete linear Lax problem (2.11) can be constructed by means of the gradient-holonomic algorithm devised in [11, 32, 54] for studying the integrability of nonlinear dynamical systems on functional manifolds. More specifically, by making use of the preliminary analytical expressions for the related compatible Poissonian structures $\vartheta, \eta\colon T^*\left(M_m^{(N)}\right) \to T\left(M_m^{(N)}\right)$ on the manifold $M_m^{(N)}$ and using the fact that the recursion operator $\Lambda := \vartheta^{-1}\eta\colon T^*\left(M_m^{(N)}\right) \to T^*\left(M_m^{(N)}\right)$ satisfies the dual Lax commutator equality (2.34), one can retrieve the standard Lax representation for it in terms of algebraic formulas. As a corollary of

Proposition 2.2 one has the existence of a nontrivial asymptotic (as $\lambda \to \infty$) solution to the Lax equation (2.7), which provides an effective Lax integrability criterion for a dynamical system (2.1) on the manifold $M_m^{(N)}$.

2.3. The Bogoyavlensky – Novikov finite-dimensional reduction scheme. In this section, we assume that our dynamical system (2.1) on the periodic manifold $M_m^{(N)}$ is Lax integrable and possesses two compatible Poissonian structures $\vartheta, \eta \colon T^*(M_m^{(N)}) \to T(M_m^{(N)})$. Thus, we have the nonlinear finite-dimensional dynamical system

$$\frac{dw}{dt} := K_n[w] = -\vartheta \operatorname{grad} H_n[w] \tag{2.35}$$

for indices $n \in \mathbb{Z}_N$, owing to its N-periodicity. The finite-dimensional dynamical system (2.35) can be equivalently considered as that on the finite-dimensional space $M_m^{(N)} \simeq (\mathbb{C}^m)^N$ parameterized by an integer index $n \in \mathbb{Z}_N$. The Liouville integrability of this system is our next concern. To study the flow (2.34) on the manifold $M_{(N)}$, we shall make use of the Bogoyavlensky – Novikov [13, 50] reduction scheme [9, 50, 51, 54].

Let $\Lambda\left(M_m^{(N)}\right) := \otimes_{j=0}^N \Lambda^j\left(M_m^{(N)}\right)$ be the standard finitely generated Grassmann algebra [6, 11 54] of differential forms on the manifold $M_{(N)}$. Then the differential complex

$$\Lambda^0(M_m^{(N)}) \xrightarrow{d} \Lambda^1(M_m^{(N)}) \to^{d} \cdots \xrightarrow{d} \Lambda^j(M_m^{(N)}) \xrightarrow{d} \Lambda^{j+1}(M_m^{(N)}) \xrightarrow{d} \cdots,$$

where $d \colon \Lambda(M_m^{(N)}) \to \Lambda(M_m^{(N)})$ is the exterior differentiation, is finite and exact. Since the discrete "derivative" $D_n := \Delta - 1$ commutes with the differentiation $d \colon \Lambda\left(M_m^{(N)}\right) \to \Lambda\left(M_m^{(N)}\right)$, $[D_n, d] = 0$ for all $n \in \mathbb{Z}_N$, and for any element $a \in \Lambda^0\left(M_m^{(N)}\right)$

$$\operatorname{grad}\left(\sum_{n \in \mathbb{Z}_N} D_n a_n[w]\right) = 0, \tag{2.36}$$

one can formulate the following Gelfand – Dikiy type [26] result.

Lemma 2.4. *Let $\mathcal{L}[w] \in \Lambda^0\left(M_m^{(N)}\right)$ be a Fréchet smooth local Lagrangian functional on the manifold $M_m^{(N)}$. Then there exists a differential 1-form $\alpha^{(1)} \in \Lambda^1(M_m^{(N)})$, such that the equality*

$$d\mathcal{L}_n[w] = \langle \operatorname{grad} \mathcal{L}_n[w], d(w)^{\intercal} \rangle + D_n \alpha_n^{(1)}[w] \tag{2.37}$$

holds for all $n \in \mathbb{Z}_N$.

Proof. One can easily see that

$$d\mathcal{L}_n[w] = \sum_{j=0}^{N-1} \left\langle \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}},\ dw_{n+j} \right\rangle = \sum_{j=0}^{N-1} \left\langle \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}},\ \Delta^j dw_n \right\rangle =$$

$$= \left\langle \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}},\ dw_n \right\rangle + D_n \left(\sum_{j=0}^{N-1} \langle p_j,\ dw_{n+j} \rangle \right),$$

where

$$p_k := \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j+k+1}}$$

for $k = 0, \ldots, N-1$. Having defined the expression

$$\operatorname{grad} \mathcal{L}_n[w] := \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}},$$

one obtains the result (2.37), where

$$\alpha_n^{(1)}[w] := \sum_{j=0}^{N-1} \langle p_j,\ dw_{n+j} \rangle \tag{2.38}$$

is the corresponding differential 1-form on the manifold $M_m^{(N)}$, thereby concluding the proof.

Lemma 2.4 is proved.

Exterior differentiating expression (2.37), we obtain that

$$-D_n \omega_n^{(2)}[w] = \langle d \operatorname{grad} \mathcal{L}_n[w],\ \wedge dw \rangle \tag{2.39}$$

for any $n \in \mathbb{Z}$, where the 2-form

$$\omega^{(2)}[w] := d\alpha^{(1)}[w] \tag{2.40}$$

is nondegenerate on $M_m^{(N)}$ if the Hessian $\partial_n^2 \mathcal{L}[w]/\partial^2 w$ is also nondegenerate.

Consider the manifold

$$\bar{M}_m^{(N)} := \left\{ \operatorname{grad} \mathcal{L}_n^{(\bar{N})}[w] = 0;\ w \in M_m^{(N)} \right\}, \tag{2.41}$$

where the Lagrangian functional is defined as

$$\mathcal{L}^{(\bar{N})} := -\gamma_{\bar{N}} + \sum_{j=0}^{\bar{N}-1} c_j \gamma_j, \tag{2.42}$$

with $\gamma_j \in \mathcal{D}\left(M_m^{(N)}\right)$, $j = 0, \ldots, \bar{N}-1$, for some $\bar{N} \in \mathbb{Z}_+$, being suitable nontrivial conservation laws for the dynamical system (2.1) as constructed above. Here $c_j \in \mathbb{C}, \leq j \leq \bar{N} - 1$, are arbitrary but fixed constants. It follows from (2.41) and (2.39) that the closed 2-form $\omega^{(2)} \in \Lambda^2\left(M_m^{(N)}\right)$ is invariant with respect to the index $n \in \mathbb{Z}_N$ on the manifold $\bar{M}_m^{(N)}$. Moreover, the submanifold (2.41) is also invariant both with respect to the index $n \in \mathbb{Z}_N$ and the evolution parameter $t \in \mathbb{R}$. In fact, for any $n \in \mathbb{Z}_N$ the Lie derivative

$$L_K \operatorname{grad} \mathcal{L}^{(\bar{N})} = \left(\operatorname{grad} \mathcal{L}^{(\bar{N})}\right)' K + K'^{,*} \left(\operatorname{grad} \mathcal{L}^{(\bar{N})}\right) = 0,$$

since the functional $\mathcal{L}_n^{(\bar{N})}[w] \in \mathcal{D}\left(\bar{M}_m^{(N)}\right)$ is a sum of conservation laws for the dynamical system (2.1), whose gradients satisfies the Lax condition (2.7). In addition, it is easy to see that if the Lie derivative $L_K \mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w] = 0, n \in \mathbb{Z}_N$, at $t = 0$, then $\mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w] = 0$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}_N$. Thus, the Bogoyavlensky – Novikov reduction of the dynamical system (2.1) upon the invariant submanifold $\bar{M}_m^{(N)}$ is completely invariantly defined.

At this point there is a natural question to ask: what is the relationship between the dynamical system (2.1) restricted to the submanifold $M_m^{(N)}$ and the dynamical system (2.1) reduced on the finite-dimensional submanifold $\bar{M}_m^{(N)} \subset M_m^{(N)}$? To further analyze the reduction, we consider the equation

$$\left\langle \mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w], K_n[w] \right\rangle = -D_n h_n^{(t)}[w], \tag{2.43}$$

for a local functional $h^{(t)}[w] \in \Lambda^0(M_m)$, which follows from the conditions (2.35) and (2.7):

$$\begin{aligned} \mathrm{grad} \left\langle \mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w], K_n[w] \right\rangle &= \left(\mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w]\right)^{\prime,*} K_n[w] + K_n^{\prime,*}[w] \mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w] = \\ &= \left(\mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w]\right)^{\prime} K_n[w] + K_n^{\prime,*}[w] \mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w] = \\ &= L_K\, \mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w] = 0. \end{aligned}$$

Since on the submanifold $\bar{M}_m^{(N)}$ the gradient $\mathrm{grad}\, \mathcal{L}_n^{(\tilde{N})}[w] = 0$ for all $n \in \mathbb{Z}_N$, we deduce from (2.43) that the local functional $h^{(t)}[w] \in \Lambda^0\left(\bar{M}_m^{(N)}\right)$ does not depend on index $n \in \mathbb{Z}_N$.

The properties of the manifold $\bar{M}_m^{(N)}$ described above, make it possible to consider it as a symplectic manifold endowed with the symplectic structure $\omega^{(2)} \in \Lambda^2\left(\bar{M}_m^{(N)}\right)$ given by expressions (2.38) and (2.40). From this point of view we can study the integrability properties of the dynamical system (2.1) reduced on the invariant finite-dimensional manifold $\bar{M}_m^{(N)} \subset \subset M_m^{(N)}$.

First, we observe that the vector field d/dt on $\bar{M}_{(N)}$ is canonically Hamiltonian [3, 6, 50] with respect to the symplectic structure $\omega^{(2)} \in \Lambda^2(\tilde{M}_{(N)})$, i.e.,

$$-i_{\frac{d}{dt}} \omega^{(2)}(u,p) = dh^{(t)}(u,p), \tag{2.44}$$

where $h^{(t)}(w,p) := h^{(t)}[w], \omega^{(2)}(w,p) := \omega^{(2)}[w]$ and $(w,p)^{\intercal} \in \bar{M}_m^{(N)}$ are canonical variables induced on the manifold $\bar{M}_m^{(N)}$ by the Liouville 1-form (2.38). More specifically, from expression (2.43) one obtains that

$$d i_{\frac{d}{dt}} \left\langle \mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w], dw_n \right\rangle = -D_n d h_n^{(t)}[w],$$

which together with the identity (2.39) in the form

$$i_{\frac{d}{dt}} d \left\langle \mathrm{grad}\, \mathcal{L}_n^{(\bar{N})}[w], dw_n \right\rangle = -D_n i_{\frac{d}{dt}} \omega_n^{(2)}[w],$$

leads to

$$\frac{d}{dt}\left\langle \operatorname{grad}\mathcal{L}_n^{(\bar N)}[w], dw_n\right\rangle = -D_n\left(dh_n^{(t)}[w] + i_{\frac{d}{dt}}\omega_n^{(2)}[w]\right). \tag{2.45}$$

Since $\operatorname{grad}\mathcal{L}^{(\bar N)}[w] = 0 = L_K \operatorname{grad}\mathcal{L}[w]$ identically on $\bar M_m^{(N)}$, from (2.45) one obtains the result (2.44).

The same is true of any of the Hamiltonian systems (2.33) commuting with (2.1) on the manifold M_m. Moreover, owing to the functional independence of invariants $\gamma_j \in \mathcal{D}\left(M_m^{(N)}\right)$, $0 \le j \le N-1$, in the Lagrangian functional (2.42), we can construct a set of functionally independent functions $h^{(j)} \in \mathcal{D}\left(\bar M_m^{(N)}\right)$, $j = 0,\dots,\bar N - 1$, as follows:

$$\left\langle \operatorname{grad}\mathcal{L}_n^{(\tilde N)}[w], \vartheta \operatorname{grad}\gamma_{j,n}[w]\right\rangle = D_n h_n^{(j)}[w].$$

It is easy to check that these functions $h^{(j)} \in \mathcal{D}\left(\bar M_m^{(N)}\right)$, $0 \le j \le \bar N - 1$, are invariant with respect to indices $n \in \mathbb{Z}_N$ and commute with each other and the Hamiltonian function $h^{(t)} \in \mathcal{D}\left(\bar M_m^{(N)}\right)$ with respect to the symplectic structure $\omega^{(2)} \in \Lambda^2\left(\bar M_m^{(N)}\right)$. Thus, if the dimension $\dim \tilde M_{(N)} = 2\tilde N$, the discrete dynamical system (2.1) reduced upon the finite-dimensional submanifold $\bar M_m^{(N)} \subset M_m^{(N)}$ is Liouville integrable. If the set of conservation laws $\gamma_j \in \mathcal{D}\left(M_m^{(N)}\right)$, $j = 0,\dots,N-1$, is functionally dependent on $M_m^{(N)}$, the scheme can be modified using the Dirac reduction technique [3, 9, 54] for determining a regular symplectic structure $\bar\omega^{(2)}[w] \in \Lambda^2\left(\bar M_m^{(N)}\right)$ on an invariant nonsingular submanifold $\bar M_m^{(N)}$.

3. The discrete nonlinear Schrödinger dynamical system analysis. ***3.1. Hamiltonian description.*** In that to follow we proceed to analyzing the properties of discrete approximation for the nonlinear integrable Schrödinger dynamical system (1.2) on a functional manifold $\tilde M \subset \subset L_2\left(\mathbb{R};\mathbb{C}^2\right)$ in the form

$$\tilde K[\psi,\psi^*] := \begin{cases} \dfrac{d}{dt}\psi = i\psi_{xx} - 2i\alpha\psi\psi\psi^*, \\[2ex] \dfrac{d}{dt}\psi^* = -i\psi^*_{xx} + 2i\alpha\psi^*\psi\psi^*, \end{cases} \tag{3.1}$$

where, by definition $(\psi,\psi^*)^\intercal \in \tilde M$, $\alpha \in \mathbb{R}$ is a constant, the subscript "x" means the partial derivative with respect to the independent variable $x \in \mathbb{R}$, $\tilde K\colon \tilde M \to T(\tilde M)$ is the corresponding vector field on $\tilde M$ and $t \in \mathbb{R}$ is the evolution parameter. The system (3.1) possesses a Lax type representation (see [50]) and is Hamiltonian

$$\frac{d}{dt}(\psi,\psi^*)^\intercal = -\tilde\theta \operatorname{grad}\tilde H[\psi,\psi^*] = \tilde K[\psi,\psi^*] \tag{3.2}$$

with respect to the canonical Poisson structure $\tilde\theta$ and the Hamiltonian function $\tilde H$, where

$$\tilde\theta := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.3}$$

is a nondegenerate mapping $\tilde{\theta}\colon T^*(\tilde{M}) \to T(\tilde{M})$ on the smooth functional manifold $\tilde{M}$, and

$$\tilde{H} := \frac{1}{2}\int\limits_{\mathbb{R}} dx \left[\psi\psi^*_{xx} + \psi_{xx}\psi^* - 2\alpha\,(\psi^*\psi)^2\right], \tag{3.4}$$

is a smooth mapping $\tilde{H}\colon \tilde{M} \to \mathbb{C}$. The corresponding symplectic structure [3, 6, 9, 11] for the Poissonian operator (3.3) is defined by

$$\tilde{\omega}^{(2)} := -\frac{i}{2}\int\limits_{\mathbb{R}} dx \left\langle (d\psi, d\psi^*)^\intercal, \wedge\tilde{\theta}^{-1}(d\psi, d\psi^*)^\intercal \right\rangle = -i\int\limits_{\mathbb{R}} dx\,[d\psi^*(x)\wedge d\psi(x)], \tag{3.5}$$

which is a nondegenerate and closed 2-form on the functional manifold $\tilde{M}$.

The simplest spatial discretizations of the dynamical system (3.1) look as the flows

$$\begin{aligned} \frac{d}{dt}\psi_n &= \frac{i}{h^2}\,(\psi_{n+1} - 2\psi_n + \psi_{n-1}) - 2i\alpha\psi_n\psi_n\psi_n^*, \\ \frac{d}{dt}\psi_n^* &= -\frac{i}{h^2}\left(\psi^*_{n+1} - 2\psi^*_n + \psi^*_{n-1}\right) + 2i\alpha\psi_n^*\psi_n\psi_n^* \end{aligned} \tag{3.6}$$

and

$$K[\psi_n, \psi_n^*] := \begin{cases} \dfrac{d}{dt}\psi_n = \dfrac{i}{h^2}\,(\psi_{n+1} - 2\psi_n + \psi_{n-1}) - i\alpha\,(\psi_{n+1} + \psi_{n-1})\,\psi_n\psi_n^*, \\[2ex] \dfrac{d}{dt}\psi_n^* = -\dfrac{i}{h^2}\left(\psi^*_{n+1} - 2\psi^*_n + \psi^*_{n-1}\right) + i\alpha\left(\psi^*_{n+1} + \psi^*_{n-1}\right)\psi_n\psi_n^*, \end{cases} \tag{3.7}$$

on some "discrete" submanifold M_h, where, by definition, $\{(\psi_n, \psi_n^*)^\intercal \in \mathbb{C}^2\colon n \in \mathbb{Z}\} \subset M_h \subset \subset l_2(\mathbb{Z};\mathbb{C}^2)$ and $K\colon M_h \to T(M_h)$ is the corresponding vector field on M_h.

Definition 3.1. *If for a function* $(\psi, \psi^*)^\intercal \in W_2^2(\mathbb{R};\mathbb{C}^2)$ *there exists the point-wise limit* $\lim_{h\to 0}(\psi_n, \psi_n^*)^\intercal = (\psi(x), \psi^*(x)))^\intercal$, *where the set of vectors* $(\psi_n, \psi_n^*)^\intercal \in \mathbb{C}^2$, $n \in \mathbb{Z}$, *solves the infinite system of equations (3.7), the set* $\{(\psi_n, \psi_n^*)^\intercal \in \mathbb{C}^2\colon n \in \mathbb{Z}\} \subset l_2(\mathbb{Z};\mathbb{C}^2)$ *will be called an approximate solution to the nonlinear Schrödinger dynamical system (3.1).*

It is well known [1, 2] that the discretization scheme (3.7) conserves the Lax type integrability [9, 11, 50] and that the scheme (3.6) does not. The integrability of (3.7) can be easily enough checked by means of either the gradient-holonomic integrability algorithm [11, 51, 53] or the well known [41] symmetry approach. In particular, the discrete dynamical system (3.7) is a Hamiltonian one [3, 6, 9, 51] on the symplectic manifold $M_h \subset l_2(\mathbb{Z};\mathbb{C}^2)$ with respect to the noncanonical symplectic structure

$$\omega_h^{(2)} = -\sum_{n\in\mathbb{Z}} \frac{ih}{2(1 - h^2\alpha\psi_n^*\psi_n)}\,\langle (d\psi_n, d\psi_n^*)^\intercal, \wedge\,(d\psi_n, d\psi_n^*)^\intercal\rangle \tag{3.8}$$

on M_h and looks as

$$\frac{d}{dt}\,(\psi_n, \psi_n^*)^\intercal = -\theta_n\,\mathrm{grad}\,H\,[\psi_n, \psi_n^*] = K\,[\psi_n, \psi_n^*], \tag{3.9}$$

where the Hamiltonian function

$$H = \sum_{n\in\mathbb{Z}} \frac{1}{h}\left(\psi_n\psi_{n+1}^* + \psi_{n+1}\psi_n^* + \frac{2}{\alpha h^2}\ln\left|1-\alpha h^2\psi_n^*\psi_n\right|\right) \tag{3.10}$$

and the related Poissonian operator $\theta_n\colon T^*(M_h) \to T(M_h)$ equals

$$\theta_n := \begin{pmatrix} 0 & -ih^{-1}\left(1-h^2\alpha\psi_n^*\psi_n\right) \\ ih^{-1}\left(1-h^2\alpha\psi_n^*\psi_n\right) & 0 \end{pmatrix}. \tag{3.11}$$

Remark 3.1. For the symplectic structure (3.8) and, respectively, the Hamiltonian function (3.10) to be suitably defined on the manifold $M_h \subset l_2(\mathbb{Z};\mathbb{C}^2)$ it is necessary to assume additionally that the finite stability condition $\lim_{N,M\to\infty}\left(\prod_{-N}^{M}(1-\alpha h^2\psi_n^*\psi_n)\right) \neq 0$ holds. The latter is simply reduced as $h \to 0$ to the equivalent integral inequality

$$\alpha \leq \int\limits_{\mathbb{R}} (x\psi^*\psi)^2\,dx\left(\int\limits_{\mathbb{R}}\psi^*\psi dx\right)^{-1},$$

which will be assumed for further to be satisfied, respectively, the manifold $\tilde{M} \subset \tilde{W}_2^2(\mathbb{R};\mathbb{C}^2)$, where $\tilde{W}_2^2(\mathbb{R};\mathbb{C}^2) := W_2^2(\mathbb{R};\mathbb{C}^2) \cap L_2^{(1)}(\mathbb{R};\mathbb{C}^2)$ with the space $L_2^{(1)}(\mathbb{R};\mathbb{C}^2) := \big\{(\psi,\psi^*)^\intercal \in \in L_2(\mathbb{R};\mathbb{C}^2)\colon \int_{\mathbb{R}} x^2(\psi^*\psi)^2dx < \infty\big\}$.

The symplectic structure (3.8) is well defined on the manifold M_h and tends as $h \to 0$ to the symplectic structure (3.5) on $\tilde{M}$, and respectively the Hamiltonian function (3.10) tends to (3.4).

In this work we have investigated the structure of the solution set to the discrete nonlinear Schrödinger dynamical system (3.7) by means of a specially devised analytical approach for invariant reducing the infinite system of ordinary differential equations (3.7) to an equivalent finite one of ordinary differential equations with respect to the evolution parameter $t \in \mathbb{R}$. As a result, there was constructed a finite set of recurrent algebraic regular relationships, allowing to expand the obtained before finite set of solutions to any discrete order $n \in \mathbb{Z}$, which makes it possible to present a wide class of the approximate solutions to the nonlinear Schrödinger dynamical system (3.1). It is worthy here to stress that the problem of constructing an effective discretization scheme for the nonlinear Schrödinger dynamical system (3.1) and its generalizations proves to be important both for applications [4, 36, 59] and for deeper understanding the nature of the related algebro-geometric and analytic structures responsible for their limiting stability and convergence properties. From these points of view we would like to mention work [40], where the standard discrete Lie-algebraic approach [9, 10] was recently applied to constructing a slightly different from (3.6) and (3.7) discretization of the nonlinear Schrödinger dynamical system (3.1). As the symplectic reduction method, devised in the present work for studying the solution sets of the discrete nonlinear Schrödinger dynamical system (3.7), is completely independent of a chosen discretization scheme, it would be reasonable and interesting to apply it to that of [40] and compare the corresponding results subject to their computational effectiveness. The discrete nonlinear Schrödinger dynamical system (2.2) is defined on the periodic manifold $M_2 \subset l^\infty(\mathbb{Z};\mathbb{C}^2)$. Its Lax type integrability was proved in [1, 12, 44] making use of the simplest discretization of the standard Zakharov – Shabat spectral problem for the well-known nonlinear Schrödinger equation. We begin this section by applying the gradient-holonomic integrability analysis described above to the discrete dynamical system (2.2). First, we shall show the existence of an infinite hierarchy of functionally independent conservation laws obtained by solving the determining Lax equation (2.7) in the asymptotic form (2.29). The following is a key result for our analysis.

Lemma 3.1. *The functional expression*

$$\varphi_n := \begin{pmatrix} 1 \\ a_n(\lambda) \end{pmatrix} \exp\left[it(2-\lambda-\lambda^{-1})\right] \prod_{j=0}^{n} \sigma_j(\lambda), \tag{3.12}$$

where

$$\begin{gathered} \sigma_j(\lambda) \sim \frac{\lambda}{h_j[u,\bar{u}]} \left(1 - \sum_{s\in\mathbb{Z}_+} \sigma_j^{(s)}[u,\bar{u}]\lambda^{-s-1}\right), \\ a_n(\lambda) \sim \sum_{s\in\mathbb{Z}_+} a_n^{(s)}[u,\bar{u}]\lambda^{-s}, \end{gathered} \tag{3.13}$$

is an asymptotic solution to the determining Lax equation

$$\frac{d\varphi_n}{dt} + K'^{,*}\varphi_n = 0 \tag{3.14}$$

as $\lambda \to \infty$ for all $n \in \mathbb{Z}_N$ with the operator $K'^{,*}\colon T^*(M_2) \to T^*(M_2)$ of the form

$$K_n'^{,*} = \begin{pmatrix} \begin{matrix} i\Delta^{-1}D_n^2 - i\bar{u}_n(u_{n+1}+u_{n-1}) - \\ -i(\Delta+\Delta^{-1})\cdot\bar{u}_n u_n \end{matrix} & i\bar{u}_n(\bar{u}_{n+1}+\bar{u}_{n-1}) \\ -iu_n(u_{n+1}+u_{n-1}) & \begin{matrix} -i\Delta^{-1}D_n^2 + iu_n(\bar{u}_{n+1}+\bar{u}_{n-1}) + \\ +i(\Delta+\Delta^{-1})\cdot\bar{u}_n u_n \end{matrix} \end{pmatrix}. \tag{3.15}$$

Proof. It suffices to find the corresponding coefficients of the asymptotic expansions (3.13). To do this, we consider the following two equations that can be easily obtained from (3.14), (3.15) and (3.12):

$$\begin{aligned} D_n^{-1}\frac{d}{dt}&\left[-\ln h_n + \ln\left(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1}\right)\right] + \\ &+ i\lambda\left[h_{n+1}^{-1}(1-\bar{u}_n u_n)\left(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1}\right)-1\right] + \\ &+ \frac{i}{\lambda}\left[(1-\bar{u}_{n-1}u_{n-1})h_n\left(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1}\right)^{-1}-1\right] - \\ &- i\bar{u}_n(u_{n+1}+u_{n-1}) + i\bar{u}_n(\bar{u}_{n+1}+\bar{u}_{n-1})\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s} \end{aligned} \tag{3.16}$$

and

$$\left(\sum_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}\right) D_n^{-1}\frac{d}{dt}\left[-\ln h_n + \ln\left(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1}\right)\right] + 4i\left(\sum_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}\right)+$$

$$+\left[i\lambda h_{n+1}(\bar u_{n+1}u_{n+1}-1)\left(\sum_{s\in\mathbb{Z}_+} a_{n+1}^{(s)}\lambda^{-s}\right)\left(\sum_{s\in\mathbb{Z}_+} a_{n+1}^{(s)}\lambda^{-s}\right)-\sum_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}\right]+$$

$$+\frac{i}{\lambda}\left[(\bar u_{n-1}u_{n-1}-1)\left(\sum_{s\in\mathbb{Z}_+} a_{n+1}^{(s)}\lambda^{-s}\right) h_n\left(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1}\right)^{-1}-\sum_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}\right]+$$

$$+\frac{d}{dt}\sum_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s} - iu_n(u_{n+1}+u_{n-1}) + iu_n(\bar u_{n+1}+\bar u_{n-1})\sum_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}.$$

Now equating the coefficients of (3.16) at the same degrees of the parameter $\lambda\in\mathbb{C}$, we recursively obtain the functional expression expression for h_n, $\sigma_n^{(s)}$ and $a_n^{(s)}$, $n\in\mathbb{Z}$, $s\in\mathbb{Z}_+$; namely,

$$h_n = (1-u_n^* u_n), \quad a_n^{(0)} = 0, \quad a_n^{(1)} = \beta,$$

$$\sigma_n^{(0)} = u_{n-1}^*(u_n+u_{n-2}) - i\Delta^{-1}D_n^2(\ln h_n)_t,$$

$$\sigma_n^{(1)} = i\frac{d}{dt}\sigma_{n-1}^{(0)} + (h_{n-1}h_{n-2}-1) + a_{n-1}^{(1)}u_{n-1}^*(u_n+u_{n-2}),$$

$$a_n^{(2)} = -3a_{n-1}^{(1)} + i\frac{d}{dt}\sigma_{n-1}^{(1)} - ia_{n-1}^{(1)}D_n^{-1}(\ln h_{n-1})_t + a_n^{(1)}\sigma_n^{(0)} - u_{n-1}\left(u_n^*+u_{n-2}^*\right)a_{n-1}^{(1)},$$

$$\frac{dh_n}{dt} = iD_n(u_{n-1}^*u_n - u_n^*u_{n-1}),\ldots,$$

whence

$$\sigma_n^{(0)} = -\left(u_n^*u_{n-1}+u_{n-1}^*u_{n-2}\right),$$

$$\sigma_n^{(1)} = i\frac{d}{dt}\sigma_{n-1}^{(0)} + \left(1-u_{n-1}^*u_{n-1}\right)\left(1-u_{n-2}^*u_{n-2}\right) + \beta u_{n-1}^*(u_n+u_{n-2}),\ldots,$$

and so on. Thus, the corresponding recursion formulas are solvable for all $s\in\mathbb{Z}_+$, so it follows that the expression (3.12) is a true asymptotic solution to the Lax equation (3.14).

Lemma 3.1 is proved.

Recalling now that the expression

$$\gamma(\lambda) := -\sum_{n=0}^{N-1}\ln h_n + \sum_{n=0}^{N-1}\ln\left(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1}\right)$$

as $\lambda \to \infty$ is a generating function of conservation laws for the dynamical system (2.2), one finds that functionals

$$\bar{\gamma}_0 = \sum_{n=0}^{N-1} \ln(1 - \bar{u}_n u_n), \gamma_0 = -\sum_{n=0}^{N-1} \sigma_n^{(0)},$$

$$\gamma_1 = -\sum_{n=0}^{N-1} \left(\sigma_n^{(1)} + \frac{1}{2}\, \sigma_n^{(0)} \sigma_n^{(0)} \right),$$

$$\gamma_2 = -\sum_{n=0}^{N-1} \left(\sigma_n^{(2)} + \frac{1}{3}\, \sigma_n^{(0)} \sigma_n^{(0)} \sigma_n^{(0)} + \sigma_n^{(0)} \sigma_n^{(1)} \right), \ldots,$$

and so on, make up an infinite hierarchy of exact conserved quantities for the discrete nonlinear Schrödinger dynamical system (2.2).

A few remarks are in order concerning the complete integrability of the discrete nonlinear Schrödinger dynamical system (2.2). First, we can easily show using the standard asymptotic small parameter approach [11, 32, 54] that the Nöther equation (2.4) on the manifold $M_2^{(N)}$ possesses [44, 51] the exact Poissonian operator solution

$$\vartheta_n = \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix}, \tag{3.17}$$

for $n \in \mathbb{Z}_N$, subject to which the dynamical the dynamical system (2.2) is Hamiltonian via

$$\frac{d}{dt}\, (u, u^*)^{\intercal} = -\vartheta \operatorname{grad} H_\vartheta \, [u, u^*]$$

on the periodic manifold $M_2^{(N)}$, where the Hamiltonian function is

$$H_\vartheta := \sum_{n=0}^{N} \ln h_n^2 - \sum_{n=0}^{N} (\bar{u}_n u_{n+1} - \bar{u}_n u_{n+1}) = 2 \ln |\gamma_0| - \frac{1}{2} (\gamma_0 + \bar{\gamma}_0).$$

Similar, but more cumbersome, calculations can be employed to find a second Poissonian operator solution to the Nöther equation (2.4) in the matrix form:

$$\eta = \begin{pmatrix} \left(h_n - u_n D_n^{-1} u_n\right) \Delta & \left(u_n^2 + u_n D_n^{-1} u_n\right) \Delta^{-1} \\ u_n^* D_n^{-1} u_n^* \Delta & -\left(1 + u_n^* D_n^{-1} u_n\right) \Delta^{-1} \end{pmatrix} \times$$

$$\times \begin{pmatrix} u_n D_n^{-1} u_n & \left(h_n - u_n D_n^{-1} u_n^*\right) \\ 1 + u_n^* D_n^{-1} u_n & -(u_n^* + u_n^* D_n^{-1} u_n^*) \end{pmatrix}. \tag{3.18}$$

where the operation

$$D_n^{-1}(\cdot) := \frac{1}{2} \left[\sum_{k=0}^{n-1} (\cdot)_k - \sum_{k=n}^{N-1} (\cdot)_k \right]$$

is quasiskew-symmetric with respect to the usual bilinear form on $T^*\left(M_2^{(N)}\right) \times T\left(M_2^{(N)}\right)$, satisfying the operator identity $\left(D_n^{-1}\right)^* = -\Delta^{-1} D_n^{-1} \Delta$, $n \in \mathbb{Z}$.

The Poissonian operators (3.17) and (3.18) are compatible, so we can obtain the related Lax representation for the dynamical system (2.2) by means of the algebraic gradient-holonomic algorithm. The corresponding result is as follows: the discrete linear spectral problem

$$\Delta f_n = l_n[u, u^*; \lambda] f_n, \tag{3.19}$$

where $f \in l^\infty(\mathbb{Z}; \mathbb{C}^2)$ and for $n \in \mathbb{Z}$

$$l_n[u, u^*; \lambda] = \begin{pmatrix} \lambda & u_n \\ u_n^* & \lambda^{-1} \end{pmatrix},$$

allows the linear Lax isospectral evolution

$$\frac{df_n}{dt} = p_n(l) f_n \tag{3.20}$$

for some matrix $p_n(l) \in \operatorname{End} \mathbb{C}^2$, $n \in \mathbb{Z}$, which is equivalent to the Hamiltonian flow

$$\frac{df_n}{dt} = \{H_\vartheta, f_n\}_\vartheta, \tag{3.21}$$

where $\{\cdot,\cdot\}_\vartheta$ is the Poissonian structure on the manifold $M_2^{(N)}$ corresponding to (3.17). The equivalence of (3.17) and (3.21) can be easily demonstrated by constructing the monodromy matrix $S_n(\lambda)$, $n \in \mathbb{Z}_N$, for all $\lambda \in \mathbb{C}$ corresponding to (3.19) and calculating the Hamiltonian evolution

$$\frac{d}{dt} S_n(\lambda) = \{H_\vartheta, S_n(\lambda)\}_\vartheta = [p_n(l), S_n(\lambda)],$$

giving rise to the same matrix $p_n(l) \in \operatorname{End} \mathbb{C}^2$, $n \in \mathbb{Z}$, as in equation (3.20).

Thus, we have shown that the nonlinear discrete Schrödinger dynamical system (2.2) is a Lax integrable bi-Hamiltonian flow on the manifold $M_2^{(N)}$. Since the solution $\varphi(\lambda) \in T^*\left(M_2^{(N)}\right)$ constructed above satisfies the gradient-like relationship

$$\lambda \vartheta \varphi(\lambda) = \eta \varphi(\lambda)$$

for all for $\lambda \in \mathbb{C}$, we showed that the conservation laws are mutually commuting with respect to both Poisson brackets $\{\cdot,\cdot\}_\vartheta$ and $\{\cdot,\cdot\}_\eta$. From whence follows the classical Liouville integrability [6, 43] of the discrete nonlinear Schrödinger dynamical system (2.2) on the periodic manifold $M_2^{(N)}$. A detailed analysis of the integrability procedure via the Bogoyavlensky – Novikov reduction [13, 50] and an explicit construction of solutions to the dynamical system (2.2) are planned for a later paper.

4. Finite dimensional reductions and their exact integrability. ***4.1. A class of Hamiltonian discretizations of the NLS dynamical system.*** The discretizations (3.6) and (3.7) can be extended to a wide classs of Hamiltonian systems, if to assume that the Poissonian structure is given by the local expression

$$\theta_n = \begin{pmatrix} 0 & -i\nu_n\left(g_n - \tilde{h}_n^2\alpha\psi_n^*\psi_n\right) \\ i\nu_n\left(g_n - \tilde{h}_n^2\alpha\psi_n^*\psi_n\right) & 0 \end{pmatrix}, \tag{4.1}$$

generalizing (3.11), and the Hamiltonian function is chosen in the form

$$H = \sum_{n\in\mathbb{Z}} h_n\left(a_n\psi_n\psi_{n+1}^* + b_n\psi_n\psi_n^* + c_n\psi_n\psi_{n-1}^* + \frac{2d_n}{\alpha}\ln\left|g_n - \alpha\tilde{h}_n^2\psi_n\psi_n^*\right|\right), \tag{4.2}$$

where $h_n, \tilde{h}_n, \nu_n, a_n, b_n, c_n, d_n$ and $g_n \in \mathbb{R}_+, n \in \mathbb{Z}$, are some parameters. The reality condition, imposed on the Hamiltonian function (4.2), yields the relationships

$$c_n h_n = a_{n-1}^* h_{n-1}, \quad b_n^* = b_n, \quad d_n^* = d_n,$$

which should be satisfied for all $n \in \mathbb{Z}$. As a result, there is obtained the corresponding generalized discrete nonlinear Schrödinger dynamical system $\frac{d}{dt}(\psi_n, \psi_n^*)^\intercal := -\theta_n \mathrm{grad}\, H[\psi_n, \psi_n^*]$, $n \in \mathbb{Z}$, equivalent to the infinite set of ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\psi_n &= i\nu_n\left(h_{n+1}c_{n+1}g_n\psi_{n+1} + \left(b_n g_n h_n - 2\tilde{h}_n^2 h_n d_n\right)\psi_n + h_{n-1}a_{n-1}g_n\psi_{n-1}\right) - \\ &\quad - i\alpha\nu_n\tilde{h}_n^2\left(h_{n+1}c_{n+1}\psi_{n+1} + h_n b_n\psi_n + h_{n-1}a_{n-1}\psi_{n-1}\right)\psi_n\psi_n^*, \\ \frac{d}{dt}\psi_n^* &= -i\nu_n\left(h_n a_n g_n\psi_{n+1}^* + \left(b_n g_n h_n - 2\tilde{h}_n^2 h_n d_n\right)\psi_n^* + h_n c_n g_n\psi_{n-1}^*\right) + \\ &\quad + i\alpha\nu_n\tilde{h}_n^2\left(h_n a_n\psi_{n+1}^* + h_n b_n\psi_n^* + h_n c_n\psi_{n-1}^*\right)\psi_n\psi_n^* \end{aligned} \tag{4.3}$$

for all $n \in \mathbb{Z}$. In the completely autonomous case, when $h_n = h$, $\tilde{h}_n = \tilde{h}$, $\nu_n = \nu$, $a_n = a$, $b_n = b$, $c_n = c$, $d_n = d$ and $g_n = g \in \mathbb{R}_+$ for all $n \in \mathbb{Z}$, the Poissonian structure (4.1) becomes

$$\theta_n = \begin{pmatrix} 0 & -i\nu\left(g - \tilde{h}^2\alpha\psi_n^*\psi_n\right) \\ i\nu\left(g - \tilde{h}^2\alpha\psi_n^*\psi_n\right) & 0 \end{pmatrix}$$

and the Hamiltonian function (4.2) becomes

$$H = \sum_{n\in\mathbb{Z}} h\left(a\psi_n\psi_{n+1}^* + b\psi_n\psi_n^* + c\psi_n\psi_{n-1}^* + \frac{2d}{\alpha}\ln\left|g - \alpha\tilde{h}^2\psi_n\psi_n^*\right|\right). \tag{4.4}$$

The corresponding reality condition for (4.4) reads as

$$c = a^*, \quad b^* = b, \quad d^* = d,$$

and the related discrete nonlinear Schrödinger dynamical systems reads as a set of the equations

$$\frac{d}{dt}\,\psi_n = i\nu h\left(cg\psi_{n+1} + \left(bg - 2\tilde{h}^2 d\right)\psi_n + ag\psi_{n-1}\right) - i\alpha\nu h\tilde{h}^2\left(c\psi_{n+1} + b\psi_n + a\psi_{n-1}\right)\psi_n\psi_n^*,$$

$$\frac{d}{dt}\,\psi_n^* = -i\nu h\left(ag\psi_{n+1}^* + \left(bg - 2\tilde{h}^2 d\right)\psi_n^* + cg\psi_{n-1}^*\right) + i\alpha\nu h\tilde{h}^2\left(a\psi_{n+1}^* + b\psi_n^* + c\psi_{n-1}^*\right)\psi_n\psi_n^*$$

for all $n \in \mathbb{Z}$. If now to make in (4.3) the substitutions

$$\nu_n = \frac{1}{h_n},\quad g_n = 1,\quad \tilde{h}_n = h_n,\quad a_n = \frac{1}{h_n^2},\quad b_n = 0,\quad c_n = \frac{1}{h_n h_{n-1}},\quad d_n = \frac{1}{h_n^4},$$

one obtains the discrete nonlinear Schrödinger dynamical system

$$K_n^{(g)}\left[\psi_n, \psi_n^*\right] := \begin{cases} \dfrac{d}{dt}\,\psi_n = \dfrac{i}{h_n^2}\left(\psi_{n+1} - 2\psi_n + h_n h_{n-1}^{-1}\psi_{n-1}\right) - \\ \qquad -i\alpha\left(\psi_{n+1} + h_n h_{n-1}^{-1}\psi_{n-1}\right)\psi_n^*\psi_n, \\ \dfrac{d}{dt}\,\psi_n^* = -\dfrac{i}{h_n^2}\left(\psi_{n+1}^* - 2\psi_n^* + h_n h_{n-1}^{-1}\psi_{n-1}^*\right) + \\ \qquad +i\alpha\left(\psi_{n+1}^* + h_n h_{n-1}^{-1}\psi_{n-1}^*\right)\psi_n^*\psi_n, \end{cases} \tag{4.5}$$

whose Hamiltonian function equals

$$H^{(g)} = \sum_{n\in\mathbb{Z}} h_n^{-1}\left(\psi_n\psi_{n+1}^* + \psi_{n+1}\psi_n^* + \frac{2}{\alpha h_n^2}\,\ln\left|1 - \alpha h_n^2\psi_n^*\psi_n\right|\right). \tag{4.6}$$

Another substitution, taken in the form

$$c = a \neq 0,\quad \nu hga = \frac{1}{h^2},\quad \left(bg - 2\tilde{h}^2 d\right)\nu h = -\frac{2}{h^2},\quad \nu h\tilde{h}^2(a + b + c) = 2, \tag{4.7}$$

is also suitable in the limit $h \to 0$ for discretization the nonlinear Schrödinger dynamical system (3.7). The corresponding discrete nonlinear Schrödinger dynamics takes the form

$$\begin{aligned} \frac{d}{dt}\,\psi_n &= \frac{i}{h^2}\left(\psi_{n+1} - 2\psi_n + \psi_{n-1}\right) - \frac{2i\alpha}{2+\mu}\left(\psi_{n+1} + \psi_{n-1} + \mu\psi_n\right)\psi_n\psi_n^*, \\ \frac{d}{dt}\,\psi_n^* &= -\frac{i}{h^2}\left(\psi_{n+1}^* - 2\psi_n^* + \psi_{n-1}^*\right) + \frac{2i\alpha}{2+\mu}\left(\psi_{n+1}^* + \psi_{n-1}^* + \mu\psi_n^*\right)\psi_n\psi_n^* \end{aligned} \tag{4.8}$$

for all for all $n \in \mathbb{Z}$, where $\mu = \dfrac{b}{a} \in \mathbb{R}_+$. Thus we obtained a one-parameter family of Hamiltonian discretizations of the NLS equation. The set of relationships (4.7) admits a lot of reductions, for instance, one can take

$$\nu = 1,\quad g = 1,\quad a = \frac{1}{h^3},\quad d = \left(\frac{\mu+2}{2}\right)^2\frac{1}{h^5},\quad \frac{\tilde{h}^2}{h^2} = \frac{2}{2+\mu},$$

not changing the infinite set of equations (4.8).

All of the constructed above discretizations of the nonlinear Schrödinger dynamical system (3.1) on the functional manifold $\tilde{M}$ can be considered as either better or worse from the computational point of view. If some of the discretization allows, except the Hamiltonian function, some extra conservation laws, it can be naturally considered as a much more suitable for numerical analysis case, allowing both to control the stability of the solution convergence, as a parameter $\mathbb{R}_+ \ni h \to 0$, and to make an invariant solution space reduction to a lower effective dimension of the related solution set.

It is worthy to observe here that the functional structure of the discretization (3.7) strongly depends both on the manifold M and on the convergent as $h \to 0$ form of the Hamiltonian function (3.11). In particular, the existence of the limit

$$\tilde{H} := \lim_{h\to 0} \sum_{n\in\mathbb{Z}} \frac{1}{h} \left(\psi_n \psi_{n+1}^* + \psi_{n+1}\psi_n^* + \frac{2}{\alpha h^2} \ln \left|1 - \alpha h^2 \psi_n^* \psi_n\right| \right), \tag{4.9}$$

coinciding with the expression (3.4), imposes a strong constraint on the functional space $\tilde{M} \subset \subset L_2(\mathbb{R};\mathbb{C}^2)$, namely, a vector-function $(\psi,\psi^*)^\intercal \in W_2^2(\mathbb{R};\mathbb{C}^2) \subset L_2(\mathbb{R};\mathbb{C}^2)$, thereby fixing a suitable functional class [7] for which the discretization conserves its physical Hamiltonian system sense. Respectively, the limiting for (4.9) symplectic structure

$$\begin{aligned} \tilde{\omega}^{(2)} &:= -\lim_{h\to 0} \sum_{n\in\mathbb{Z}} \frac{i}{2} \left\langle (d\psi_n, d\psi_n^*)^\intercal, \wedge \theta_n^{-1} (d\psi_n, d\psi_n^*)^\intercal \right\rangle = \\ &= -\lim_{h\to 0} i \sum_{n\in\mathbb{Z}} h \left(1 - \alpha h^2 \psi_n^* \psi_n\right)^{-1} d\psi_n^* \wedge d\psi_n = -i \int_{\mathbb{R}} dx \, [d\psi^*(x) \wedge d\psi(x)] \end{aligned} \tag{4.10}$$

on the manifold $\tilde{M}$ coincides exactly with the canonical symplectic structure (3.5) for the dynamical system (3.2).

If now to assume that a vector function $(\psi,\psi^*)^\intercal \in W_2^1(\mathbb{R};\mathbb{C}^2) \subset L_2(\mathbb{R};\mathbb{C}^2)$, the Hamiltonian function (3.11) can be taken only as

$$H^{(s)} = \sum_{n\in\mathbb{Z}} \left(\psi_n \psi_{n+1}^* + \psi_{n+1}\psi_n^* + \frac{2}{\alpha h^2} \ln \left|1 - \alpha h^2 \psi_n^* \psi_n\right| \right), \tag{4.11}$$

and the corresponding Poissonian structure as

$$\theta_n^{(s)} := \begin{pmatrix} 0 & ih^{-2}\left(h^2 \alpha \psi_n^* \psi - 1\right) \\ ih^{-2}\left(1 - h^2 \alpha \psi_n^* \psi\right) & 0 \end{pmatrix}. \tag{4.12}$$

The limiting for (4.11) Hamiltonian function

$$\tilde{H}^{(s)} := \lim_{h\to 0} H^{(s)} = \int_{\mathbb{R}} dx \, (\psi \psi_x^* + \psi_x \psi^*) = 0$$

becomes trivial and, simultaneously, the limiting for (4.12) symplectic structure

$$\tilde{\omega}^{(2)}_{(s)} := \lim_{h\to 0}\sum_{n\in\mathbb{Z}}\frac{i}{2}\left\langle (d\psi_n, d\psi_n^*)^{\intercal}, \wedge\theta_n^{(s),-1}(d\psi_n, d\psi_n^*)^{\intercal}\right\rangle =$$

$$= \lim_{h\to 0} i\sum_{n\in\mathbb{Z}} h^2\left(1-\alpha h^2\psi_n^*\psi_n\right)^{-1} d\psi_n^*\wedge d\psi_n = 0$$

becomes trivial too. Thus, the functional space $W_2^1\left(\mathbb{R};\mathbb{C}^2\right)\subset L_2\left(\mathbb{R};\mathbb{C}^2\right)$ is not suitable for the discretization (3.7) of the nonlinear integrable Schrödinger dynamical system (3.1).

It is important here to stress that the discretization parameter $h\in\mathbb{R}_+$ can be taken as depending on the node $n\in\mathbb{Z}$: $h\to h_n\in\mathbb{R}_+$, which satisfies the condition $\sup_{n\in\mathbb{Z}} h_n\le\varepsilon$, where the condition $\varepsilon\to 0$ should be later imposed. For instance, one can replace the dynamical system (3.7) by (4.5), the Poissonian structure (3.8) by

$$\theta_n^{(g)} := \begin{pmatrix} 0 & ih_n^{-1}\left(h_n^2\alpha\psi_n^*\psi - 1\right) \\ ih_n^{-1}\left(1-h_n^2\alpha\psi_n^*\psi\right) & 0\end{pmatrix} \tag{4.13}$$

and, respectively, the Hamiltonian function (3.11) becomes exactly (4.6).

It is easy to check that the modified discrete dynamical system (4.5) can be equivalently rewritten as

$$\frac{d}{dt}\left(\psi_n,\psi_n^*\right)^{\intercal} = -\theta_n^{(g)}\operatorname{grad} H^{(g)}\left[\psi_n,\psi_n^*\right]$$

for all $n\in\mathbb{Z}$, meaning, in particular, that the Hamiltonian function (4.6) is conservative. The latter follows from the fact that the skewsymmetric operator (4.13) is Poissonian on the discretized manifold M_h. Moreover, if to impose the constraint that uniformly in $n\in\mathbb{Z}$ the limit $\lim_{\varepsilon\to 0}\left(h_n h_{n-1}^{-1}\right) = 1$, the dynamical system (4.5) reduces to (3.1) and the corresponding limiting symplectic structure

$$\tilde{\omega}^{(2)}_{(g)} := -\lim_{\varepsilon\to 0}\sum_{n\in\mathbb{Z}}\frac{i}{2}\left\langle (d\psi_n, d\psi_n^*)^{\intercal}, \wedge\theta_n^{(g),-1}(d\psi_n, d\psi_n^*)^{\intercal}\right\rangle =$$

$$= -\lim_{\varepsilon\to 0} i\sum_{n\in\mathbb{Z}} h_n\left(1-\alpha h_n^2\psi_n^*\psi_n\right)^{-1} d\psi_n^*\wedge d\psi_n = -i\int_{\mathbb{R}} dx\left[d\psi^*(x)\wedge d\psi(x)\right],$$

coincides exactly with the symplectic structure (4.10).

Remark 4.1. It is, by now, a not solved, but interesting, problem whether the modified discrete Hamiltonian dynamical system (4.5) sustains to be Lax type integrable. It is left for studying in a separate work.

4.2. Conservation laws for the integrable discrete NLS system. Taking into account that the discrete dynamical system (3.7) is well posed in the space $M_h := w_{h,2}^2\left(\mathbb{Z};\mathbb{C}^2\right)\subset l_2\left(\mathbb{Z};\mathbb{C}^2\right)$, suitably approximating the Sobolev space of functions $W_2^2\left(\mathbb{R};\mathbb{C}^2\right)$, we can go further and to approximate the space $w_{h,2}^2\left(\mathbb{Z};\mathbb{C}^2\right)$ by means of an infinite hierarchy of strictly invariant finite dimensional subspaces $M_h^{(N)}\simeq\bar{w}_{h,2}^2\left(\mathbb{Z}_{(N)};\mathbb{C}^2\right)$, $N\in\mathbb{Z}_+$. In particular, as it was before shown both in [1, 2] by means of the inverse scattering transform method [1, 50] and in [12, 51, 53] by means of the gradient-holonomic approach [54], the discrete nonlinear Schrödinger dynamical system (3.7) possesses on the manifold M_h an infinite hierarchy of the functionally independent conservation laws:

$$\bar{\gamma}_0 = \frac{1}{\alpha h^3} \sum_{n\in\mathbb{Z}} \ln \left|1 - \alpha h^2 \psi_n^* \psi_n\right|, \quad \gamma_0 = \sum_{n\in\mathbb{Z}_+} \sigma_n^{(0)},$$

$$\gamma_1 = \sum_{n\in\mathbb{Z}} \left(\sigma_n^{(1)} + \frac{1}{2}\, \sigma_n^{(0)} \sigma_n^{(0)} \right),$$

$$\gamma_2 = \sum_{n\in\mathbb{Z}} \left(\sigma_n^{(2)} + \frac{1}{3}\, \sigma_n^{(0)} \sigma_n^{(0)} \sigma_n^{(0)} + \sigma_n^{(0)} \sigma_n^{(1)} \right), \ldots,$$

where the quantities $\sigma_n^{(j)}$, $n \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, are defined as follows:

$$\begin{aligned} \sigma_n^{(0)} &= -\frac{1}{\alpha h^2} \left(\psi_n^* \psi_{n-1} + \psi_{n-1}^* \psi_{n-2} \right), \\ \sigma_n^{(1)} &= i\, \frac{d}{dt}\, \sigma_{n-1}^{(0)} + \left(1 - \alpha h^2 \psi_{n-1}^* \psi_{n-1}\right) \left(1 - \alpha h^2 \psi_{n-2}^* \psi_{n-2}\right) + \\ &\quad + \beta\, \frac{\alpha}{h^2} \psi_{n-1}^* \left(\psi_n + \psi_{n-1}\right), \ldots, \end{aligned} \tag{4.14}$$

and $\beta \in \mathbb{R}$ is an arbitrary constant parameter. As a result of (4.14) one finds the following infinite hierarchy of smooth conservation laws:

$$\begin{aligned} \bar{H}_0 &= \sum_{n\in\mathbb{Z}} \ln \left|1 - \alpha h^2 \psi_n^* \psi_n\right|, \\ H_0 &= \sum_{n\in\mathbb{Z}} \psi_n^* \psi_{n+1}, \quad H_0^* = \sum_{n\in\mathbb{Z}} \psi_n \psi_{n+1}^*, \\ H_1 &= \sum_{n\in\mathbb{Z}} \left(\frac{1}{2}\, \psi_n^2 \psi_{n-1}^{*,2} + \psi_n \psi_{n+1} \psi_{n-1}^* \psi_n^* - \frac{\psi_n \psi_{n-2}^*}{\alpha h^2} \right), \\ H_1^* &= \sum_{n\in\mathbb{Z}} \left(\frac{1}{2} \psi_{n-1}^2 \psi_n^{*,2} + \psi_{n-1} \psi_n \psi_{n+1}^* \psi_n^* - \frac{\psi_{n-2} \psi_n^*}{\alpha h^2} \right), \\ H_2 &= \sum_{n\in\mathbb{Z}} \left[\frac{1}{3}\, \psi_n^3 \psi_{n-1}^{*,3} + \psi_n \psi_{n+1} \psi_{n-1}^* \psi_n^* \left(\psi_n \psi_{n-1}^* + \psi_{n+1} \psi_n^* + \psi_{n+2} \psi_{n+1}^* \right) - \right. \\ &\quad - \frac{\psi_n \psi_{n-1}^*}{\alpha h^2} \left(\psi_n \psi_{n-2}^* + \psi_{n+1} \psi_{n-1}^* \right) - \\ &\quad \left. - \frac{\psi_n \psi_n^*}{\alpha h^2} \left(\psi_{n+1} \psi_{n-2}^* + \psi_{n+2} \psi_{n-1}^* \right) + \frac{\psi_n \psi_{n-3}^*}{\alpha^2 h^4} \right], \end{aligned} \tag{4.15}$$

$$H_2^* = \sum_{n\in\mathbb{Z}} \left[\frac{1}{3}\, \psi_n^{*,3}\psi_{n-1}^3 + \psi_n^*\psi_{n+1}^*\psi_{n-1}\psi_n \left(\psi_n^*\psi_{n-1} + \psi_{n+1}^*\psi_n + \psi_{n+2}^*\psi_{n+1}\right) - \right.$$

$$- \frac{\psi_n^*\psi_{n-1}}{\alpha h^2} \left(\psi_n^*\psi_{n-2} + \psi_{n+1}^*\psi_{n-1}\right) -$$

$$\left. - \frac{\psi_n\psi_n^*}{\alpha h^2} \left(\psi_{n+1}^*\psi_{n-2} + \psi_{n+2}^*\psi_{n-1}\right) + \frac{\psi_n^*\psi_{n-3}}{\alpha^2 h^4} \right],$$

and so on.

Taking into account the functional structure of the equations (3.6) or (3.7), one can define the space $\mathcal{D}(M_h)$ of smooth functions $\gamma\colon M_h \to \mathbb{C}$ on M_h as that invariant with respect to the phase transformation $\mathbb{C}^2 \ni (\psi_n, \psi_n^*) \to (e^{\alpha}\psi_n, e^{-\alpha}\psi_n^*) \in \mathbb{C}^2$ for any $n \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$. Equivalently, a function $\gamma \in \mathcal{D}(M_h)$ iff the condition

$$\sum_{n\in\mathbb{Z}} \langle \operatorname{grad}\gamma\,[\psi_n, \psi_n^*]\,, (\psi_n, -\psi_n^*)^{\intercal} \rangle \,=\, 0 \tag{4.16}$$

holds on M_h. Note that conserved quantities (4.15) belong to $\mathcal{D}(M_h)$.

The conservation law $\bar{H}_0 \in \mathcal{D}(M_h)$ is a Casimir function for the Poissonian structure (3.8) on the manifold M_h, that is for any $\gamma \in \mathcal{D}(M_h)$ the Poisson bracket

$$\{\gamma, \bar{H}_0\} := \sum_{n\in\mathbb{Z}} \left\langle \operatorname{grad}\gamma\,[\psi_n, \psi_n^*]\,, \theta_n \operatorname{grad}\bar{H}_0\,[\psi_n, \psi_n^*] \right\rangle =$$

$$= i\alpha h \sum_{n\in\mathbb{Z}} \langle \operatorname{grad}\gamma\,[\psi_n, \psi_n^*]\,, (\psi_n, -\psi_n^*) \rangle \,=\, 0, \tag{4.17}$$

owing to the condition (4.16). The Hamiltonian function (3.11) is obtained from the first three invariants of (4.15) as

$$H \,=\, \frac{2}{\alpha h^3}\, \bar{H}_0 + \frac{1}{h}\, \left(H_0 + H_0^*\right).$$

Remark 4.2.. Similarly to the limiting condition (4.9), the same limiting expression one obtains from the discrete invariant function

$$H^{(w)} \,=\, \frac{1}{2\alpha h^3}\, \bar{H}_0 - \frac{\alpha h}{4}\, \left(H_1 + H_1^*\right),$$

that is

$$\lim_{h-0} H^{(w)} \,=\, \tilde{H} := \frac{1}{2} \int\limits_{\mathbb{R}} dx \left[\psi\psi_{xx}^* + \psi_{xx}\psi^* - 2\alpha\left(\psi^*\psi\right)^2\right].$$

As one can observe, some combinations of the discrete conservation laws allow well defined and finite limiting expressions in the functional form, coinciding with the corresponding conservation laws of the continuous nonlinear Schrödinger dynamical system (3.1), yet almost all other ones fail to possess such well defined limiting functional expressions. This phenomenon appears to be strictly connected with the mathematical properties of the basic manifold $M_2^{(N)}$, on which these disscrete conservation laws are defined. As in general, the sequences of these conservation laws are often not bounded as the discretization parameter $h \to 0$, their limiting functional expression naturally does not exists. Another effect can happen when these sequences are in reality bounded as $h \to 0$, yet compiling only compact subsets, possessing having no limiting functional expression. In general, all these phenomena are deeply related with the well known mathematical fact, which states that a given continuous function possesses a not countable set of different discrete approximations, and many of them can be not convergable as $h \to 0$ to the function under regard.

Based on these results, one can apply to the discrete dynamical system (3.7) the Bogoyavlensky – Novikov type reduction scheme, devised before in [50, 53] and obtain a completely Liouville integrable finite dimensional dynamical system on the manifold $M_h^{(N)}$. Namely, we consider the critical submanifold $M_h^{(N)} \subset M_h$ of the following real-valued action functional:

$$\mathcal{L}_h^{(N)} := \sum_{n\in\mathbb{Z}} \mathcal{L}_h^{(N)}[\psi_n, \psi_n^*] = \bar{c}_0(h)\bar{H}_0 + \sum_{j=0}^{N} c_j(h)\left(H_j + H_j^*\right),$$

where, by definition, $\bar{c}_0, c_j\colon \mathbb{R}_+ \to \mathbb{R}$, $j = \overline{0,N}$, are suitably defined functions for arbitrary but fixed $N \in \mathbb{Z}_+$, and

$$M_h^{(N)} := \left\{(\psi, \psi^*)^\intercal \in M_h\colon \operatorname{grad}\mathcal{L}_h^{(N)}[\psi_n, \psi_n^*] = 0,\ n \in \mathbb{Z}\right\}.$$

As one can easily show, the submanifold $M_h^{(N)} \subset M_h$ is finite-dimensional and for any $N \in \mathbb{Z}_+$ is invariant with respect to the vector field $K\colon M_h \to T(M_h)$. This property makes it possible to reduce it on the submanifold $M_h^{(N)} \subset M_h$ and to obtain a resulting finite-dimensional system of ordinary differential equations on $M_h^{(N)}$, whose solution manifold coincides with an subspace of exact solutions to the initial dynamical system (3.7). The latter proves to be canonically Hamiltonian on the manifold $M_h^{(N)}$ and, moreover, completely Liouville – Arnold integrable. If the mappings $\bar{c}_0, c_j\colon \mathbb{R}_+ \to \mathbb{R}$, $j = \overline{0,N}$, are chosen in such a way that the flow (3.7), invariantly reduced on the finite dimensional submanifold $M_h^{(N)} \subset M_h$, is nonsingular as $h \to 0$ and complete, then the corresponding solutions to the discrete dynamical system (3.7) will respectively approach those to the nonlinear integrable Schrödinger dynamical system (3.1).

Below we will proceed to realizing this scheme for the most simple cases $N = 1$ and $N = 2$. Another way of analyzing the discrete dynamical system (3.7), being interesting enough, consists in applying the approaches recently devised in [15, 45] and based on the long-time behavior of the chosen discretization subject to a fixed Hamiltonian function structure.

4.3. The finite dimensional reduction scheme: the case $N = 1$. Consider the following nondegenerate action functional:

$$\mathcal{L}_h^{(1)} = \sum_{n\in\mathbb{Z}} \bar{c}_0(h) \ln\left|1 - \alpha h^2 \psi_n^* \psi_n\right| + \sum_{n\in\mathbb{Z}} c_0(h) \left(\psi_n^* \psi_{n+1} + \psi_n \psi_{n+1}^*\right) +$$
$$+ \sum_{n\in\mathbb{Z}} c_1(h) \left(\frac{1}{2} \psi_n^2 \psi_{n-1}^{*,2} + \psi_n \psi_{n+1} \psi_{n-1}^* \psi_n^* - \frac{\psi_n \psi_{n-2}^*}{\alpha h^2} + \right.$$
$$\left. + \frac{1}{2} \psi_n^2 \psi_{n+1}^{*,2} + \psi_n \psi_{n+1} \psi_{n+1}^* \psi_{n+2}^* - \frac{\psi_{n-1} \psi_{n+1}^*}{\alpha h^2} \right)$$

with mappings $\bar{c}_0, c_j \colon \mathbb{R}_+ \to \mathbb{R}$, $j = \overline{0,1}$, taken as

$$\bar{c}_0(h) = \frac{4\xi + 1}{2\alpha h^3}, \quad c_0(h) = \frac{\xi}{h}, \quad c_1(h) = \frac{\alpha h}{4},$$

and being easily determined for any $\xi \in \mathbb{R}$ from the condition for existence of a limit as $h \to 0$:

$$\tilde{\mathcal{L}}^{(1)} := \lim_{h\to 0} \mathcal{L}_h^{(1)}.$$

The corresponding invariant critical submanifold

$$M_h^{(1)} := \left\{ (\psi, \psi^*)^\intercal \in M_h \colon \operatorname{grad} \mathcal{L}_h^{(1)} [\psi_n, \psi_n^*] = 0,\ n \in \mathbb{Z} \right\}$$

is equivalent to the following system of discrete up-recurrent relationships with respect to indices $n \in \mathbb{Z}$:

$$\begin{aligned}
\psi_{n+2} = {} & -\frac{-\bar{c}_0(h)/c_1(h)\psi_n}{(\frac{1}{\alpha h^2} - \psi_{n+1}\psi_{n+1}^*)(\frac{1}{\alpha h^2} - \psi_n \psi_n^*)} + \\
& + \frac{2\psi_{n-1} c_0(h)/c_1(h) + \psi_n(\psi_{n+1}\psi_{n-1}^* + \psi_{n-1}\psi_{n+1}^*)}{(\frac{1}{\alpha h^2} - \psi_{n+1}\psi_{n+1}^*)} + \\
& + \frac{(\psi_{n+1}^2 + \psi_{n-1}^2)\psi_n^* - \psi_{n-2}(\frac{1}{\alpha h^2} - \psi_{n-1}\, \psi_{n-1}^*)}{(\frac{1}{\alpha h^2} - \psi_{n+1}\, \psi_{n+1}^*)} := \\
& := \Phi_+(\psi_{n+1}, \psi_{n+1}^*; \psi_n, \psi_n^*; \psi_{n-1}, \psi_{n-1}^*), \\
\psi_{n+2}^* = {} & -\frac{-\bar{c}_0(h)/c_1(h)\psi_n^*}{(\frac{1}{\alpha h^2} - \psi_{n+1}\psi_{n+1}^*)(\frac{1}{\alpha h^2} - \psi_n \psi_n^*)} + \\
& + \frac{2\psi_{n-1}^* c_0(h)/c_1(h) + \psi_n^*\, (\psi_{n+1}\psi_{n-1}^* + \psi_{n-1}\psi_{n+1}^*)}{(\frac{1}{\alpha h^2} - \psi_{n+1}\, \psi_{n+1}^*)} + \\
& + \frac{(\psi_{n+1}^{*,2} + \psi_{n-1}^{*,2})\psi_n - \psi_{n-2}^*(\frac{1}{\alpha h^2} - \psi_{n-1}\psi_{n-1}^*)}{(\frac{1}{\alpha h^2} - \psi_{n+1}\psi_{n+1}^*)} := \\
& := \Phi_+^*(\psi_{n+1}, \psi_{n+1}^*; \psi_n, \psi_n^*; \psi_{n-1}, \psi_{n-1}^*).
\end{aligned} \tag{4.18}$$

The latter can be also rewritten as the system of down-recurrent mappings

$$
\begin{aligned}
\psi_{n-2} &= -\frac{-\bar{c}_0(h)/c_1(h)\psi_n}{(\frac{1}{\alpha h^2}-\psi_{n-1}\psi^*_{n-1})(\frac{1}{\alpha h^2}-\psi_n\psi^*_n)}+\\
&\quad+\frac{2\psi_{n-1}c_0(h)/c_1(h)+\psi_n(\psi_{n+1}\psi^*_{n-1}+\psi_{n-1}\psi^*_{n+1})}{(\frac{1}{\alpha h^2}-\psi_{n-1}\psi^*_{n-1})}+\\
&\quad+\frac{(\psi^2_{n+1}+\psi^2_{n-1})\psi^*_n-\psi_{n+2}(\frac{1}{\alpha h^2}-\psi_{n+1}\psi^*_{n+1})}{(\frac{1}{\alpha h^2}-\psi_{n-1}\psi^*_{n-1})}:=\\
&:=\Phi_-(\psi_{n-1},\psi^*_{n-1};\psi_n,\psi^*_n;\psi_{n+1},\psi^*_{n+1}),\\
\psi^*_{n-2} &= -\frac{-\bar{c}_0(h)/c_1(h)\psi^*_n}{(\frac{1}{\alpha h^2}-\psi_{n-1}\psi^*_{n-1})(\frac{1}{\alpha h^2}-\psi_n\psi^*_n)}+\\
&\quad+\frac{2\psi^*_{n-1}c_0(h)/c_1(h)+\psi^*_n(\psi_{n+1}\psi^*_{n-1}+\psi_{n-1}\psi^*_{n+1})}{(\frac{1}{\alpha h^2}-\psi_{n-1}\psi^*_{n-1})}+\\
&\quad+\frac{(\psi^{*,2}_{n+1}+\psi^{*,2}_{n-1})\psi_n-\psi^*_{n+2}(\frac{1}{\alpha h^2}-\psi_{n+1}\psi^*_{n+1})}{(\frac{1}{\alpha h^2}-\psi_{n-1}\psi^*_{n-1})}:=\\
&:=\Phi^*_-(\psi_{n-1},\psi^*_{n-1};\psi_n,\psi^*_n;\psi_{n+1},\psi^*_{n+1}),
\end{aligned}
\tag{4.19}
$$

which also hold for all $n\in\mathbb{Z}$. The relationships (4.18) (or, the same, relationships (4.19)) mean that the whole submanifold $M_h^{(1)}\subset M_h$ is retrieved by means of the initial values

$$(\bar{\psi}_{-1},\bar{\psi}^*_{-1};\bar{\psi}_0,\bar{\psi}^*_0;\bar{\psi}_1,\bar{\psi}^*_1;\bar{\psi}_2,\bar{\psi}^*_2)^{\intercal}\in M_h^{(1)}\simeq\mathbb{C}^8.$$

Thereby, the submanifold $M_h^{(1)}\subset M_h^8$ is naturally diffeomorphic to the finite dimensional complex space M_h^8. Taking into account the canonical symplecticity [51, 53] of the submanifold $M_h^{(1)}\simeq M_h^8$ and its invariance with respect to the vector field (3.7) one can easily reduce it on this submanifold $M_h^{(1)}\simeq M_h^8$ and obtain the following equivalent finite dimensional flow on the manifold M_h^8:

$$
\begin{aligned}
\frac{d}{dt}\psi_2 &= \frac{i}{h^2}\left[\Phi_+(\psi_2,\psi^*_2;\psi_1,\psi^*_1;\psi_0,\psi^*_0)-2\psi_2+\psi_1\right]-\\
&\quad-i\alpha\left[\Phi_+(\psi_2,\psi^*_2;\psi_1,\psi^*_1;\psi_0,\psi^*_0)+\psi_1\right]\psi_2\psi^*_2,\\
\frac{d}{dt}\psi_1 &= \frac{i}{h^2}\left[\psi_2-2\psi_1+\psi_0\right]-i\alpha\left(\psi_2+\psi_0\right)\psi_1\psi^*_1,\\
\frac{d}{dt}\psi_0 &= \frac{i}{h^2}\left[\psi_1-2\psi_0+\psi_{-1}\right]-i\alpha\left(\psi_1+\psi_{-1}\right)\psi_0\psi^*_0,
\end{aligned}
\tag{4.20}
$$

$$\frac{d}{dt}\psi_{-1} = \frac{i}{h^2}\left[\psi_0 - 2\psi_{-1} + \Phi_-\left(\psi_{-1},\psi_{-1}^*;\psi_0,\psi_0^*;\psi_1,\psi_1^*\right)\right] - \\ - i\alpha\left[\psi_0 + \Phi_-\left(\psi_{-1},\psi_{-1}^*;\psi_0,\psi_0^*;\psi_1,\psi_1^*\right)\right]\psi_{-1}\psi_{-1}^*,$$

and

$$\begin{aligned} \frac{d}{dt}\psi_{-1}^* &= -\frac{i}{h^2}\left[\psi_0^* - 2\psi_{-1}^* + \Phi_-^*\left(\psi_{-1},\psi_{-1}^*;\psi_0,\psi_0^*;\psi_1,\psi_1^*\right)\right] + \\ &\quad + i\alpha\left[\psi_0^* + \Phi_-^*\left(\psi_{-1},\psi_{-1}^*;\psi_0,\psi_0^*;\psi_1,\psi_1^*\right)\right]\psi_{-1}\psi_{-1}^*, \\ \frac{d}{dt}\psi_0^* &= -\frac{i}{h^2}\left[\psi_1^* - 2\psi_0^* + \psi_{-1}^*\right] + i\alpha\left(\psi_1^* + \psi_{-1}^*\right)\psi_0\psi_0^*, \\ \frac{d}{dt}\psi_1^* &= -\frac{i}{h^2}\left[\psi_2^* - 2\psi_1^* + \psi_0^*\right] + i\alpha\left(\psi_2^* + \psi_0^*\right)\psi_1\psi_1^*, \\ \frac{d}{dt}\psi_2^* &= -\frac{i}{h^2}\left[\Phi_+^*\left(\psi_2,\psi_2^*;\psi_1,\psi_1^*;\psi_0,\psi_0^*\right) - 2\psi_2^* + \psi_1^*\right] + \\ &\quad + i\alpha\left[\Phi_+^*\left(\psi_2,\psi_2^*;\psi_1,\psi_1^*;\psi_0,\psi_0^*\right) + \psi_1^*\right]\psi_2\psi_2^*. \end{aligned} \tag{4.21}$$

The next proposition, characterizing the Hamiltonian structure of the reduced dynamical system (4.20) and (4.21), holds.

Proposition 4.1. *The eight-dimensional complex dynamical system (4.20) and (4.21) is Hamiltonian on the manifold $M_h^{(1)} \simeq M_h^8$ with respect to the canonical symplectic structure*

$$\omega_h^{(2)} = \sum_{j=\overline{-2,1}}\left(dp_{-j}\wedge d\psi_{-j} + dp_{-j}^*\wedge d\psi_{-j}^*\right), \tag{4.22}$$

where, by definition,

$$p_{-j} := \mathcal{L}_{h,\psi_{n-j+1}}^{(1)\prime,*}[\psi_n,\psi_n^*]\cdot 1, \quad p_{-j}^* := \mathcal{L}_{h,\psi_{n-j+1}^*}^{(1)\prime,*}[\psi_n,\psi_n^*]\cdot 1 \tag{4.23}$$

for $j = \overline{-2,1}$ modulo the constraint $\operatorname{grad}\mathcal{L}_h^{(1)}[\psi_n,\psi_n^*] = 0$, $n\in\mathbb{Z}$, *on the submanifold $M_h^{(1)} \simeq \simeq M_h^8$, and the sign $''\prime,*''$ means the corresponding discrete Frechét up-directed derivative and its natural conjugation with respect to the convolution mapping on $T^*(M_h^{(1)})\times T(M_h^{(1)})$.*

Proof. The symplectic structure (4.22) easily follows [11, 51, 53] from the discrete version of the Gelfand – Dikiy [26] differential relationship:

$$d\mathcal{L}_h^{(1)}[\psi_n,\psi_n^*] = \left\langle \operatorname{grad}\mathcal{L}_h^{(1)}[\psi_{n-1},\psi_{n-1}^*], \left(d\psi_{n-1}, d\psi_{n-1}^*\right)^\intercal\right\rangle + \\ + \frac{d}{dn}\alpha_h^{(1)}\left(\psi_{n-1},\psi_{n-1}^*;\psi_n,\psi_n^*;\psi_{n+1},\psi_{n+1}^*;\psi_{n+2},\psi_{n+2}^*\right),$$

where $\alpha_h^{(1)} \in \Lambda^1(M_h^{(1)})$ is, owing to the condition $\operatorname{grad}\mathcal{L}_h^{(1)}[\psi_n,\psi_n^*] = 0$, $n\in\mathbb{Z}$, on the submanifold $M_h^{(1)}$, not depending on the index $n \in \mathbb{Z}$ and suitably defined one-form on the manifold M_h^8, allowing the following canonical representation:

$$\alpha_h^{(1)} = \sum_{j=\overline{-2,1}} \left(p_{-j}(h)d\psi_{-j} + p_{-j}^*(h)d\psi_{-j}^*\right)$$

with functions $p_{-j}, p_{-j}^* \colon M_h^{(1)} \times \mathbb{R} \to \mathbb{C}$, $j = \overline{-2,1}$. The latter, being defined by the expressions (4.23), compile jointly with variables $\psi_{-j}, \psi_{-j}^* \colon M_h^{(1)} \times \mathbb{R} \to \mathbb{C}$, $j = \overline{-2,1}$, the global coordinates on the finite dimensional symplectic manifold M_h^8, proving the proposition.

Proposition 4.1 is proved.

The dynamical system (4.20) and (4.21) on the manifold M_h^8 possesses, except its Hamiltonian function, additionally exactly four mutually commuting functionally independent conservation laws $\mathcal{H}_k, \mathcal{H}_k^* \colon M_h^8 \to \mathbb{R}$, $k = \overline{0,1}$, and one Casimir function $\bar{\mathcal{H}}_0 \colon M_h^8 \to \mathbb{R}$, which can be calculated [53] from the following functional relationships:

$$\langle \operatorname{grad} H_k\left[\psi_n, \psi_n^*\right], K\left[\psi_n, \psi_n^*\right]\rangle := -\frac{d}{dn}\,\mathcal{H}_k\left(\psi_{n-1}, \psi_{n-1}^*; \psi_n, \psi_n^*; \psi_{n+1}, \psi_{n+1}^*; \psi_{n+2}, \psi_{n+2}^*\right),$$
$$\langle \operatorname{grad} H_k^*\left[\psi_n, \psi_n^*\right], K\left[\psi_n, \psi_n^*\right]\rangle := -\frac{d}{dn}\,\mathcal{H}_k^*\left(\psi_{n-1}, \psi_{n-1}^*; \psi_n, \psi_n^*; \psi_{n+1}, \psi_{n+1}^*; \psi_{n+2}, \psi_{n+2}^*\right), \tag{4.24}$$
$$\left\langle \operatorname{grad} \bar{H}_0\left[\psi_n, \psi_n^*\right], K\left[\psi_n, \psi_n^*\right]\right\rangle := -\frac{d}{dn}\,\bar{\mathcal{H}}_0\left(\psi_{n-1}, \psi_{n-1}^*; \psi_n, \psi_n^*; \psi_{n+1}, \psi_{n+1}^*; \psi_{n+2}, \psi_{n+2}^*\right),$$

for $k = \overline{0,1}$ modulo the constraint $\operatorname{grad} \mathcal{L}_h^{(1)}\left[\psi_{n-2}, \psi_{n-2}^*\right] = 0$, $n \in \mathbb{Z}$, on the manifold $M_h^{(1)} \simeq \simeq M_h^8$, where $\frac{d}{dn} := \Delta - 1$ is a discrete analog of the differentiation and the shift operator Δ acts as $\Delta f_n := f_{n+1}$, $n \in \mathbb{Z}$, for any mapping $f \colon \mathbb{Z} \to \mathbb{C}$. From (4.24) one can obtain by menas of simple but tedious calculations analytical expressions for the invariants $\mathcal{H}_k^* \colon M_h^8 \to \mathbb{R}$, which give rise to the corresponding Hamiltonian function for the dynamical system (4.20) and (4.21), owing to the relationship (4.17):

$$\mathcal{H} = \frac{2}{\alpha h^3}\,\bar{\mathcal{H}}_0 + \frac{1}{h}\,\left(\mathcal{H}_0 + \mathcal{H}_0^*\right),$$

satisfying the following canonical Hamiltonian system with respect to the symplectic structure (4.22):

$$\frac{d\psi_{-j}}{dt} = \frac{\partial \mathcal{H}}{\partial p_{-j}}, \quad \frac{d\psi_{-j}^*}{dt} = \frac{\partial \mathcal{H}}{\partial p_{-j}^*}, \quad \frac{dp_{-j}}{dt} = -\frac{\partial \mathcal{H}}{\partial \psi_{-j}}, \quad \frac{dp_{-j}^*}{dt} = -\frac{\partial \mathcal{H}}{\partial \psi_{-j}^*},$$

where $j = \overline{-2,1}$, which is a Liouville – Arnold integrable on the symplectic manifold M_h^8.

Remark 4.3. The same way on can construct the finite dimensional reduction of the discrete Schrödinger dynamical system (3.7) in the case $N = 2$. Making use of the calculated before conservation laws (4.15), one can take the corresponding action functional as

$$\mathcal{L}_h^{(2)} = \bar{c}_0(h)\sum_{n\in\mathbb{Z}} \ln\left|1-\alpha h^2\psi_n^*\psi_n\right| + c_0(h)\sum_{n\in\mathbb{Z}}\left(\psi_n^*\psi_{n-1}+\psi_n\psi_{n-1}^*\right) +$$

$$+ c_1(h)\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\,\psi_n^2\psi_{n-1}^{*,2}+\psi_n\psi_n^*\left(\psi_{n+1}\psi_{n-1}^*+\psi_{n-1}\psi_{n+1}^*\right)+\right.$$

$$\left.+\frac{1}{2}\,\psi_{n-1}^2\psi_n^{*,2}-\frac{\psi_n\psi_{n-2}^*}{\alpha h^2}-\frac{\psi_{n-2}\psi_n^*}{\alpha h^2}\right)+$$

$$+ c_2(h)\sum_{n\in\mathbb{Z}}\left(\frac{1}{3}\,\psi_n^3\psi_{n-1}^{*,3}+\psi_n\psi_{n+1}\psi_{n-1}^*\psi_n^*\left(\psi_n\psi_{n-1}^*+\psi_{n+1}\psi_n^*+\psi_{n+2}\psi_{n+1}^*\right)-\right.$$

$$-\frac{\psi_n\psi_{n-1}^*}{\alpha h^2}\left(\psi_n\psi_{n-2}^*+\psi_{n+1}\psi_{n-1}^*\right)-\frac{\psi_n\psi_n^*}{\alpha h^2}\left(\psi_{n+1}\psi_{n-2}^*+\psi_{n+2}\psi_{n-1}^*\right)+$$

$$+\frac{\psi_n\psi_{n-3}^*}{\alpha^2h^4}+\frac{1}{3}\,\psi_n^{*,3}\psi_{n-1}^3+\psi_n^*\psi_{n+1}^*\psi_{n-1}\psi_n\left(\psi_n^*\psi_{n-1}+\psi_{n+1}^*\psi_n+\psi_{n+2}^*\psi_{n+1}\right)-$$

$$\left.-\frac{\psi_n^*\psi_{n-1}}{\alpha h^2}\left(\psi_n^*\psi_{n-2}+\psi_{n+1}^*\psi_{n-1}\right)-\frac{\psi_n\psi_n^*}{\alpha h^2}\left(\psi_{n+1}^*\psi_{n-2}+\psi_{n+2}^*\psi_{n-1}\right)+\frac{\psi_n^*\psi_{n-3}}{\alpha^2h^4}\right)$$

with mappings $\bar{c}_0, c_j\colon \mathbb{R}_+\to\mathbb{R},\ j=\overline{0,2}$, defined as before from the condition that there exists the limit

$$\tilde{\mathcal{L}}^{(2)} := \lim_{h\to 0}\mathcal{L}_h^{(2)}.$$

The respectively defined critical submanifold

$$M_h^{(2)} := \left\{(\psi,\psi^*)^\intercal \in M_h\colon \operatorname{grad}\mathcal{L}_h^{(2)}[\psi_n,\psi_n^*]=0,\quad n\in\mathbb{Z}\right\}$$

becomes diffeomorphic to a finite dimensional canonically symplectic manifold M_h^{12} on which the suitably reduced discrete Schrödinger dynamical system (3.7) becomes a Liouville – Arnold integrable Hamiltonian system. The details of the related calculations are planned to be presented in a separate work under preparation.

4.4. The Fourier analysis of the integrable discrete NLS system. It easy to observe that the linearized Schrödinger system (3.1) admits the following Fourier type solution:

$$\psi(x,t)=\int\limits_{\mathbb{R}} ds\xi(s,t)\exp(ixs),\quad \psi^*(x,t)=\int\limits_{\mathbb{R}} ds\xi^*(s,t)\exp(-ixs) \tag{4.25}$$

for all $x,t\in\mathbb{R}$, where $\dfrac{d\xi}{dt}=-is^2\xi,\ \dfrac{d\xi^*}{dt}=is^2\xi^*$, i. e.,

$$\xi(s,t)=\bar{\xi}(s)e^{-is^2t},\xi^*(s,t)=\bar{\xi}^*(s)e^{is^2t}$$

and $\bar{\xi}, \bar{\xi}^* : \mathbb{R} \to \mathbb{C}$ are prescribed functions (the Fourier transforms of initial conditions). Likewise, the linearized discrete Schrödinger dynamical system (3.7) allows the following general discrete Fourier type solution:

$$\psi_n = \int\limits_{\mathbb{R}} ds \xi_h(s,t) \exp(ihns), \quad \psi_n^* = \int\limits_{\mathbb{R}} ds \xi_h^*(s,t) \exp(-ihns) \tag{4.26}$$

for all $n \in \mathbb{Z}$, where the evolution parameter $t \in \mathbb{R}$, $(\psi_n, \psi_n^*)^\intercal \in w_{h,2}^2(\mathbb{Z}; \mathbb{C}^2)$ and

$$\xi_h(s,t) = \bar{\xi}_h(s) \exp\left(-i \frac{4t}{h^2} \sin^2 \frac{sh}{2}\right), \quad \xi_h^*(s,t) = \bar{\xi}_h^*(s) \exp\left(i \frac{4t}{h^2} \sin^2 \frac{sh}{2}\right).$$

Here the function $(\bar{\xi}_h, \bar{\xi}_h^*)^\intercal \in W_{h,2}^2(\mathbb{R}; \mathbb{C}^2) \subset L_2(\mathbb{R}; \mathbb{C}^2)$, where the functional space $W_{h,2}^2(\mathbb{R}; \mathbb{C}^2)$ is yet to be determined. From the boundary condition $(\psi_n, \psi_n^*)^\intercal \in w_{h,2}^2(\mathbb{Z}; \mathbb{C}^2)$ it follows that expressions

$$\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \psi_n^* \psi_n = \int\limits_{\mathbb{R}} ds \xi_h^*(s) \xi_h(s) < \infty,$$

$$\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\psi_{n+1}^* \psi_n + \psi_n^* \psi_{n+1}\right) = 2 \int\limits_{\mathbb{R}} ds \cos(hs) \xi_h^*(s) \xi_h(s) < \infty,$$

ensure the boundedness of the Hamiltonian function (3.11), thereby determining a functional space $W_{h,2}^2(\mathbb{R}; \mathbb{C}^2)$ to which belong the vector function $(\xi_h, \xi_h^*)^\intercal \in L_2(\mathbb{R}; \mathbb{C}^2)$. However the discrete evolution is not following along the continuous trajectory.

Being motivated by works [14, 16], we modify the discrete system as folows in order to obtain the exact discretization:

$$\begin{aligned} \frac{d}{dt} \psi_n &= \frac{i}{\delta^2} \left(\psi_{n+1} - 2\psi_n + \psi_{n-1}\right) - i\alpha \left(\psi_{n+1} + \psi_{n-1}\right) \psi_n \psi_n^*, \\ \frac{d}{dt} \psi_n^* &= -\frac{i}{\delta^2} \left(\psi_{n+1}^* - 2\psi_n^* + \psi_{n-1}^*\right) + i\alpha \left(\psi_{n+1}^* + \psi_{n-1}^*\right) \psi_n \psi_n^*. \end{aligned} \tag{4.27}$$

Substituting (4.26) into the linearization of (4.27) we obtain

$$\xi_h(s,t) = \bar{\xi}_h(s) \exp\left(-i \frac{4t}{\delta^2} \sin^2 \frac{sh}{2}\right), \quad \xi_h^*(s,t) = \bar{\xi}_h^*(s) \exp\left(i \frac{4t}{\delta^2} \sin^2 \frac{sh}{2}\right).$$

Therefore, linearization of the discretization (4.27) is exact (i.e., $\psi(nh,t) = \psi_n(t), n \in \mathbb{Z}$, if we assume

$$\delta = \frac{2}{s} \sin \frac{hs}{2}$$

for any $h \in \mathbb{R}$. Thus, the parameter $\delta > 0$ depends on $s \in \mathbb{R}$ yet for small $h \to 0$ one gets $\delta = h(1 + O(h^2 s^2)$.

The nonlinear case is more difficult. In the continuous nonlinear case (4.25) we have

$$\frac{d\xi}{dt} = -is^2\xi - 2i\alpha\beta[s;\xi], \quad \frac{d\xi^*}{dt} = is^2\xi^* + 2i\alpha\beta^*[s;\xi^*], \tag{4.28}$$

where the functionals $\beta, \beta^*\colon \mathbb{R}\times L_2(\mathbb{R};\mathbb{C}) \to L_2(\mathbb{R};\mathbb{C})$, determined as

$$\beta[s;\xi] := \int\limits_{\mathbb{R}^2} ds'ds''\xi\left(s+s'-s''\right)\xi\left(s''\right)\xi^*\left(s'\right),$$

$$\beta^*[s;\xi] := \int\limits_{\mathbb{R}^2} ds'ds''\xi^*\left(s+s'-s''\right)\xi^*\left(s''\right)\xi\left(s'\right),$$

depend both on $s \in \mathbb{R}$ and on the element $\xi \in L_2(\mathbb{R};\mathbb{C})$, as well as depends parametrically on the evolution parameter $t \in \mathbb{R}$ through (4.28). In the nonlinear discrete case (4.27) we, respectively, obtain

$$\frac{d\xi_h}{dt} = -i\xi_h\frac{4}{\delta^2}\sin^2\frac{sh}{2} + 2i\alpha\beta_h[s;\xi_h], \quad \frac{d\xi_h^*}{dt} = i\xi_h^*\frac{4}{\delta^2}\sin^2\frac{sh}{2} + 2i\alpha\beta_h^*[s;\xi_h],$$

where the functionals $\beta_h, \beta_h^*\colon \mathbb{R}\times L_2(\mathbb{R};\mathbb{C}) \to L_2(\mathbb{R};\mathbb{C})$ are determined as

$$\beta_h[s;\xi_h] := \int\limits_{\mathbb{R}^2} ds'ds''\cos\left[h\left(s+s'-s''\right)\right]\xi_h\left(s+s'-s''\right)\xi_h\left(s''\right)\xi_h^*\left(s'\right),$$

$$\beta_h^*[s;\xi_h] := \int\limits_{\mathbb{R}^2} ds'ds''\cos\left[h\left(s+s'-s''\right)\right]\xi_h^*\left(s+s'-s''\right)\xi_h^*\left(s''\right)\xi_h(s')$$

for any $s \in \mathbb{R}$. To the regret, proceeding further with the truly nonlinear case still presists to be a nontrivial problem, yet we hope to obtain a suitable procedure analogous to that of [15, 17].

Instead of it one can analyze the related functional space constraints on the space of functions $(\bar{\xi}_h, \bar{\xi}_h^*)^\intercal \in W_{h,2}^2(\mathbb{R};\mathbb{C}^2)$, representing solutions to the discrete nonlinear equation (3.7) via the expressions (4.26), being imposed by the corresponding finite dimensional reduction scheme of Section 4.3. This procedure actually may be realized, if to consider the derived before recurrence relationships (4.18) (or similarly, (4.19)) allowing to obtain the related constraints on the space of functions $\left(\bar{\xi}_h, \bar{\xi}_h^*\right)^\intercal \in W_{h,2}^2\left(\mathbb{R};\mathbb{C}^2\right)$, but the resulting relationships prove to be much complicated and cumbersome expressions.

Thus, one can suggest the following practical numerical-analytical scheme of constructing solutions to the discrete nonlinear Schrödinger dynamical system (3.7): first to solve the Cauchy problem to the finite-dimensional system of ordinary differential equations (4.20) and (4.21), and next to substitute them into the system of recurrent algebraic relationships (4.18) and (4.19), obtaining this way the whole infinite hierarchy of the sought for solutions.

5. Conclusion. Within the presented investigation of solutions to the discrete nonlinear Schrödinger dynamical system (3.7) we have succeeded in two important points. First, we have developed an effective enough scheme of invariant reducing the infinite system of ordinary differential equations (3.7) to an equivalent finite one of ordinary differential equations with respect to the evolution parameter $t \in \mathbb{R}$. Second, we constructed a finite set of recurrent algebraic regular relationships, allowing to expand the obtained before solutions to any discrete order $n \in \mathbb{Z}$ and giving rise to the sought for solutions of the system (3.7).

It is important to mention here that within the presented analysis we have not used the Lax type representation for the discrete nonlinear Schrödinger dynamical system (3.7), whose existence was stated many years ago in [1] and whose complete solution set analysis was done in works [1, 2, 12, 50] by means of both the inverse scattering transform and the algebraic-geometric methods. Concerning the set of recurrent relationships for exact solutions to the finite-dimensional reduction of the discrete nonlinear Schrödinger dynamical system (3.7), obtained both in the presented work and in work [12], based on the corresponding Lax type representation, an interesting problem of finding between them relationship arises, and an answer to it would explain the hidden structure of the complete Liouville – Arnold integrability of the related set of the reduced ordinary differential equations.

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