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PERTURBATION THEOREMS FOR A MULTIFREQUENCY SYSTEM WITH IMPULSES

ТЕОРЕМИ ПРО ЗБУРЕННЯ ДЛЯ БАГАТОЧАСТОТНИХ ІМПУЛЬСНИХ СИСТЕМ

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The problem of preservation of a piecewise continuous invariant toroidal set for a class of multifrequency systems with impulses at nonfixed moments under perturbations of the right-hand side is considered. New theorems set constraints on perturbation terms not in the whole phase space, but only in a nonwandering set of dynamical system, to guarantee the existence of exponentially stable invariant toroidal set.

Розглянуто задачу збереження кусково-неперервної інваріантної тороїдальної множини для деякого класу багаточастотних систем з імпульсами в нефіксовані моменти часу та зі збуренням у правій частині. Нові теореми, що задають обмеження на члени збурення не на всьому фазовому просторі, а лише на неблукаючій множині динамічної системи, встановлюють існування експоненціально стійкої інваріантної тороїдальної множини.

1. Introduction. There are several mathematical frameworks to model processes that combine continuous and discontinuous behavior simultaneously. Among them we would like to emphasize dynamic equation on time scales [1] and hybrid dynamical systems [3, 4, 16]. However throughout the paper we will use the framework of impulsive differential equations proposed in [12]. This framework was first adapted to problems of qualitative analysis of multifrequency oscillations [10] and is most convenient to consider such problems (see [5, 11] and references therein for details). It also has a wide application to control theory and a variety of stability-related problems [2].

In recent years a considerable attention is paid to relaxing the conditions for preservation of invariant toroidal manifolds of multifrequency systems under perturbations of the right-hand side [8, 9]. In this paper we will develop new theorems for preservation of the invariant toroidal set of multifrequency systems with impulses at nonfixed moments. The main object of investigation is a systems of differential equation defined in the direct product of an n-dimensional torus $\mathcal{T}_m$ and an m-dimensional Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi),$$

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$$\frac{dx}{dt} = A(\varphi)x + f(\varphi), \quad \varphi \in \mathcal{T}_m \setminus \Gamma, \tag{1}$$

$$\Delta x|_{\varphi\in\Gamma} = B(\varphi)x + g(\varphi),$$

where $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$, $A, B \in C(\mathcal{T}_m)$, $f, g \in C_\Gamma(\mathcal{T}_m)$, $a \in C_{\text{Lip}}(\mathcal{T}_m)$, $C(\mathcal{T}_m)$ $(C_\Gamma(\mathcal{T}_m))$ is a space of continuous (piecewise continuous with first-kind discontinuities in Γ) functions that are 2π-periodic with respect to each of the variables φ_j, $j = 1, \ldots, m$, Γ is a smooth compact submanifold of the torus $\mathcal{T}_m$ of codimension 1.

To the best of our knowledge the problem of existence of invariant sets for such systems was first considered by Perestyuk in [10] for the case $a(\varphi) = \omega \equiv$ const. Later these results were extended by Tkachenko to a more general case [13]. In [14, 15] a concept of exponential dichotomy was considered and a problem of its preservation under small perturbation was investigated. A problem of existence of invariant sets and their smoothness properties were treated in [13–15] as well. Sufficient conditions for existence of exponential dichotomy for such class of systems were developed in [17].

In this paper we consider a narrower class of systems of type (1) in the case where the system has an exponentially stable invariant toroidal set. We prove theorems on preservation of an exponentially stable invariant set under perturbations of the right-hand side which are less restrictive than in [15]. We use the techniques proposed in [8] for multifrequency systems without impulses and extend the results of [6, 7]. The rest of the paper is organized as follows. In Section 2 we give a short introduction to the invariant tori problem for systems defined in the direct product of a torus and an Euclidean space. The main results are stated in Section 3. An example and a short discussion completes the paper.

2. Systems defined in $\mathcal{T}_m \times \mathbb{R}^n$. In this section we recall the basic approach to consider a problem of existence of invariant sets for systems (1) based on [6, 13]. Let $\varphi_t(\varphi)$ be a solution of the first equation from (1) that satisfies the initial condition $\varphi_0(\varphi) = \varphi$. A Lipschitz condition for the function $a(\varphi)$ guarantees the existence and uniqueness of such a solution.

We assume that the set Γ is a smooth submanifold of a torus of codimension 1 and is defined be the equation $\Phi(\varphi) = 0$, where Φ is a continuous scalar function. Denote by $t_i(\varphi)$ the solutions of equation $\Phi(\varphi_t(\varphi)) = 0$ that are the moments of impulsive perturbations in system (1). Assume that there exists $\theta > 0$ such that

$$t_i(\varphi) - t_{i-1}(\varphi) > \theta. \tag{2}$$

Along with system (1), consider a linear system,

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x + f(\varphi_t(\varphi)), \quad \varphi \in \mathcal{T}_m \setminus \Gamma,$$

$$\Delta x|_{\varphi\in\Gamma} = B(\varphi_t(\varphi))x + g(\varphi_t(\varphi)) \tag{3}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter. We get system (3) by substituting φ with $\varphi_t(\varphi)$ in the second and the third equations of (1).

Definition 1. *By an invariant toroidal manifold of system (1) we call a manifold that is defined by a function $u(\varphi) \in C_\Gamma(\mathcal{T}_m)$ such that the function $x(t, \varphi) = u(\varphi_t(\varphi))$ is a solution of system (3) for every $\varphi \in \mathcal{T}^m$.*

Suppose $C(\varphi) \in C_\Gamma(\mathcal{T}_m)$ is an arbitrary square matrix and $X_\tau^t(\varphi)$ is a fundamental matrix of the linear system

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x, \quad \varphi \in \mathcal{T}_m \setminus \Gamma,$$
$$\Delta x|_{\varphi\in\Gamma} = B(\varphi_t(\varphi))x \tag{4}$$

that depends on φ as a parameter.

Definition 2. *The function*

$$G_0(\tau,\varphi) = \begin{cases} X_\tau^0(\varphi)C(\varphi_\tau(\varphi)), & \tau \le 0, \\ -X_\tau^0(\varphi)(E - C(\varphi_\tau(\varphi))), & \tau > 0, \end{cases}$$

is called a Green – Samoilenko function of the invariant tori problem for system (1) if the following inequality holds:

$$\int_{-\infty}^{+\infty} \|G_0(\tau,\varphi)\|\, d\tau \le K < \infty.$$

The existence of a Green – Samoilenko function along with (2) guarantees existence of an invariant toroidal set of system (1) of the form

$$x = u(\varphi) = \int_{-\infty}^{+\infty} G_0(\tau,\varphi)f(\varphi_\tau(\varphi))d\tau + \sum_{-\infty<t_i(\varphi)<\infty} G_0(t_i(\varphi)+0,\varphi)g(\varphi_{t_i(\varphi)}(\varphi)), \quad \varphi \in \mathcal{T}_m,$$

for arbitrary $f, g \in C_\Gamma(\mathcal{T}_m)$.

Throughout this paper we will consider a special case where a fundamental matrix $X_\tau^t(\varphi)$ of system (4) satisfies the estimate

$$\left\|X_\tau^t(\varphi)\right\| \le Ke^{-\gamma(t-\tau)} \quad \text{for} \quad t \ge \tau. \tag{5}$$

From (5) it directly follows that a Green – Samoilenko function exists and has the form

$$G_0(\tau,\varphi) = \begin{cases} X_\tau^0(\varphi)C(\varphi_\tau(\varphi)), & \tau \le 0, \\ 0, & \tau > 0. \end{cases} \tag{6}$$

An invariant toroidal set then is called asymptotically stable as it is stable and attracts all trajectories from a vicinity.

3. Main results. Along with system (1), we consider a perturbed system,

$$\frac{d\varphi}{dt} = a(\varphi),$$
$$\frac{dx}{dt} = [A(\varphi) + A_1(\varphi)]x + f(\varphi), \quad \varphi \in \mathcal{T}_m \setminus \Gamma, \tag{7}$$
$$\Delta x|_{\varphi\in\Gamma} = [B(\varphi) + B_1(\varphi)]x + g(\varphi),$$

where perturbation terms $A_1, B_1 \in C(\mathcal{T}_m)$. Since the function A_1 and B_1 are continuous on compact manifold there exist $\sup_{\varphi\in\mathcal{T}_m} A_1(\varphi) = a_1$ and $\sup_{\varphi\in\mathcal{T}_m} B_1(\varphi) = b_1$. In [15] a more general problem was considered for a system that possesses an invariant toroidal set (without exponential stability property) and with a perturbation term in the first equation of (7). It was proven that the perturbation terms should be sufficiently small to guarantee the existence of an invariant toroidal set of the perturbed system. In this paper we consider the case where system (1) has an exponentially stable invariant toroidal set and develop theorems with less restrictive constraints.

Theorem 1. *Let the fundamental matrix $X_\tau^t(\varphi)$ satisfy estimate (5)*

$$\left\|X_\tau^t(\varphi)\right\| \le Ke^{-\gamma(t-\tau)} \quad \textit{for} \quad t \ge \tau$$

with some $K \ge 1, \gamma > 0$. If

$$Ka_1 + \frac{1}{\theta}\ln(1+Kb_1) < \gamma, \tag{8}$$

then system (7) has an exponentially stable invariant toroidal manifold for arbitrary $f, g \in C_\Gamma(\mathcal{T}_m)$.

Proof. The fundamental matrix of the perturbed system can be represented as

$$\Omega_0^t(\varphi) = X_0^t(\varphi) + \int_0^t X_s^t(\varphi)A_1(\varphi_s(\varphi))\Omega_0^s(\varphi)ds + \sum_{0\le t_i(\varphi)<t} X_{t_i(\varphi)}^t(\varphi)B_1(\varphi_{t_i(\varphi)}(\varphi))\Omega_0^{t_i(\varphi)}(\varphi).$$

Taking estimate (5) into account we have

$$\left\|\Omega_0^t(\varphi)\right\| \le Ke^{-\gamma t} + \int_0^t Ke^{-\gamma(t-s)}a_1\left\|\Omega_0^s(\varphi)\right\|ds + \sum_{0\le t_i(\varphi)<t} Ke^{-\gamma(t-t_i(\varphi))}b_1\left\|\Omega_0^{t_i(\varphi)}(\varphi)\right\|,$$

$$e^{\gamma t}\left\|\Omega_0^t(\varphi)\right\| \le K + \int_0^t Ke^{\gamma s}a_1\left\|\Omega_0^s(\varphi)\right\|ds + \sum_{0\le t_i(\varphi)<t} Ke^{\gamma t_i(\varphi)}b_1\left\|\Omega_0^{t_i(\varphi)}(\varphi)\right\|.$$

Utilizing a Gronwall–Bellmann type inequality for piecewise continuous functions [12] (Lemma 2) we get

$$e^{\gamma t}\left\|\Omega_0^t(\varphi)\right\| \le K(1+Kb_1)^{i(0,t)}e^{Ka_1 t},$$

where $i(a,b)$ is the number of impulsive perturbation in the interval (a,b). Finally, from (2),

$$\left\|\Omega_0^t(\varphi)\right\| \le Ke^{-(\gamma - Ka_1 - \frac{1}{\theta}\ln(1+Kb_1))t} \quad \text{for} \quad t > 0.$$

From condition (8) it directly follows that the fundamental matrix of the perturbed system satisfies an estimate of type (5) with the same constant K and possibly different $\tilde{\gamma} = \gamma - Ka_1 - \frac{1}{\theta}\ln(1+Kb_1)$. It means that there exists a Green–Samoilenko function of the form (6) and an exponentially stable invariant toroidal set for arbitrary functions $f, g \in C_\Gamma(\mathcal{T}_m)$.

Theorem 1 is proved.

Next we will relax the dwell-time condition (8). In particular we will show that it is sufficient to set restrictions on perturbations not on the whole surface of the torus $\mathcal{T}_m$, but only in a nonwandering set of trajectories of the dynamical system $\frac{d\varphi}{dt} = a(\varphi)$.

Definition 3. *A point φ is called wandering if there exist its neighbourhood $U(\varphi)$ and a positive number $T > 0$ such that*

$$U(\varphi) \cap \varphi_t(U(\varphi)) = 0 \quad for \quad t \geq T. \tag{9}$$

Let W be a set of all wandering points of a dynamical system and $\Omega = \mathcal{T}_m \setminus W$ be a set of nonwandering points. From compactness of a torus it follows that the set Ω is nonempty and compact. Since the function A_1 and B_1 are continuous on a compact set there exist $\sup_{\varphi\in\Omega} A_1(\varphi) = \tilde{a}_1$ *and* $\sup_{\varphi\in\Omega} B_1(\varphi) = \tilde{b}_1$.

Theorem 2. *Let the fundamental matrix $X_\tau^t(\varphi)$ satisfy estimate (5),*

$$\left\| X_\tau^t(\varphi) \right\| \leq K e^{-\gamma(t-\tau)} \quad for \quad t \geq \tau$$

with some $K \geq 1, \gamma > 0$. If the following dwell-time condition holds

$$K\tilde{a}_1 + \frac{1}{\theta} \ln\left(1 + K\tilde{b}_1\right) < \gamma, \tag{10}$$

then system (7) has an exponentially stable invariant toroidal set for arbitrary $f, g \in C_\Gamma(\mathcal{T}_m)$.

Proof. The fundamental matrix of the perturbed system can be represented as

$$\Omega_0^t(\varphi) = X_0^t(\varphi) + \int_0^t X_s^t(\varphi) A_1(\varphi_s(\varphi)) \Omega_0^s(\varphi) ds + \sum_{0 \leq t_i(\varphi) < t} X_{t_i(\varphi)}^t(\varphi) B_1(\varphi_{t_i(\varphi)}(\varphi)) \Omega_0^{t_i(\varphi)}(\varphi).$$

Taking estimate (5) into account we have

$$\begin{aligned} \left\| \Omega_0^t(\varphi) \right\| &\leq K e^{-\gamma t} + \int_0^t K e^{-\gamma(t-s)} \|A_1(\varphi_s(\varphi))\| \|\Omega_0^s(\varphi)\| ds + \\ &+ \sum_{0 \leq t_i(\varphi) < t} K e^{-\gamma(t - t_i(\varphi))} \|B_1(\varphi_{t_i(\varphi)}(\varphi))\| \|\Omega_0^{t_i(\varphi)}(\varphi)\|, \\ e^{\gamma t} \left\| \Omega_0^t(\varphi) \right\| &\leq K + \int_0^t K e^{\gamma s} \|A_1(\varphi_s(\varphi))\| \|\Omega_0^s(\varphi)\| ds + \\ &+ \sum_{0 \leq t_i(\varphi) < t} K e^{\gamma t_i(\varphi)} \|B_1(\varphi_{t_i(\varphi)}(\varphi))\| \|\Omega_0^{t_i(\varphi)}(\varphi)\|. \end{aligned} \tag{11}$$

Now we will employ an approach used in [8]. Let $U_\varepsilon(\Omega)$ be an ε-neighbourhood of the set Ω. We will show that for any fixed $\varepsilon > 0$ there exists a finite time $T > 0$ that does not depend on φ such that $\varphi_t(\varphi) \in U_\varepsilon(\Omega)$ for $t > T$.

Indeed, since $\mathcal{T}_m$ is a compact set and $U_\varepsilon(\Omega)$ is an open set, the set $\mathcal{T}_m \backslash U_\varepsilon(\Omega)$ is compact and consists of wandering points. Thus, for every point $\varphi \in \mathcal{T}_m \backslash U_\varepsilon(\Omega)$, there exists a neighbourhood $U(\varphi)$ satisfying condition (9) for $t \leq T(\varphi)$. Since the phase space is compact, we can choose finitely many neighbourhoods of this kind, $U_1, U_2, \ldots, U_N$, such that

$$\bigcup_{k=1,\ldots,n} U_k = \mathcal{T}_m \setminus U_\varepsilon(\Omega)$$

and denote the corresponding numbers $T(\varphi)$ by $T_1, T_2, \ldots, T_N$.

Let $\varphi \in \mathcal{T}_m \setminus U_\varepsilon(\Omega)$ be an arbitrary point from the neighbourhood U_{n_1}. According to (9), for a period of time not larger than T_{n_1}, it leaves this neighbourhood forever. Assume that it then appears in the neighbourhood U_{n_2} and leaves it for a time that does not exceed T_{n_2}, etc. Finally, for a time not greater than $\sum_{k=1}^N T_k$, the point necessarily appears in $U_\varepsilon(\Omega)$ because, according to (9), it cannot return to any of the neighbourhoods U_k, $k = 1, \ldots, N$.

Thus the time of stay of the point $\varphi \in \mathcal{T}_m \setminus U_\varepsilon(\Omega)$ is limited to

$$T = \sum_{k=1}^N T_k.$$

Since the matrices $A_1, B_1 \in C(\mathcal{T}_m)$, for any fixed $\varepsilon_a, \varepsilon_b > 0$ there exist a positive constant $\varepsilon > 0$ and a finite time T that does not depend on φ such that, for every $\varphi \in \mathcal{T}_m \setminus U_\varepsilon(\Omega)$,

$$\|A_1(\varphi_t(\varphi))\| \leq \tilde{a} + \varepsilon_a, \quad \|B_1(\varphi_t(\varphi))\| \leq \tilde{b} + \varepsilon_b \quad \text{for} \quad t \geq T.$$

Then from (11) we get

$$e^{\gamma t}\|\Omega_0^t(\varphi)\| \leq K + \int_0^T K e^{\gamma s}\|A_1(\varphi_s(\varphi))\|\|\Omega_0^s(\varphi)\|ds+$$

$$+ \sum_{0 \leq t_i(\varphi) < T} K e^{\gamma t_i(\varphi)}\|B_1(\varphi_{t_i(\varphi)}(\varphi))\|\|\Omega_0^{t_i(\varphi)}(\varphi)\| + \int_T^t K e^{\gamma s}(\tilde{a}_1 + \varepsilon_a)\|\Omega_0^s(\varphi)\|ds+$$

$$+ \sum_{T \leq t_i(\varphi) < t} K e^{\gamma t_i(\varphi)}(\tilde{b}_1 + \varepsilon_b)\|\Omega_0^{t_i(\varphi)}(\varphi)\|. \tag{12}$$

Estimating

$$K + \int_0^T K e^{\gamma s}\left\|A_1(\varphi_s(\varphi))\right\|\left\|\Omega_0^s(\varphi)\right\|ds + \sum_{0 \leq t_i(\varphi) < T} K e^{\gamma t_i(\varphi)}\left\|B_1(\varphi_{t_i(\varphi)}(\varphi))\right\|\left\|\Omega_0^{t_i(\varphi)}(\varphi)\right\| \leq \tilde{K}$$

we have

$$e^{\gamma t}\left\|\Omega_0^t(\varphi)\right\| \leq \tilde{K} + \int_T^t K e^{\gamma s}(\tilde{a}_1 + \varepsilon_a)\left\|\Omega_0^s(\varphi)\right\|ds + \sum_{T \leq t_i(\varphi) < t} K e^{\gamma t_i(\varphi)}(\tilde{b}_1 + \varepsilon_a)\left\|\Omega_0^{t_i(\varphi)}(\varphi)\right\| \leq$$

$$\leq \tilde{K} + \int_0^t Ke^{\gamma s}(\tilde{a}_1 + \varepsilon_a)\|\Omega_0^s(\varphi)\|ds + \sum_{0\leq t_i(\varphi)<t} Ke^{\gamma t_i(\varphi)}(\tilde{b}_1 + \varepsilon_a)\|\Omega_0^{t_i(\varphi)}(\varphi)\|.$$

Utilizing a Gronwall – Bellmann type inequality for a piecewise continuous function [12] ([Lemma 2) we obtain

$$e^{\gamma t}\left\|\Omega_0^t(\varphi)\right\| \leq \tilde{K}(1 + K(\tilde{b}_1 + \varepsilon_b))^{i(0,t)}e^{K(\tilde{a}_1+\varepsilon_b)t},$$

$$\left\|\Omega_0^t(\varphi)\right\| \leq \tilde{K}e^{-(\gamma - K(\tilde{a}_1+\varepsilon_a) - \frac{1}{\theta}\ln(1+K(\tilde{b}_1+\varepsilon_b)))t} \quad \text{for} \quad t > 0, \varphi \in \mathcal{T}_m \setminus U_\varepsilon(\Omega).$$

Now consider the case where $\varphi \in U_\varepsilon(\Omega)$. It means that for every initial value $\varphi \in U_\varepsilon(\Omega)$ there exists a constant $T_1(\varphi)$ such that

$$\|A_1(\varphi_t(\varphi))\| \leq \tilde{a} + \varepsilon_a, \quad \|B_1(\varphi_t(\varphi))\| \leq \tilde{b} + \varepsilon_b \quad \text{for} \quad t \in [0, T_1(\varphi)] \cup [T_1(\varphi) + T, +\infty). \tag{13}$$

Remark 1. If $\varphi \in \Omega$ it means that the trajectory never leaves the nonwandering set of the dynamical system and estimates (13) are valid for any $t \geq 0$. The same situation can happen when $\varphi \in U_\varepsilon(\Omega)$, but the trajectory $\varphi_t(\varphi)$ never leaves the neighbourhood $U_\varepsilon(\Omega)$. However next we will treat the worst case, where the trajectory that starts in $U_\varepsilon(\Omega)$ leaves it in a time $T_1(\varphi)$ that depends on φ.

Then from (11), considering sufficiently large $t > T_1(\varphi) + T$ and utilizing a Gronwall-Bellmann type inequality for piecewise continuous functions [12](Lemma 2), we get

$$e^{\gamma t}\left\|\Omega_0^t(\varphi)\right\| \leq Ke^{\int_0^t K\|A_1(\varphi_s(\varphi))\|ds} \prod_{0<t_i(\varphi)<t}\left(1 + K\left\|B_1(\varphi_{t_i(\varphi)}(\varphi))\right\|\right) \leq$$

$$\leq Ke^{K(\tilde{a}_1+\varepsilon_a)t}e^{\int_{T_1(\varphi)}^{T_1(\varphi)+T} K\|A_1(\varphi_s(\varphi))\|ds} \prod_{0\leq t_i(\varphi)<t}\Big(1 + K(\tilde{b}_1 + \varepsilon_b)\Big) \times$$

$$\times \prod_{T_1(\varphi)\leq t_i(\varphi)<T_1(\varphi)+T}\left(1 + K\left\|B_1(\varphi_{t_i(\varphi)}(\varphi))\right\|\right) \leq \bar{K}e^{K(\tilde{a}_1+\varepsilon_a)t}e^{\frac{1}{\theta}\ln(1+K(\tilde{b}_1+\varepsilon_b))t},$$

where the constant $\bar{K}$ does not depend on φ. Indeed,

$$Ke^{\int_{T_1(\varphi)}^{T_1(\varphi)+T} K\|A_1(\varphi_s(\varphi))\|ds} \prod_{T_1(\varphi)\leq t_i(\varphi)<T_1(\varphi)+T}\left(1 + K\left\|B_1(\varphi_{t_i(\varphi)}(\varphi))\right\|\right) \leq$$

$$\leq Ke^{\int_{T_1(\varphi)}^{T_1(\varphi)+T} Ka_1 ds} \prod_{T_1(\varphi)\leq t_i(\varphi)<T_1(\varphi)+T}(1 + Kb_1) \leq Ke^{Ka_1 T}(1 + Kb_1)^{\frac{1}{\theta}} = \bar{K}.$$

Then the estimate for a fundamental matrix has the form

$$\left\|\Omega_0^t(\varphi)\right\| \leq \bar{K}e^{-(\gamma - K(\tilde{a}_1+\varepsilon_a) - \frac{1}{\theta}\ln(1+K(\tilde{b}_1+\varepsilon_b)))t} \quad \text{for} \quad t > 0, \varphi \in U_\varepsilon(\Omega).$$

Finally, denoting $\hat{K} = \max\{\tilde{K}, \bar{K}\}$ we arrive at the estimate

$$\left\|\Omega_0^t(\varphi)\right\| \leq \hat{K}e^{-(\gamma - K(\tilde{a}_1+\varepsilon_a) - \frac{1}{\theta}\ln(1+K(\tilde{b}_1+\varepsilon_b)))t} \quad \text{for} \quad t > 0, \varphi \in \mathcal{T}_m.$$

If the dwell-time condition (10) holds then the fundamental matrix of the perturbed system satisfies the estimate of a type (5) but with possibly different constants K and γ since we can choose ε_a and ε_b to be arbitrarily small. It means that there exists a Green – Samoilenko function of the form (6) and an exponentially stable invariant toroidal set for arbitrary functions $f, g \in C_\Gamma(\mathcal{T}_m)$.

Theorem 2 is proved.

4. Example and discussion. Consider an example that shows the advantages of the proposed theorems.

Example.

$$\frac{d\varphi_1}{dt} = -\sin^2\frac{\varphi_1}{2}, \quad \frac{d\varphi_2}{dt} = \omega, \quad \varphi \in \mathcal{T}_2,$$
$$\frac{dx}{dt} = -x + f(\varphi), \quad \varphi \in \mathcal{T}_2 \setminus \Gamma, \quad \Delta x|_{\varphi\in\Gamma} = g(\varphi), \tag{14}$$

where ω is a constant. System (14) has an invariant toroidal manifold and the fundamental matrix $X_\tau^t(\varphi)$ satisfy the estimate (5) with constants $K = \gamma \equiv 1$

$$\|X_\tau^t(\varphi)\| \le e^{-(t-\tau)} \quad \text{for} \quad t \ge \tau.$$

Now perturb system (14) to get

$$\frac{d\varphi_1}{dt} = -\sin^2\frac{\varphi_1}{2}, \quad \frac{d\varphi_2}{dt} = \omega, \quad \varphi \in \mathcal{T}_2,$$
$$\frac{dx}{dt} = (-1 + A\sin\varphi_1)x + f(\varphi), \quad \varphi \in \mathcal{T}_2 \setminus \Gamma, \tag{15}$$
$$\Delta x\big|_{\varphi\in\Gamma} = B\sin\varphi_1 \cdot x + g(\varphi),$$

where A and B are arbitrary constants. Suppose that the set Γ is such that the estimate (2) for the moments of impulsive perturbation holds. The question is does the perturbed system (15) has an exponentially stable invariant set for arbitrary functions $f, g \in C_\Gamma(\mathcal{T}_2)$?

The following estimates for the perturbation terms hold:

$$\sup_{\varphi\in\mathcal{T}_2} A\sin\varphi_1 = A, \quad \sup_{\varphi\in\mathcal{T}_2} B\sin\varphi_1 = B.$$

The previously known perturbation theorem for a more general class of systems [15] demands the norms of the perturbations to be not more than some particular value. However by adjusting constants A and B one could make it bigger than any fixed δ. So we cannot conclude about the existence of invariant set of the perturbed system (15). However the unperturbed system (14) possesses an exponentially stable invariant toroidal set. So we can try to use Theorem 1 or Theorem 2.

A dwell-time conditions (8) has the form

$$A + \frac{1}{\theta}\ln(1 + Kb) < 1.$$

It is obvious that by adjusting the constants A and B one could make it false. It means that Theorem 1 couldn't answer in the affirmative to the example's question.

However a dynamical system on a two-dimensional torus has a very simple structure of the limit sets and recurrent trajectories. In particular the nonwandering set Ω consists of only one meridian $\varphi_1 = 0$,

$$\Omega = \{\varphi \in \mathcal{T}_2 : \varphi_1 = 0, \varphi_2 \in \mathcal{T}_1\}.$$

A point that starts on the meridian is spinning with a constant speed, all other trajectories tend to Ω by spirals. The estimates for the perturbation terms are

$$\sup_{\varphi\in\Omega} A \sin\varphi_1 = 0, \quad \sup_{\varphi\in\Omega} B \sin\varphi_1 = 0.$$

Then the dwell-time condition has the form $0 + \dfrac{1}{\theta}\ln 1 < 1$. It is obvious that for every $\dfrac{1}{\theta} < \infty$ there exist sufficiently small constants $\varepsilon_a, \varepsilon_b > 0$ such that the dwell-time condition $\varepsilon_a + \varepsilon_b < 1$ holds. From Theorem 2 it follows that system (15) has an exponentially stable invariant toroidal set for arbitrary $f, g \in C_\Gamma(\mathcal{T}_2)$ if only the time sequence of impulsive moments is such that estimate (2) holds.

Proved theorems allow to investigate the qualitative behavior of solutions of a class of impulsive systems that have a simple structure of limit sets and recurrent trajectories. The constraints of Theorem 2 are less restrictive than those of Theorem 1. A perturbed system should satisfy the dwell-time condition not for every $\varphi \in \mathcal{T}_m$, but only for $\varphi \in \Omega$. However it is worth to note that if the first equation of system (7) is $\dot{\varphi} = \omega = \mathrm{const}$, that is very frequent in applications, then its nonwandering set Ω coincides with a whole torus and Theorem 2 has no advantages compared to Theorem 1.

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