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A COUPLED SYSTEM OF NONLOCAL FRACTIONAL DIFFERENTIAL EQUATIONS WITH COUPLED AND UNCOUPLED SLIT-STRIPS INTEGRAL BOUNDARY CONDITIONS

З'ЄДНАНІ СИСТЕМИ НЕЛОКАЛЬНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРОБОВОГО ПОРЯДКУ ЗІ З'ЄДНАНИМИ ТА НЕЗ'ЄДНАНИМИ РОЗЩЕПЛЕНИМИ СМУГАМИ В ІНТЕГРАЛЬНИХ ГРАНИЧНИХ УМОВАХ

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This paper is concerned with the existence and uniqueness of solutions for a coupled system of fractional differential equations with coupled and uncoupled slit-strips integral boundary conditions. The existence and uniqueness of solutions is established by Banach's contraction principle, while the existence of solutions is derived by using Leray – Schauder's alternative. The results are explained with the aid of examples.

Розглядається існування та єдиність розв'язків з'єднаних систем нелокальних диференціальних рівнянь дробового порядку зі з'єднаними та нез'єднаними розщепленими смугами в інтегральних граничних умовах. Існування та єдиність розв'язків встановлено за допомогою теореми Банаха про стискаючі відображення. Існування розв'язків доведено з використанням альтернативи Лерея – Шаудера. Результати пояснено за допомогою прикладів.

1. Introduction. The study of boundary-value problems for linear and nonlinear differential equations is a popular field of research and finds extensive applications in a variety of disciplines of pure and applied sciences. The investigation of boundary-value problems of fractional-order has recently picked up a great momentum and a variety of results of diverse interest, ranging from theoretical to application aspects, are available in the literature on the topic. In particular, the tools of fractional calculus have revolutionized the field of mathematical modelling and the integer-order models in many physical and engineering phenomena have been transformed to their fractional-order counterparts. One of the salient features accounting for this trend is probably the nonlocal characteristic of fractional-order operators, which can describe the hereditary properties of many important materials and processes. For examples and applications in physics, chemistry, biology, biophysics, blood flow phenomena, control theory, wave propagation, signal and image processing, viscoelasticity, percolation, identification, fitting of experimental data, economics etc., we refer the reader to the books [1–3]. For some recent work on the topic, see [4–24] and the references therein. In a recent paper [25], the authors discussed some new fractional boundary-value problems with slit-strips conditions.

The investigation of coupled systems of fractional order differential equations is also very significant as such systems appear in a variety of problems of applied nature, especially in biosciences. For details and examples, the reader is referred to the papers [26–32] and the references cited therein.

In this paper, motivated by [25], we study a coupled system of nonlinear fractional differential equations:

$$\begin{aligned} {}^cD^q x(t) &= f(t, x(t), y(t)), \quad t \in [0,1], \quad 1 < q \leq 2, \\ {}^cD^p y(t) &= g(t, x(t), y(t)), \quad t \in [0,1], \quad 1 < p \leq 2, \end{aligned} \tag{1.1}$$

supplemented with coupled and uncoupled slit-strips type integral boundary conditions respectively given by

$$\begin{aligned} x(0) = 0, \quad x(\zeta) &= a\int_0^\eta y(s)ds + b\int_\xi^1 y(s)ds, \quad 0 < \eta < \zeta < \xi < 1, \\ y(0) = 0, \quad y(\zeta) &= a\int_0^\eta x(s)ds + b\int_\xi^1 x(s)ds, \quad 0 < \eta < \zeta < \xi < 1, \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} x(0) = 0, \quad x(\zeta) &= a\int_0^\eta x(s)ds + b\int_\xi^1 x(s)ds, \quad 0 < \eta < \zeta < \xi < 1, \\ y(0) = 0, \quad y(\zeta) &= a\int_0^\eta y(s)ds + b\int_\xi^1 y(s)ds, \quad 0 < \eta < \zeta < \xi < 1, \end{aligned} \tag{1.3}$$

where ${}^cD^q$, ${}^cD^p$ denote the Caputo fractional derivative of order q and p respectively, $f, g\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and a, b are real constants.

Here we remark that the differential equations with integral boundary conditions constitute an important class of boundary-value problems. The concept of coupled and uncoupled integral boundary conditions introduced in this paper is new. We can interpret these conditions physically as the contribution due to finite strips of arbitrary lengths on the given interval is related to the value of the unknown function at an arbitrary (nonlocal) position in the region off these strips. The applications of strip-slit boundary conditions, for instance, can be found in the works [33–36].

The paper is organized as follows. In Section 2, we present the main results for a coupled system of nonlinear fractional differential equations with coupled slit-strips integral boundary conditions while the results for uncoupled integral boundary conditions are discussed in Section 3. Our results rely on the standard tools of the fixed point theory and are well illustrated with the aid of examples.

2. Coupled slit-strips integral boundary conditions case. First of all, we recall definitions of fractional integral and derivative [1, 2].

Definition 2.1. *The Riemann–Liouville fractional integral of order q for a continuous function g is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\, ds, \quad q > 0,$$

provided the integral exists.

Definition 2.2. *For at least n-times continuously differentiable function $g\colon [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, \quad n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Now we prove an auxiliary result which is pivotal to define the solution for the problem (1.1), (1.2).

Lemma 2.1 (Auxiliary lemma). *Given $\phi, \psi \in C([0,1],\mathbb{R})$, the following system:*

$$\begin{gathered} {}^cD^q x(t) = \phi(t), \quad t \in [0,1], \quad 1 < q \le 2, \\ {}^cD^p y(t) = \psi(t), \quad t \in [0,1], \quad 1 < p \le 2, \\ x(0) = 0, \quad x(\zeta) = a\int_0^\eta y(s)ds + b\int_\xi^1 y(s)ds, \quad 0 < \eta < \zeta < \xi < 1, \\ y(0) = 0, \quad y(\zeta) = a\int_0^\eta x(s)ds + b\int_\xi^1 x(s)ds, \quad 0 < \eta < \zeta < \xi < 1, \end{gathered} \tag{2.1}$$

can be written in the equivalent integral equations

$$x(t) = \frac{t}{\zeta^2 - \Delta^2}\Bigg[\zeta\Bigg\{a\int_0^\eta\int_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds + b\int_\xi^1\int_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds -$$

$$- \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)}\,\phi(s)\,ds\Bigg\} + \Delta\Bigg\{a\int_0^\eta\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)\,d\tau ds +$$

$$+b\int\limits_{\xi}^{1}\int\limits_{0}^{s}\frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)\,d\tau ds-\int\limits_{0}^{\zeta}\frac{(\zeta-s)^{p-1}}{\Gamma(p)}\,\psi(s)\,ds\Bigg\}\Bigg]+$$

$$+\int\limits_{0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\,\phi(s)ds, \tag{2.2}$$

$$y(t)=\frac{t}{\zeta^2-\Delta^2}\Bigg[\Delta\Bigg\{a\int\limits_{0}^{\eta}\int\limits_{0}^{s}\frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds+b\int\limits_{\xi}^{1}\int_{0}^{s}\frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds-$$

$$-\int\limits_{0}^{\zeta}\frac{(\zeta-s)^{q-1}}{\Gamma(q)}\,\phi(s)\,ds\Bigg\}+\zeta\Bigg\{a\int\limits_{0}^{\eta}\int\limits_{0}^{s}\frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)\,d\tau ds+$$

$$+b\int\limits_{\xi}^{1}\int\limits_{0}^{s}\frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)\,d\tau ds-\int\limits_{0}^{\zeta}\frac{(\zeta-s)^{p-1}}{\Gamma(p)}\psi(s)ds\Bigg\}\Bigg]+$$

$$+\int\limits_{0}^{t}\frac{(t-s)^{p-1}}{\Gamma(p)}\,\psi(s)\,ds, \tag{2.3}$$

where

$$\Delta=\left[a\eta^2+b\left(1-\xi^2\right)\right]/2\neq 0. \tag{2.4}$$

Proof. It is well known that the general solution of the fractional differential equations in (2.1) can be written as

$$x(t)=c_0+c_1t+\int\limits_{0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\,\phi(s)\,ds, \tag{2.5}$$

$$y(t)=c_2+c_3t+\int\limits_{0}^{t}\frac{(t-s)^{p-1}}{\Gamma(p)}\,\psi(s)\,ds, \tag{2.6}$$

where $c_0, c_1\in\mathbb{R}$ are arbitrary constants.

Applying the conditions $x(0)=0$, $y(0)=0$, it is found that $c_0=0$, $c_2=0$. In view of the nonlocal conditions

$$x(\zeta)=a\int\limits_{0}^{\eta}y(s)ds+b\int\limits_{\xi}^{1}y(s)ds,\quad y(\zeta)=a\int\limits_{0}^{\eta}x(s)ds+b\int\limits_{\xi}^{1}x(s)ds,$$

we obtain a system of equations

$$\zeta c_1 - \Delta c_3 = a\int\limits_0^\eta\int\limits_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds + b\int\limits_\xi^1\int\limits_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds -$$

$$-\int\limits_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)}\,\phi(s)ds,$$

$$-\Delta c_1 + \zeta c_3 = a\int\limits_0^\eta\int\limits_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)\,d\tau ds + b\int\limits_\xi^1\int\limits_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)\,d\tau ds -$$

$$-\int\limits_0^\zeta \frac{(\zeta-s)^{p-1}}{\Gamma(p)}\,\psi(s)\,ds,$$

where

$$\Delta = \left[a\eta^2 + b\left(1-\xi^2\right)\right]/2.$$

Solving the system (2.7), (2.8), we have

$$c_1 = \frac{1}{\zeta^2-\Delta^2}\left[\zeta\left\{a\int\limits_0^\eta\int\limits_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds + b\int\limits_\xi^1\int\limits_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds -\right.\right.$$

$$\left.-\int\limits_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)}\,\phi(s)ds\right\} + \Delta\left\{a\int\limits_0^\eta\int\limits_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)\,d\tau ds +\right.$$

$$\left.\left.+b\int\limits_\xi^1\int\limits_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)\,d\tau ds - \int\limits_0^\zeta \frac{(\zeta-s)^{p-1}}{\Gamma(p)}\,\psi(s)\,ds\right\}\right],$$

$$c_3 = \frac{1}{\zeta^2-\Delta^2}\left[\Delta\left\{a\int\limits_0^\eta\int\limits_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds + b\int\limits_\xi^1\int\limits_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)}\,\psi(\tau)\,d\tau ds -\right.\right.$$

$$\left.-\int\limits_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)}\phi(s)ds\right\} + \zeta\left\{a\int\limits_0^\eta\int\limits_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)\,d\tau ds +\right.$$

$$\left.\left.+b\int\limits_\xi^1\int\limits_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}\,\phi(\tau)d\tau\,ds - \int\limits_0^\zeta \frac{(\zeta-s)^{p-1}}{\Gamma(p)}\,\psi(s)\,ds\right\}\right].$$

Substituting the values of c_0, c_1, c_2, c_3 in (2.5) and (2.6), we get (2.2) and (2.3). The converse follows by direct computation.

Lemma 2.1 is proved.

2.1. Existence results. Let us introduce the space $X = \{x(t)|x(t) \in C([0,1])\}$ endowed with the norm $\|x\| = \max\{|x(t)|, t \in [0,1]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{y(t)|y(t) \in C([0,1])\}$ be endowed with the norm $\|y\| = \max\{|y(t)|, t \in [0,1]\}$. Obviously the product space $(X \times Y, \|(x,y)\|)$ is a Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 2.1, we define an operator $T\colon X \times Y \to X \times Y$ by

$$T(x,y)(t) = \begin{pmatrix} T_1(x,y)(t) \\ \\ T_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned} T_1(x,y)(t) = \frac{t}{\zeta^2 - \Delta^2} \Bigg[\zeta \Bigg\{ a \int_0^{\eta} \int_0^{s} \frac{(s-\tau)^{p-1}}{\Gamma(p)} g(\tau, x(\tau), y(\tau))\, d\tau ds + \\ + b \int_{\xi}^{1} \int_0^{s} \frac{(s-\tau)^{p-1}}{\Gamma(p)} g(\tau, x(\tau), y(\tau))\, d\tau ds - \int_0^{\zeta} \frac{(\zeta - s)^{q-1}}{\Gamma(q)} f(s, x(s), y(s))\, ds \Bigg\} + \\ + \Delta \Bigg\{ a \int_0^{\eta} \int_0^{s} \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau, x(\tau), y(\tau))\, d\tau ds + \\ + b \int_{\xi}^{1} \int_0^{s} \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau, x(\tau), y(\tau))\, d\tau ds - \\ - \int_0^{\zeta} \frac{(\zeta - s)^{p-1}}{\Gamma(p)} g(s, x(s), y(s))\, ds \Bigg\} \Bigg] + \int_0^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), y(s))\, ds, \end{aligned}$$

$$\begin{aligned} T_2(x,y)(t) = \frac{t}{\zeta^2 - \Delta^2} \Bigg[\Delta \Bigg\{ a \int_0^{\eta} \int_0^{s} \frac{(s-\tau)^{p-1}}{\Gamma(p)} g(\tau, x(\tau), y(\tau))\, d\tau ds + \\ + b \int_{\xi}^{1} \int_0^{s} \frac{(s-\tau)^{p-1}}{\Gamma(p)} g(\tau, x(\tau), y(\tau))\, d\tau ds - \int_0^{\zeta} \frac{(\zeta - s)^{q-1}}{\Gamma(q)} f(s, x(s), y(s)) ds \Bigg\} + \\ + \zeta \Bigg\{ a \int_0^{\eta} \int_0^{s} \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau, x(\tau), y(\tau))\, d\tau ds + \end{aligned}$$

$$+ b \int_{\xi}^{1} \int_{0}^{s} \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau, x(\tau), y(\tau))\, d\tau ds -$$

$$- \int_{0}^{\zeta} \frac{(\zeta - s)^{p-1}}{\Gamma(p)} g(s, x(s), y(s)) ds \Bigg\} \Bigg] + \int_{0}^{t} \frac{(t-s)^{p-1}}{\Gamma(p)} g(s, x(s), y(s))\, ds.$$

For the sake of convenience, we set

$$M_1 = \frac{1}{\Gamma(p+1)} + \frac{1}{|\zeta^2 - \Delta^2|} \left[\frac{\zeta^{q+1}}{\Gamma(q+1)} + |\Delta||a| \frac{\eta^{q+1}}{\Gamma(q+2)} + |\Delta||b| \frac{1-\xi^{q+1}}{\Gamma(q+2)} \right], \tag{2.7}$$

$$M_2 = \frac{1}{|\zeta^2 - \Delta^2|} \left[\zeta |a| \frac{\eta^{p+1}}{\Gamma(p+2)} + \zeta |b| \frac{1-\xi^{p+1}}{\Gamma(p+2)} + |\Delta| \frac{\zeta^p}{\Gamma(p+1)} \right], \tag{2.8}$$

$$M_3 = \frac{1}{|\zeta^2 - \Delta^2|} \left[\zeta |a| \frac{\eta^{q+1}}{\Gamma(q+2)} + \zeta |b| \frac{1-\xi^{q+1}}{\Gamma(q+2)} + |\Delta| \frac{\zeta^q}{\Gamma(q+1)} \right], \tag{2.9}$$

$$M_4 = \frac{1}{\Gamma(p+1)} + \frac{1}{|\zeta^2 - \Delta^2|} \left[\frac{\zeta^{p+1}}{\Gamma(p+1)} + |\Delta||a| \frac{\eta^{p+1}}{\Gamma(p+2)} + |\Delta||b| \frac{1-\xi^{p+1}}{\Gamma(p+2)} \right], \tag{2.10}$$

and

$$M_0 = \min \Big\{ 1 - (M_1 + M_3)k_1 - (M_2 + M_4)\lambda_1, 1 - (M_1 + M_3)k_2 - (M_2 + M_4)\lambda_2 \Big\}, \tag{2.11}$$

$$k_i, \lambda_i \geq 0, \quad i = 1, 2.$$

The first result is concerned with the existence and uniqueness of solutions for the problem (1.1), (1.2) and is based on Banach's contraction mapping principle.

Theorem 2.1. *Assume that* $f, g\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions and there exist constants* $m_i, n_i, i = 1, 2,$ *such that for all* $t \in [0,1]$ *and* $u_i, v_i \in \mathbb{R}, i = 1, 2,$

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq m_1|u_1 - v_1| + m_2|u_2 - v_2|$$

and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq n_1|u_1 - v_1| + n_2|u_2 - v_2|.$$

In addition, assume that

$$(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2) < 1,$$

where $M_i, i = 1, 2, 3, 4,$ *are given by (2.7) – (2.10). Then the boundary-value problem (1.1), (1.2) has a unique solution.*

Proof. Define $\sup_{t \in [0,1]} f(t, 0, 0) = N_1 < \infty$ and $\sup_{t \in [0,1]} g(t, 0, 0) = N_2 < \infty$ such that

$$r \geq \frac{(M_1 + M_3)N_1 + (M_2 + M_4)N_2}{1 - [(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2)]}.$$

We show that $TB_r \subset B_r$, where $B_r = \{(x,y) \in X \times Y \colon \|(x,y)\| \le r\}$.

For $(x,y) \in B_r$, we have

$$|T_1(x,y)(t)| = \max_{t\in[0,1]} \left[\frac{t}{|\zeta^2 - \Delta^2|} \left[\zeta \left\{ |a| \int_0^{\eta} \int_0^{s} \frac{(s-\tau)^{p-1}}{\Gamma(p)} (|g(\tau, x(\tau), y(\tau)) - g(\tau,0,0)| + \right.\right.\right.$$

$$+ |g(\tau,0,0)|) d\tau ds + |b| \int_{\xi}^{1} \int_0^{s} \frac{(s-\tau)^{p-1}}{\Gamma(p)} (|g(\tau, x(\tau), y(\tau)) - g(\tau,0,0)| +$$

$$\left. + |g(\tau,0,0)|) d\tau ds + \int_0^{\zeta} \frac{(\zeta - s)^{q-1}}{\Gamma(q)} (|f(s, x(s), y(s)) - f(s,0,0)| + |f(s,0,0)|) ds \right\} +$$

$$+ |\Delta| \left\{ |a| \int_0^{\eta} \int_0^{s} \frac{(s-\tau)^{q-1}}{\Gamma(q)} (|f(\tau, x(\tau), y(\tau)) - f(\tau,0,0)| + |f(\tau,0,0)|) d\tau ds + \right.$$

$$+ |b| \int_{\xi}^{1} \int_0^{s} \frac{(s-\tau)^{q-1}}{\Gamma(q)} t (|f(\tau, x(\tau), y(\tau)) - f(\tau,0,0)| + |f(\tau,0,0)|) \, d\tau ds +$$

$$\left.\left. + \int_0^{\zeta} \frac{(\zeta - s)^{p-1}}{\Gamma(p)} (|g(s, x(s), y(s)) - g(s,0,0)| + |g(s,0,0)|) \, ds \right\}\right] +$$

$$\left. + \int_0^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} (|f(s, x(s), y(s)) - f(s,0,0)| + |f(s,0,0)|) ds \right] \le$$

$$\le \frac{1}{|\zeta^2 - \Delta^2|} \left[\zeta \left\{ |a| \int_0^{\eta} \int_0^{s} \frac{(s-\tau)^{p-1}}{\Gamma(p)} (n_1\|x\| + n_2\|y\| + N_2) \, d\tau ds + \right.\right.$$

$$+ |b| \int_{\xi}^{1} \int_0^{s} \frac{(s-\tau)^{p-1}}{\Gamma(p)} (n_1\|x\| + n_2\|y\| + N_2) d\tau ds +$$

$$\left. + \int_0^{\zeta} \frac{(\zeta - s)^{q-1}}{\Gamma(q)} (m_1\|x\| + m_2\|y\| + N_1) ds \right\} +$$

$$+ |\Delta| \left\{ |a| \int_0^{\eta} \int_0^{s} \frac{(s-\tau)^{q-1}}{\Gamma(q)} (m_1\|x\| + m_2\|y\| + N_1) \, d\tau ds + \right.$$

$$+ |b| \int_{\xi}^{1} \int_{0}^{s} \frac{(s-\tau)^{q-1}}{\Gamma(q)} (m_1\|x\| + m_2\|y\| + N_1)\, d\tau ds +$$

$$+ \int_{0}^{\zeta} \frac{(\zeta - s)^{p-1}}{\Gamma(p)} (n_1\|x\| + n_2\|y\| + N_2) ds \Bigg\}\Bigg] +$$

$$+ \int_{0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} (m_1\|x\| + m_2\|y\| + N_1)\, ds \le$$

$$\le \frac{1}{|\zeta^2 - \Delta^2|} \left[\zeta |a| \frac{\eta^{p+1}}{\Gamma(p+2)} + \zeta |b| \frac{1-\xi^{p+1}}{\Gamma(p+2)} + |\Delta| \frac{\zeta^p}{\Gamma(p+1)}\right] (n_1\|x\| + n_2\|y\| + N_2) +$$

$$+ \left\{ \frac{1}{|\zeta^2 - \Delta^2|} \left[\frac{\zeta^{q+1}}{\Gamma(q+1)} + |\Delta||a| \frac{\eta^{q+1}}{\Gamma(q+2)} + |\Delta||b| \frac{1-\xi^{q+1}}{\Gamma(q+2)}\right] + \frac{1}{\Gamma(q+1)} \right\} \times$$

$$\times (m_1\|x\| + m_2\|y\| + N_1) = M_2(n_1\|x\| + n_2\|y\| + N_2) + M_1(m_1\|x\| + m_2\|y\| + N_1) =$$

$$= (M_2 n_1 + M_1 m_1)\|x\| + (M_2 n_2 + M_1 m_2)\|y\| + M_2 N_2 + M_1 N_1 \le$$

$$\le (M_2 n_1 + M_1 m_1 + M_2 n_2 + M_1 m_2) r + M_2 N_2 + M_1 N_1.$$

In the same way, we can obtain that

$$|T_2(x,y)(t)| \le \left\{ \frac{1}{|\zeta^2 - \Delta^2|} \left[|\Delta||a| \frac{\eta^{p+1}}{\Gamma(p+2)} + |\Delta||b| \frac{1-\xi^{p+1}}{\Gamma(p+2)} + \zeta \frac{\zeta^{p+1}}{\Gamma(p+1)}\right] + \frac{1}{\Gamma(q+1)} \right\} \times$$

$$\times (n_1\|x\| + n_2\|y\| + N_2) + \frac{1}{|\zeta^2 - \Delta^2|} \times$$

$$\times \left[|\Delta| \frac{\zeta^q}{\Gamma(q+1)} + \zeta |a| \frac{\eta^{q+1}}{\Gamma(q+2)} + \zeta |b| \frac{1-\xi^{q+1}}{\Gamma(q+2)}\right] (m_1\|x\| + m_2\|y\| + N_1) =$$

$$= M_4(n_1\|x\| + n_2\|y\| + N_2) + M_3(m_1\|x\| + m_2\|y\| + N_1) =$$

$$= (M_4 n_1 + M_3 m_1)\|x\| + (M_4 n_2 + M_3 m_2)\|y\| + M_4 N_2 + M_3 N_1 \le$$

$$\le (M_4 n_1 + M_3 m_1 + M_4 n_2 + M_3 m_2) r + M_4 N_2 + M_3 N_1.$$

Consequently, $\|T(x,y)(t)\| \le r$.

Now for $(x_2, y_2), (x_1, y_1) \in X \times Y$, and for any $t \in [0,1]$, we get

$$|T_1(x_2,y_2)(t) - T_1(x_1,y_1)(t)| \le$$

$$\le \frac{t}{|\zeta^2 - \Delta^2|} \left[\zeta \left\{ |a| \int_0^{\eta} \int_0^{s} \frac{(s-\tau)^{p-1}}{\Gamma(p)} |g(\tau, x_2(\tau), y_2(\tau)) - g(\tau, x_1(\tau), y_1(\tau))|\, d\tau ds + \right.\right.$$

$$+|b|\int_{\xi}^{1}\int_{0}^{s}\frac{(s-\tau)^{p-1}}{\Gamma(p)}\,|g(\tau,x_2(\tau),y_2(\tau))-g(\tau,x_1(\tau),y_1(\tau))|d\tau ds+$$

$$\left.+\int_{0}^{\zeta}\frac{(\zeta-s)^{q-1}}{\Gamma(q)}|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|ds\right\}+$$

$$+|\Delta|\left\{|a|\int_{0}^{\eta}\int_{0}^{s}\frac{(s-\tau)^{q-1}}{\Gamma(q)}\,|f(\tau,x_2(\tau),y_2(\tau))-f(\tau,x_1(\tau),y_1(\tau))|d\tau ds+\right.$$

$$+|b|\int_{\xi}^{1}\int_{0}^{s}\frac{(s-\tau)^{q-1}}{\Gamma(q)}\,|f(\tau,x_2(\tau),y_2(\tau))-f(\tau,x_1(\tau),y_1(\tau))|\,d\tau ds+$$

$$\left.\left.+\int_{0}^{\zeta}\frac{(\zeta-s)^{p-1}}{\Gamma(p)}\,|g(s,x_2(s),y_2(s))-g(s,x_1(s),y_1(s))|ds\right\}\right]+$$

$$+\int_{0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\,|g(s,x_2(s),y_2(s))-g(s,x_1(s),y_1(s))|\,ds\le\frac{1}{|\zeta^2-\Delta^2|}\times$$

$$\times\left[\zeta|a|\,\frac{\eta^{p+1}}{\Gamma(p+2)}+\zeta|b|\,\frac{1-\xi^{p+1}}{\Gamma(p+2)}+|\Delta|\,\frac{\zeta^{p}}{\Gamma(p+1)}\right](n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)+$$

$$+\left\{\frac{1}{|\zeta^2-\Delta^2|}\left[\frac{\zeta^{q+1}}{\Gamma(q+1)}+|\Delta||a|\,\frac{\eta^{q+1}}{\Gamma(q+2)}+|\Delta||b|\,\frac{1-\xi^{q+1}}{\Gamma(q+2)}\right]+\frac{1}{\Gamma(q+1)}\right\}\times$$

$$\times(m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)\le M_2(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)+$$

$$+M_1(m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)=(M_2n_1+M_1m_1)\|x_2-x_1\|+$$

$$+(M_2n_2+M_1m_2)\|y_2-y_1\|,$$

and consequently we obtain

$$\|T_1(x_2,y_2)(t)-T_1(x_1,y_1)\|\le(M_2n_1+M_1m_1+M_2n_2+M_1m_2)[\|x_2-x_1\|+\|y_2-y_1\|]. \quad (2.12)$$

Similarly,

$$\|T_2(x_2,y_2)(t)-T_2(x_1,y_1)\|\le(M_4n_1+M_3m_1+M_4n_2+M_3m_2)[\|x_2-x_1\|+\|y_2-y_1\|]. \quad (2.13)$$

It follows from (2.12) and (2.13) that

$$\|T(x_2,y_2)(t)-T(x_1,y_1)(t)\|\le[(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)](\|u_2-u_1\|+\|v_2-v_1\|).$$

Since $(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)<1$, therefore, T is a contraction operator. So, by Banach's fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1.1), (1.2).

Theorem 2.1 is proved.

In the next result, we prove the existence of solutions for the problem (1.1), (1.2) by applying Leray–Schauder alternative.

Lemma 2.2 (Leray–Schauder alternative, [37, p. 4]). *Let $F\colon E\to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F)=\{x\in E\colon x=\lambda F(x)\quad \text{for some}\quad 0<\lambda<1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 2.2. *Assume that there exist real constants $k_i,\lambda_i\geq 0$, $i=1,2$, and $k_0>0$, $\lambda_0>0$ such that for any $x_i\in\mathbb{R}$, $i=1,2$, we have*

$$|f(t,x_1,x_2)|\leq k_0+k_1|x_1|+k_2|x_2|,$$

$$|g(t,x_1,x_2)|\leq \lambda_0+\lambda_1|x_1|+\lambda_2|x_2|.$$

In addition it is assumed that

$$(M_1+M_3)k_1+(M_2+M_4)\lambda_1<1\quad\text{and}\quad (M_1+M_3)k_2+(M_2+M_4)\lambda_2]<1,$$

where M_i, $i=1,2,3,4$, are given by (2.7)–(2.10). Then there exists at least one solution for the boundary-value problem (1.1), (1.2).

Proof. First, we show that the operator $T\colon X\times Y\to X\times Y$ is completely continuous. By continuity of functions f and g, the operator T is continuous.

Let $\Omega\subset X\times Y$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t,x(t),y(t))|\leq L_1,\quad |g(t,x(t),y(t))|\leq L_2\quad \forall (x,y)\in\Omega.$$

Then for any $(x,y)\in\Omega$ we have

$$|T_1(x,y)(t)|\leq \frac{t}{\zeta^2-\Delta^2}\left[\zeta\left\{|a|\int\limits_0^\eta\int\limits_0^s\frac{(s-\tau)^{p-1}}{\Gamma(p)}|g(\tau,x(\tau),y(\tau))|d\tau ds+\right.\right.$$

$$+|b|\int\limits_\xi^1\int\limits_0^s\frac{(s-\tau)^{p-1}}{\Gamma(p)}\,|g(\tau,x(\tau),y(\tau))|d\tau ds+$$

$$\left.+\int\limits_0^\zeta\frac{(\zeta-s)^{q-1}}{\Gamma(q)}\,|f(s,x(s),y(s))|ds\right\}+$$

$$+|\Delta|\left\{|a|\int\limits_0^\eta\int\limits_0^s\frac{(s-\tau)^{q-1}}{\Gamma(q)}\,|f(\tau,x(\tau),y(\tau))|\,d\tau ds+\right.$$

$$+|b|\int_{\xi}^{1}\int_{0}^{s}\frac{(s-\tau)^{q-1}}{\Gamma(q)}\,|f(\tau,x(\tau),y(\tau))|d\tau ds+\int_{0}^{\zeta}\frac{(\zeta-s)^{p-1}}{\Gamma(p)}\,|g(s,x(s),y(s))|ds\Bigg\}\Bigg]+$$

$$+\int_{0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\,|f(s,x(s),y(s))|ds\le$$

$$\le\frac{1}{|\zeta^2-\Delta^2|}\left[\zeta|a|\,\frac{\eta^{p+1}}{\Gamma(p+2)}+\zeta|b|\,\frac{1-\xi^{p+1}}{\Gamma(p+2)}+|\Delta|\,\frac{\zeta^p}{\Gamma(p+1)}\right]\,L_2+$$

$$+\left\{\frac{1}{|\zeta^2-\Delta^2|}\left[\frac{\zeta^{q+1}}{\Gamma(q+1)}+|\Delta||a|\,\frac{\eta^{q+1}}{\Gamma(q+2)}+|\Delta||b|\,\frac{1-\xi^{q+1}}{\Gamma(q+2)}\right]+\frac{1}{\Gamma(q+1)}\right\}\,L_1,$$

which implies that

$$\|T_1(x,y)\|\le\frac{1}{|\zeta^2-\Delta^2|}\left[\zeta|a|\,\frac{\eta^{p+1}}{\Gamma(p+2)}+\zeta|b|\,\frac{1-\xi^{p+1}}{\Gamma(p+2)}+|\Delta|\,\frac{\zeta^p}{\Gamma(p+1)}\right]L_2+$$

$$+\left\{\frac{1}{|\zeta^2-\Delta^2|}\;\left[\frac{\zeta^{q+1}}{\Gamma(q+1)}+|\Delta||a|\,\frac{\eta^{q+1}}{\Gamma(q+2)}+|\Delta||b|\,\frac{1-\xi^{q+1}}{\Gamma(q+2)}\right]+\frac{1}{\Gamma(q+1)}\right\}L_1=$$

$$=M_2L_2+M_1L_1.$$

Similarly, we get

$$\|T_2(x,y)\|\le\left\{\frac{1}{|\zeta^2-\Delta^2|}\left[|\Delta||a|\,\frac{\eta^{p+1}}{\Gamma(p+2)}+|\Delta||b|\,\frac{1-\xi^{p+1}}{\Gamma(p+2)}+\zeta\,\frac{\zeta^{p+1}}{\Gamma(p+1)}\right]+\frac{1}{\Gamma(q+1)}\right\}L_2+$$

$$+\frac{1}{|\zeta^2-\Delta^2|}\left[|\Delta|\,\frac{\zeta^q}{\Gamma(q+1)}+\zeta|a|\,\frac{\eta^{q+1}}{\Gamma(q+2)}+\zeta|b|\,\frac{1-\xi^{q+1}}{\Gamma(q+2)}\right]L_1=M_4L_2+M_3L_1.$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded. Next, we show that T is equicontinuous. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$. Then we have

$$|T_1(x(t_2),y(t_2))-T_1(x(t_1),y(t_1))|\le$$

$$\le\frac{1}{\Gamma(q)}\int_{0}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]|f(s,x(s),y(s))|ds+$$

$$+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}|f(s,x(s),y(s))|ds+$$

$$+\frac{t_2-t_1}{\zeta^2-\Delta^2}\left[\zeta\left\{|a|\int_{0}^{\eta}\int_{0}^{s}\frac{(s-\tau)^{p-1}}{\Gamma(p)}\,|g(\tau,x(\tau),y(\tau))|d\tau ds+\right.\right.$$

$$+|b|\int_{\xi}^{1}\int_{0}^{s}\frac{(s-\tau)^{p-1}}{\Gamma(p)}\,|g(\tau,x(\tau),y(\tau))|\,d\tau ds+\int_{0}^{\zeta}\frac{(\zeta-s)^{q-1}}{\Gamma(q)}\,|f(s,x(s),y(s))|ds\Bigg\}+$$

$$+|\Delta|\Bigg\{|a|\int_{0}^{\eta}\int_{0}^{s}\frac{(s-\tau)^{q-1}}{\Gamma(q)}\,|f(\tau,x(\tau),y(\tau))|d\tau ds+$$

$$+|b|\int_{\xi}^{1}\int_{0}^{s}\frac{(s-\tau)^{q-1}}{\Gamma(q)}\,|f(\tau,x(\tau),y(\tau))|d\tau ds+\int_{0}^{\zeta}\frac{(\zeta-s)^{p-1}}{\Gamma(p)}\,|g(s,x(s),y(s))|ds\Bigg\}\Bigg]\leq$$

$$\leq\frac{L_1}{\Gamma(q)}\int_{0}^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]ds+\frac{L_1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds+$$

$$+\frac{t_2-t_1}{|\zeta^2-\Delta^2|}\Bigg\{\left[\zeta|a|\frac{\eta^{p+1}}{\Gamma(p+2)}+\zeta|b|\,\frac{1-\xi^{p+1}}{\Gamma(p+2)}+|\Delta|\,\frac{\zeta^{p}}{\Gamma(p+1)}\right]L_2+$$

$$+\left[\frac{\zeta^{q+1}}{\Gamma(q+1)}+|\Delta||a|\,\frac{\eta^{q+1}}{\Gamma(q+2)}+|\Delta||b|\,\frac{1-\xi^{q+1}}{\Gamma(q+2)}\right]L_1\Bigg\}.$$

Analogously, we can obtain

$$|T_2(x(t_2),y(t_2))-T_2(x(t_1),y(t_1))|\leq$$

$$\leq\frac{L_2}{\Gamma(p)}\int_{0}^{t_1}\left[(t_2-s)^{p-1}-(t_1-s)^{p-1}\right]ds+\frac{L_2}{\Gamma(p)}\int_{t_1}^{t_2}(t_2-s)^{p-1}ds+$$

$$+\frac{t_2-t_1}{|\zeta^2-\Delta^2|}\Bigg\{\left[|\Delta||a|\,\frac{\eta^{p+1}}{\Gamma(p+2)}+|\Delta||b|\,\frac{1-\xi^{p+1}}{\Gamma(p+2)}+\zeta\,\frac{\zeta^{p+1}}{\Gamma(p+1)}\right]L_2+$$

$$+\left[|\Delta|\frac{\zeta^{q}}{\Gamma(q+1)}+\zeta|a|\,\frac{\eta^{q+1}}{\Gamma(q+2)}+\zeta|b|\,\frac{1-\xi^{q+1}}{\Gamma(q+2)}\right]L_1\Bigg\}.$$

Therefore, the operator $T(x,y)$ is equicontinuous, and thus the operator $T(x,y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E}=\{(x,y)\in X\times Y|(x,y)=\lambda T(x,y),0\leq\lambda\leq1\}$ is bounded. Let $(x,y)\in\mathcal{E}$, then $(x,y)=\lambda T(x,y)$. For any $t\in[0,1]$ we have

$$x(t)=\lambda T_1(x,y)(t),\quad y(t)=\lambda T_2(x,y)(t).$$

Then

$$|x(t)|\leq\frac{1}{\zeta^2-\Delta^2}\Bigg[\zeta\Bigg\{|a|\int_{0}^{\eta}\int_{0}^{s}\frac{(s-\tau)^{p-1}}{\Gamma(p)}\,|g(\tau,x(\tau),y(\tau))|d\tau ds+$$

$$+ |b| \int_\xi^1 \int_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)} |g(\tau, x(\tau), y(\tau))| d\tau ds + \int_0^\zeta \frac{(\zeta - s)^{q-1}}{\Gamma(q)} |f(s, x(s), y(s))| ds \Bigg\} +$$

$$+ |\Delta| \Bigg\{ |a| \int_0^\eta \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} |f(\tau, x(\tau), y(\tau))| d\tau ds +$$

$$+ |b| \int_\xi^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} |f(\tau, x(\tau), y(\tau))| d\tau ds + \int_0^\zeta \frac{(\zeta - s)^{p-1}}{\Gamma(p)} |g(s, x(s), y(s))| ds \Bigg\} \Bigg] +$$

$$+ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s), y(s))| ds \le \frac{1}{|\zeta^2 - \Delta^2|} \times$$

$$\times \left[\zeta |a| \frac{\eta^{p+1}}{\Gamma(p+2)} + \zeta |b| \frac{1 - \xi^{p+1}}{\Gamma(p+2)} + |\Delta| \frac{\zeta^p}{\Gamma(p+1)} \right] (\lambda_0 + \lambda_1 \|x\| + \lambda_2 \|y\|) +$$

$$+ \left\{ \frac{1}{|\zeta^2 - \Delta^2|} \left[\frac{\zeta^{q+1}}{\Gamma(q+1)} + |\Delta||a| \frac{\eta^{q+1}}{\Gamma(q+2)} + |\Delta||b| \frac{1 - \xi^{q+1}}{\Gamma(q+2)} \right] + \frac{1}{\Gamma(q+1)} \right\} \times$$

$$\times (k_0 + k_1 \|x\| + k_2 \|y\|)$$

and

$$|y(t)| \le \left\{ \frac{1}{|\zeta^2 - \Delta^2|} \left[|\Delta||a| \frac{\eta^{p+1}}{\Gamma(p+2)} + |\Delta||b| \frac{1 - \xi^{p+1}}{\Gamma(p+2)} + \zeta \frac{\zeta^{p+1}}{\Gamma(p+1)} \right] + \frac{1}{\Gamma(q+1)} \right\} \times$$

$$\times (\lambda_0 + \lambda_1 \|x\| + \lambda_2 \|y\|) + \frac{1}{|\zeta^2 - \Delta^2|} \left[|\Delta| \frac{\zeta^q}{\Gamma(q+1)} + \zeta |a| \frac{\eta^{q+1}}{\Gamma(q+2)} + \zeta |b| \frac{1 - \xi^{q+1}}{\Gamma(q+2)} \right] \times$$

$$\times (k_0 + k_1 \|x\| + k_2 \|y\|).$$

Hence we have

$$\|x\| \le M_1(k_0 + k_1 \|x\| + k_2 \|y\|) + M_2(\lambda_0 + \lambda_1 \|x\| + \lambda_2 \|y\|)$$

and

$$\|y\| \le M_3(k_0 + k_1 \|x\| + k_2 \|y\|) + M_4(\lambda_0 + \lambda_1 \|x\| + \lambda_2 \|y\|),$$

which imply that

$$\|x\| + \|y\| \le (M_1 + M_3) k_0 + (M_2 + M_4) \lambda_0 + [(M_1 + M_3) k_1 + (M_2 + M_4) \lambda_1] \|x\| +$$

$$+ [(M_1 + M_3) k_2 + (M_2 + M_4) \lambda_2] \|y\|.$$

Consequently,

$$\|(x,y)\| \leq \frac{(M_1+M_3)k_0+(M_2+M_4)\lambda_0}{M_0},$$

for any $t \in [0,1]$, where M_0 is defined by (2.11), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 2.2, the operator T has at least one fixed point. Hence the boundary-value problem (1.1), (1.2) has at least one solution.

Theorem 2.2 is proved.

2.2. Examples. Example 2.1. Consider the following system of coupled fractional differential equations with slit-strips integral boundary conditions:

$$\begin{aligned}
{}^cD^{5/4}x(t) &= \frac{2}{55}\,x(t) + \frac{3}{61}\,\frac{|y(t)|}{(1+|y(t)|)} + \frac{3}{2}, \quad t \in [0,1],\\
{}^cD^{3/2}y(t) &= \frac{1}{27}\,\frac{|\cos x(t)|}{(1+|\cos x(t)|)} + \frac{2}{41}\,\sin y(t) + 3, \quad t \in [0,1],\\
x(0) = 0, &\quad x(1/2) = \int\limits_0^{1/3} y(s)ds + \int\limits_{2/3}^{1} y(s)ds,\\
y(0) = 0, &\quad y(1/2) = \int\limits_0^{1/3} x(s)ds + \int\limits_{2/3}^{1} x(s)ds.
\end{aligned} \tag{2.14}$$

Here $q = 5/4$, $p = 3/2$, $a = 1$, $b = 1$, $\zeta = 1/2$, $\eta = 1/3$, $\xi = 2/3$. With the given values, it is found that $\Delta = 1/3$, $m_1 = 2/55$, $m_2 = 3/61$, $n_1 = 1/27$, $n_2 = 2/41$, $M_1 \simeq 2.731029$, $M_2 \simeq 1.397944$, $M_3 \simeq 1.854888$, $M_4 \simeq 2.216142$, and

$$(M_1+M_3)(m_1+m_2) + (M_2+M_4)(n_1+n_2) \simeq 0.702454 < 1.$$

Thus all the conditions of Theorem 2.1 are satisfied. Therefore, by the conclusion of Theorem 2.1, the problem (2.14) has a unique solution on $[0,1]$.

Example 2.2. Let us consider the problem (2.14) with the following values:

$$f(t,x(t),y(t)) = \frac{1}{2} + \frac{2}{41}\,\sin x(t) + \frac{2}{43\pi}\,y(t)\tan^{-1}x(t),$$

$$g(t,x(t),y(t)) = \frac{2}{3} + \frac{1}{11}\,x(t) + \frac{1}{17}\,\sin y(t).$$

Clearly $|f(t,x,y)| \leq k_0 + k_1|x| + k_2|y|$, $|g(t,x,y)| = \lambda_0 + \lambda_1|x| + \lambda_2|y|$, where $k_0 = 1/2$, $k_1 = 2/41$, $k_2 = 1/43$, $\lambda_0 = 2/3$, $\lambda_1 = 1/11$, $\lambda_2 = 1/17$. Furthermore,

$$(M_1+M_3)k_1+(M_2+M_4)\lambda_1 \simeq 0.552257 < 1, \quad (M_1+M_3)k_2+(M_2+M_4)\lambda_2 \simeq 0.319243 < 1.$$

Thus all the conditions for Theorem 2.2 hold true and consequently the conclusion of Theorem 2.2 applies to the problem (2.14) with the given values of $f(t,x,y)$ and $g(t,x,y)$.

3. Uncoupled slit-strips integral boundary conditions case. In relation to the problem (1.1)–(1.3), we consider the following lemma.

Lemma 3.1 (Auxiliary lemma). *For* $\chi \in C([0,1],\mathbb{R})$, *the unique solution of the problem*

$$
\begin{gathered}
{}^cD^q x(t) = \chi(t), \quad 1 < q \le 2, \quad t \in [0,1], \\
x(0) = 0, \quad x(\zeta) = a\int_0^\eta x(s)ds + b\int_\xi^1 x(s)ds, \quad 0 < \eta < \zeta < \xi < 1,
\end{gathered} \tag{3.1}
$$

is given by

$$
x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\chi(s)ds + \frac{t}{A}\Bigg\{a\int_0^\eta\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}\chi(\tau)\,d\tau ds +
$$

$$
+ b\int_\xi^1\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}\chi(\tau)\,d\tau ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)}\chi(s)\,d\,ds\Bigg\}, \tag{3.2}
$$

where

$$
A = \zeta - \frac{a\eta^2}{2} - \frac{b(1-\xi^2)}{2} \neq 0. \tag{3.3}
$$

Proof. We just provide the outline of the proof. The general solution of the fractional differential equation in (3.1) can be written as

$$
x(t) = e_0 + e_1 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}y(s)\,ds, \tag{3.4}
$$

where $e_0, e_1 \in \mathbb{R}$ are arbitrary constants. Applying the given boundary conditions, we find that $e_0 = 0$, and

$$
e_1 = \frac{1}{A}\Bigg\{a\int_0^\eta\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}y(\tau)\,d\tau\,ds + b\int_\xi^1\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}y(\tau)d\tau ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)}y(s)\,d\,ds\Bigg\}.
$$

Substituting the values of e_0, e_1 in (3.4), we get (3.2).

Lemma 3.1 is proved.

3.1. Existence results for uncoupled case. In view of Lemma 3.1, we define an operator $\mathfrak{T}\colon X\times Y \to X\times Y$ by

$$
\mathfrak{T}(u,v)(t) = \begin{pmatrix} \mathfrak{T}_1(u,v)(t) \\ \\ \mathfrak{T}_2(u,v)(t) \end{pmatrix},
$$

where

$$\mathfrak{T}_1(u,v)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s),v(s))\,ds + \frac{t}{A}\Bigg\{ a\int_0^\eta\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau,u(\tau),v(\tau))\,d\tau ds +$$

$$+\, b\int_\xi^1\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau,u(\tau),v(\tau))\,d\tau ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} f(s,u(s),v(s))\,ds \Bigg\}$$

and

$$\mathfrak{T}_2(u,v)(t) = \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} h(s,u(s),v(s))\,ds + \frac{t}{A}\Bigg\{ a\int_0^\eta\int_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)} h(\tau,u(\tau),v(\tau))\,d\tau ds +$$

$$+\, b\int_\xi^1\int_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p)} h(\tau,u(\tau),v(\tau))\,d\tau ds - \int_0^\zeta \frac{(\zeta-s)^{p-1}}{\Gamma(p)} f(s,u(s),v(s))ds \Bigg\}.$$

In the sequel, we set

$$\mu_1 = \frac{1}{\Gamma(q+1)} + \frac{1}{|A|}\left\{ |a|\frac{\eta^{q+1}}{\Gamma(q+2)} + |b|\frac{1-\xi^{q+1}}{\Gamma(q+2)} + \frac{\zeta^q}{\Gamma(q+1)} \right\}, \tag{3.5}$$

$$\mu_2 = \frac{1}{\Gamma(p+1)} + \frac{1}{|A|}\left\{ |a|\frac{\eta^{p+1}}{\Gamma(p+2)} + |b|\frac{1-\xi^{p+1}}{\Gamma(p+2)} + \frac{\zeta^p}{\Gamma(p+1)} \right\}. \tag{3.6}$$

Now we present the existence and uniqueness result for the problem (1.1)–(1.3). We do not provide the proof of this result as it is similar to the one for Theorem 2.1.

Theorem 3.1. *Assume that $f,g\colon [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $\bar{m}_i, \bar{n}_i$, $i=1,2$, such that for all $t\in[0,1]$ and $u_i, v_i\in\mathbb{R}$, $i=1,2$,*

$$|f(t,u_1,u_2) - g(t,v_1,v_2)| \le \bar{m}_1|u_1-v_1| + \bar{m}_2|u_2-v_2|$$

and

$$|g(t,u_1,u_2) - h(t,v_1,v_2)| \le \bar{n}_1|u_1-v_1| + \bar{n}_2|u_2-v_2|.$$

In addition, assume that

$$\mu_1(\bar{m}_1+\bar{m}_2) + \mu_2(\bar{n}_1+\bar{n}_2) < 1,$$

where μ_1 and μ_2 are given by (3.5) and (3.6) respectively. Then the boundary-value problem (1.1)–(1.3) has a unique solution.

Example 3.1. Consider the following system of coupled fractional differential equations with uncoupled slit-strips integral boundary conditions

$$
\begin{gathered}
{}^cD^{5/4}x(t) = \frac{|x(t)|}{24(1+|x(t)|)} + \frac{1}{20}\tan^{-1}y + 1, \quad t \in [0,1], \\
{}^cD^{3/2}y(t) = \frac{1}{35}\sin x(t) + \frac{1}{25}y(t) + 4, \quad t \in [0,1], \\
x(0) = 0, \quad x(1/2) = \int_0^{1/3} x(s)ds + \int_{2/3}^{1} x(s)ds, \\
y(0) = 0, \quad y(1/2) = \int_0^{1/3} y(s)ds + \int_{2/3}^{1} y(s)ds.
\end{gathered}
\tag{3.7}
$$

Here $q = 5/4$, $p = 3/2$, $a = 1$, $b = 1$, $\zeta = 1/2$, $\eta = 1/3$, $\xi = 2/3$. With the given values, it is found that $A = 1/6$, $\bar{m}_1 = 1/24$, $\bar{m}_2 = 1/20$, $\bar{n}_1 = 1/35$, $\bar{n}_2 = 1/25$, $\mu_1 \simeq 4.716276$, $\mu_2 \simeq$ $\simeq 3.614087$. In consequence, $\mu_1(\bar{m}_1+\bar{m}_2)+\mu_2(\bar{n}_1+\bar{n}_2) \simeq 0.680148 < 1$. Thus all the conditions of Theorem 3.1 are satisfied. Therefore, there exists a unique solution for the problem (3.7) on $[0, 1]$.

The second result dealing with the existence of solutions for the problem (1.1) – (1.3) is analogous to Theorem 2.2 and is given below.

Theorem 3.2. *Assume that there exist real constants $\rho_i, \nu_i \geq 0$, $i = 1, 2$, and $\rho_0 > 0$, $\nu_0 > 0$ such that for any $x_i \in \mathbb{R}$, $i = 1, 2$, we have*

$$
|f(t, x_1, x_2)| \leq \rho_0 + \rho_1|x_1| + \rho_2|x_2|,
$$

$$
|g(t, x_1, x_2)| \leq \nu_0 + \nu_1|x_1| + \nu_2|x_2|.
$$

In addition it is assumed that

$$
\mu_1\rho_1 + \mu_2\nu_1 < 1 \quad \textit{and} \quad \mu_1\rho_2 + \mu_2\nu_2 < 1,
$$

where μ_1 and μ_2 are given by (3.5) and (3.6) respectively. Then the boundary-value problem (1.1) – (1.3) has at least one solution.

Proof. Setting

$$
\mu_0 = \min\{1 - (\mu_1\rho_1 + \mu_2\nu_1),\ 1 - (\mu_1\rho_2 + \mu_2\nu_2)\}, \ \ \rho_i,\ \nu_i \geq 0, i = 1, 2,
$$

with μ_1 and μ_2 given by (3.5) and (3.6) respectively, the proof is similar to that of Theorem 2.2. So we omit it.

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