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EXPONENTIAL DICHOTOMY AND EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS

ЕКСПОНЕНЦІАЛЬНА ДИХОТОМІЯ ТА ІСНУВАННЯ МАЙЖЕ ПЕРІОДИЧНИХ РОЗВ'ЯЗКІВ ІМПУЛЬСНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

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We obtain conditions for existence of piecewise continuous almost periodic solutions of a system of impulsive differential equations with exponentially dichotomous linear part. The robustness of exponential dichotomy and exponential contraction for linear systems with small perturbations of right-hand sides and points of impulsive action are studied.

Отримано умови існування кусково-неперервних майже періодичних розв'язків систем диференціальних рівнянь з імпульсною дією та експоненціально дихотомічною лінійною частиною. Вивчено грубість експоненціальної дихотомії та експоненціальної стійкості лінійних систем при малих збуреннях правих частин та точок імпульсної дії.

1. Introduction. We investigate the problem of existence of a piecewise continuous almost periodic solution for the semilinear impulsive differential equation

$$\frac{du}{dt} = A(t)u + f(t,u), \quad t \neq \tau_j, \tag{1}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j) + g_j(u(t_j)), \quad j \in \mathbb{Z}, \tag{2}$$

where $u : \mathbb{R} \to \mathbb{R}^n$. We use the concept of discontinuous almost periodic functions in the sense of [1, 2]. There are many works (see, e.g., [3–6] and references given there) devoted to a study of almost periodic solutions of impulsive systems.

We assume that the corresponding linear homogeneous equation (if $f \equiv 0, g_j \equiv 0$) has an exponential dichotomy. Matrices $(I + B_j)$ may degenerate, $\det(I + B_j) = 0$, for some (or all) $j \in \mathbb{Z}$ therefore, solutions of the system are not extendable to the negative semiaxis or are ambiguously extendable. Defining exponential dichotomy we require that only solutions of linear system from the unstable manifold can be unambiguously extended to the negative semiaxis. This corresponds to the definition of exponential dichotomy for evolution equations in an infinite dimensional Banach space [7–9].

Robustness is an impotent property of exponential dichotomy [8–10]. We mention articles [11–14] where the robustness of exponential dichotomy for impulsive systems by small perturbations in the right-hand sides is proved. In this article we prove robustness of exponential dichotomy also by small perturbation of points of the impulsive action. We use change of time in the system. Then approximation of the impulsive system by difference systems (see [7]) can be used.

2. Preliminaries and main results. Let X be an abstract Banach space and $\mathbb{R}$ and $\mathbb{Z}$ be the sets of real and integer numbers, respectively.

We will consider the space $\mathcal{PC}(J, X)$, $J \subset \mathbb{R}$, of all piecewise continuous functions $x : J \to \to X$ such that

i) the set $T = \{\tau_j \in J : \tau_{j+1} > \tau_j, j \in \mathbb{Z}\}$ is the set of discontinuities of x;

ii) $x(t)$ is left-continuous, $x(t_j - 0) = x(t_j)$, and there exists $\lim_{t \to t_j + 0} x(t) = x(t_j + 0) < \infty$.

Definition 1. *A strictly increasing sequence $\{\tau_k\}$ of real numbers has uniformly almost periodic sequences of differences if for any $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods common for all the sequences $\{\tau_k^j\}$, where $\tau_k^j = \tau_{k+j} - \tau_k$, $j \in \mathbb{Z}$.*

Recall that an integer p is called an ε-almost period of a sequence $\{x_k\}$ if $\|x_{k+p} - x_k\| < \varepsilon$ for any $k \in \mathbb{Z}$. A sequence $\{x_k\}$ is almost periodic if for any $\varepsilon > 0$ there exists a relatively dense set of its ε-almost periods.

Definition 2. *A function $\varphi(t) \in \mathcal{PC}(\mathbb{R}, X)$ is said to be W-almost periodic if*

i) the sequence $\{\tau_k\}$ of discontinuities of $\varphi(t)$ has uniformly almost periodic sequences of differences;

ii) for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity and $|t' - t''| < \delta$ then $\|\varphi(t') - \varphi(t'')\| < \varepsilon$;

iii) for any $\varepsilon > 0$ there exists a relatively dense set Γ of ε-almost periods such that if $\tau \in \Gamma$, then $\|\varphi(t + \tau) - \varphi(t)\| < \varepsilon$ for all $t \in \mathbb{R}$ which satisfy the condition $|t - \tau_k| \geq \varepsilon$, $k \in \mathbb{Z}$.

We consider the impulsive equation (1), (2) with the following assumptions:

(H_1) the matrix-valued function $A(t)$ is Bohr almost periodic,

(H_2) the sequence of real numbers τ_k has uniformly almost periodic sequences of differences, and there exists $\theta > 0$ such that $\inf_k \tau_k^1 = \theta > 0$,

(H_3) the sequence $\{B_j\}$ of $(n \times n)$-matrices is almost periodic,

(H_4) we shell use the notation $U_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$; the function $f(t, u) : \mathbb{R} \times \mathbb{R}^n \to \to \mathbb{R}^n$ is continuous in u and is W-almost periodic in t uniformly with respect to $u \in U_\rho$ with some $\rho > 0$,

(H_5) the sequence $\{g_j(u)\}$ of continuous functions $U_\rho \to \mathbb{R}^n$ is almost periodic uniformly with respect to $u \in U_\rho$.

By Lemma 22 [9, p. 192] for a sequence $\{\tau_j\}$ with uniformly almost periodic sequences of differences there exists the limit

$$\lim_{T \to \infty} \frac{\mathfrak{i}(t, t + T)}{T} = p$$

uniformly with respect to $t \in \mathbb{R}$, where $\mathfrak{i}(s, t)$ is the number of the points τ_k lying in the interval (s, t).

The next lemma is proved in [9].

Lemma 1. *Assume that the sequence of real numbers $\{\tau_j\}$ has uniformly almost periodic sequences of differences, the sequence $\{B_j\}$ is almost periodic and the function $f(t) : \mathbb{R} \to \mathbb{R}^n$ is W-almost periodic. Then for any $\varepsilon > 0$ there exist a real number ν, $0 < \nu < \varepsilon$, and relatively dense sets of real numbers Γ and integers Q such that the following relations hold:*

$$\|f(t + r) - f(t)\| < \varepsilon, \quad t \in \mathbb{R}, \quad |t - \tau_j| > \varepsilon, \quad j \in \mathbb{Z},$$

$$\|B_{k+q} - B_k\| < \varepsilon, \quad \|\tau_k^q - r\| < \nu,$$

for $k \in \mathbb{Z}$, $r \in \Gamma$, $q \in Q$.

Definition 3. *A function* $x(t) : [t_0, t_1] \to \mathbb{R}^n$ *is said to be a solution of the initial problem* $u(t_0) = u_0 \in \mathbb{R}^n$ *for the equation (1), (2) on* $[t_0, t_1]$ *if*

(i) it is continuous in $[t_0, \tau_k], (\tau_k, \tau_{k+1}], \ldots, (t_{k+s}, t_1]$ *with discontinuities of the first kind at the moments* $t = \tau_j$,

(ii) $x(t)$ *is continuously differentiable in each of the intervals* $(t_0, \tau_k), (\tau_k, \tau_{k+1}), \ldots, (t_{k+s}, t_1)$ *and satisfies the equations (1) and (2) if* $t \in (t_0, t_1)$, $t \neq \tau_j$ *and* $t = \tau_j$ *respectively,*

(iii) the initial value condition $u(t_0) = u_0$ *is fulfilled.*

Together with equation (1), (2) we consider the linear homogeneous equation

$$\frac{du}{dt} = A(t)u, \quad t \neq \tau_j, \tag{3}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j), \quad j \in \mathbb{Z}. \tag{4}$$

Denote by $X(t, s)$ the evolution operator of the linear equation without impulses (3). It satisfies $X(\tau, \tau) = I$, $X(t, s)X(s, \tau) = X(t, \tau)$, $t, s, \tau \in \mathbb{R}$.

We define an evolution operator for equation (3), (4) by

$$U(t, s) = X(t, s) \quad \text{if} \quad \tau_k < s \leq t \leq \tau_{k+1},$$

and

$$U(t, s) = X(t, \tau_k)(I + B_k)X(\tau_k, \tau_{k-1}) \ldots (I + B_m)X(\tau_m, s), \tag{5}$$

if $\tau_{m-1} < s \leq \tau_m < \tau_{m+1} < \ldots < \tau_k < t \leq \tau_{k+1}$.

Definition 4. *We say that system (3), (4) has an exponential dichotomy on* $\mathbb{R}$ *with exponent* $\beta > 0$ *and bound* $M \geq 1$ *if there exist projections* $P(t)$, $t \in \mathbb{R}$, *such that*

(i) $U(t, s)P(s) = P(t)U(t, s)$, $t \geq s$;

(ii) $U(t, s)|_{\mathrm{Im}\,(P(s))}$ *for* $t \geq s$ *is an isomorphism on* $\mathrm{Im}\,(P(s))$, *then* $U(s, t)$ *is defined as an inverse map from* $\mathrm{Im}\,(P(t))$ *to* $\mathrm{Im}\,(P(s))$;

(iii) $\|U(t, s)(1 - P(s))\| \leq Me^{-\beta(t-s)}$, $t \geq s$;

(iv) $\|U(t, s)P(s)\| \leq Me^{\beta(t-s)}$, $t \leq s$.

Now we formulate our main result.

Theorem 1. *Suppose that system (1), (2) satisfies assumptions* (H_1) – (H_5), *linear system (3), (4) is exponentially dichotomous with constants* β *and* $M \geq 1$.

Assume that the functions $f(t, u)$ *and* $g_j(u)$ *satisfy the Lipschitz condition*

$$\|f(t, u_1) - f(t, u_2)\| \leq L\|u_1 - u_2\|, \quad \|g_j(u_1) - g_j(u_2)\| \leq L\|u_1 - u_2\|, \quad j \in \mathbb{Z},$$

with a positive constant L and are uniformly bounded in the region $t \in \mathbb{R}$, $u \in U_\rho$:

$$\sup_{(t,u)} \|f(t, u)\| \leq H < \infty, \quad \sup_{u} \|g_j(u)\| \leq H < \infty, \quad j \in \mathbb{Z}.$$

Then for a sufficiently small L the system (1), (2) has a unique piecewise continuous W-almost periodic solution.

3. Robustness of exponential dichotomy. If system (3), (4) has an exponential dichotomy on $\mathbb{R}$, then the nonhomogeneous equation

$$\frac{du}{dt} = A(t)u + f(t), \quad t \neq \tau_j, \tag{6}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j) + g_j, \quad j \in \mathbb{Z}, \tag{7}$$

has a unique solution bounded on $\mathbb{R}$,

$$u_0(t) = \int\limits_{-\infty}^{\infty} G(t,s) f(s)(x) ds + \sum_{j \in \mathbb{Z}} G(t, \tau_j) g_j, \tag{8}$$

where

$$G(t,s) = \begin{cases} U(t,s)(I - P(s)), & t \geq s, \\ -U(t,s)P(s), & t < s, \end{cases}$$

is a Green function such that

$$\|G(t,s)\| \leq Me^{-\beta|t-s|}, \quad t, s \in \mathbb{R}. \tag{9}$$

Analogously to [7, p. 250] it can be proved that a function $u(t)$ is a bounded solution on the semiaxis $[t_0, +\infty)$ if and only if

$$u(t) = U(t,t_0)(I - P(t_0))u(t_0) + \int\limits_{t_0}^{+\infty} G(t,s) f(s) ds + \sum_{t_0 \leq \tau_j} G(t, \tau_j) g_j, \quad t \geq t_0. \tag{10}$$

A function $u(t)$ is a bounded solution on the semiaxis $(-\infty, t_0]$ if and only if

$$u(t) = U(t,t_0)P(t_0)u(t_0) + \int\limits_{-\infty}^{t_0} G(t,s) f(s) ds + \sum_{t_0 > \tau_j} G(t, \tau_j) g_j, \quad t \leq t_0. \tag{11}$$

Lemma 2. *Let the impulsive system (3), (4) be exponentially dichotomous with positive constants β and M. Then there exists $\delta_0 > 0$ such that the perturbed systems*

$$\frac{du}{dt} = \tilde{A}(t)u, \quad t \neq \tilde{\tau}_j, \tag{12}$$

$$\Delta u|_{t=\tilde{\tau}_j} = u(\tilde{\tau}_j + 0) - u(\tilde{\tau}_j) = \tilde{B}_j u(\tilde{\tau}_j), \quad j \in \mathbb{Z}, \tag{13}$$

with $\sup_j |\tau_j - \tilde{\tau}_j| \leq \delta_0$, $\sup_j \|B_j - \tilde{B}_j\| \leq \delta_0$, $\sup_t \|A(t) - \tilde{A}(t)\| \leq \delta_0$, *are also exponentially dichotomous with some constants $\beta_1 \leq \beta$ and $M_1 \geq M$.*

Proof. In system (3), (4), we introduce the change of time $t = \alpha(t')$ such that $\tau_j = \alpha(\tilde{\tau}_j)$, $j \in \mathbb{Z}$, and the function α is continuously differentiable and monotone on each interval $(\tilde{\tau}_j, \tilde{\tau}_{j+1})$.

The function α can be chosen in a piecewise linear form,

$$t = a_j t' + b_j, \quad a_j = \frac{\tau_{j+1} - \tau_j}{\tilde{\tau}_{j+1} - \tilde{\tau}_j}, \quad b_j = \frac{\tau_{j+1}\tilde{\tau}_j - \tau_j \tilde{\tau}_{j+1}}{\tilde{\tau}_{j+1} - \tilde{\tau}_j} \quad \text{if} \quad t' \in (\tilde{\tau}_j, \tilde{\tau}_{j+1}).$$

The function $\alpha(t')$ satisfies the conditions

$$|\alpha(t') - t'| \leq \delta_0, \quad \left| \frac{d\alpha(t')}{dt'} - 1 \right| \leq a\delta_0$$

with some positive constant a independent of j and δ_0.

System (3), (4) in the new coordinates $v(t') = u(\alpha(t'))$ has the form

$$\frac{dv}{dt'} = A_1(t')v, \quad t \neq \tilde{\tau}_j, \tag{14}$$

$$\Delta v|_{t'=\tilde{\tau}_j} = v(\tilde{\tau}_j + 0) - v(\tilde{\tau}_j) = B_j v(\tilde{\tau}_j), \quad j \in \mathbb{Z}, \tag{15}$$

where $A_1(t') = \dfrac{d\alpha(t')}{dt'} A(\alpha(t'))$. System (14), (15) has evolution the operator $U_1(t', s') = U(\alpha(t'), \alpha(s'))$. If system (3), (4) has exponential dichotomy with a projection $P(t)$ at point t, then system (14), (15) has exponential dichotomy with the projection $P_1(t') = P(\alpha(t'))$ at point t'. Indeed,

$$\|U_1(t', s')(1 - P_1(s'))\| = \|U(\alpha(t'), \alpha(s'))(1 - P(\alpha(s'))\| \leq$$

$$\leq M e^{-\beta(\alpha(t') - \alpha(s'))} \leq M_1 e^{-\beta(t' - s')}, \quad t' \geq s',$$

where $M_1 = M e^{2\delta_0}$. The inequality for an unstable manifold is proved analogously.

The linear systems (14), (15) and (12), (13) have the same points of impulsive actions $\tilde{\tau}_j$, $j \in \mathbb{Z}$, and

$$\|A_1(t') - \tilde{A}(t')\| \leq \left\| \frac{d\alpha(t')}{dt'} A(\alpha(t')) - A(\alpha(t')) \right\| + \|A(\alpha(t')) - A(t')\| +$$

$$+ \|A(t') - \tilde{A}(t')\| \leq \delta_0 \left(a \sup_t \|A\| + \sup_t \left\| \frac{dA}{dt} \right\| + 1 \right).$$

Let $\tilde{U}(t', s')$ be an evolution operator for system (12), (13). To show that for a sufficiently small δ_0 system (12), (13) is exponentially dichotomous we use the following version of Theorem 7.6.10 [7]:

Assume that the evolution operator $U_1(t', s')$ has an exponential dichotomy on $\mathbb{R}$ and satisfies

$$\sup_{0 \leq t' - s' \leq d} \|U_1(t', s')\| < \infty \tag{16}$$

for some positive d. Then there exists $\eta > 0$ such that

$$\|\tilde{U}(t', s') - U_1(t', s')\| < \eta, \quad \text{whenever} \quad t - s \leq d$$

and the evolution operator $\tilde{U}(t', s')$ has an exponential dichotomy on $\mathbb{R}$.

For proving this statement we set

$$t_n = s' + dn, \;\; T_n = U_1(s' + d(n+1), s' + dn), \;\; \tilde{T}_n = \tilde{U}(s' + d(n+1), s' + dn) \;\text{ for }\; n \in \mathbb{Z}.$$

If the evolution operator $U_1(t, s)$ has an exponential dichotomy, then $\{T_n\}$ has a discrete dichotomy in the sense of [7] (Definition 7.6.4).

By [7] (Theorem 7.6.7), there exists $\eta > 0$ such that $\{\tilde{T}_n\}$ with $\sup_n \|T_n - \tilde{T}_n\| \leq \eta$ has a discrete dichotomy.

Now we are in the conditions of [7, p. 229, 230], Excersise 10 (see also a more general statement [8], Theorem 4.1), that finishes the proof.

The exponentially dichotomous system (12), (13) has the Green function

$$\tilde{G}(t, s) = \begin{cases} \tilde{U}(t, s)(I - \tilde{P}(s)), & t \geq s, \\ -\tilde{U}(t, s)\tilde{P}(s), & t < s, \end{cases}$$

such that

$$\|\tilde{G}(t, s)\| \leq M_1 e^{-\beta_1 |t-s|}, \quad t, s \in \mathbb{R}.$$

Lemma 3. *The difference of the Green functions of the exponentially dichotomous linear systems (12), (13) and (14), (15) satisfies*

$$\tilde{G}(t, \tau) - G_1(t, \tau) = \int\limits_{-\infty}^{\infty} G_1(t, s)(\tilde{A}(s) - A_1(s))\tilde{G}(s, \tau)ds +$$

$$+ \sum_j G_1(t, \tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j, \tau), \quad t, \tau \in \mathbb{R}, \tag{17}$$

where $G_1(t, \tau) = G(\alpha(t), \alpha(\tau))$.

Proof. $\tilde{G}(t, \tau)$ satisfies the equation

$$\frac{du}{dt} = A_1(t)u + (\tilde{A}(t) - A_1(t))\tilde{G}(t, \tau),$$

$$\Delta u|_{t=\tilde{\tau}_j} = B_j u + (\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j, \tau).$$

By (10), we have, for $t \geq \tau$,

$$\tilde{G}(t, \tau) = U_1(t, \tau)(I - P_1(\tau))\tilde{G}(\tau, \tau) + \int\limits_{\tau}^{+\infty} G_1(t, s)(\tilde{A}(s) - A_1(s))\tilde{G}(s, \tau)ds +$$

$$+ \sum_{\tau \leq \tilde{\tau}_j} G_1(t, \tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j, \tau). \tag{18}$$

Analogously, by (11), we have, for $t < \tau$,

$$\tilde{G}(t,\tau) = U_1(t,\tau)P_1(\tau)\tilde{G}(\tau-0,\tau) + \int\limits_{-\infty}^{\tau} G_1(t,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds +$$

$$+ \sum_{\tau > \tilde{\tau}_j} G_1(t,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau). \tag{19}$$

Putting $t = \tau$ in (18), we get

$$P_1(\tau)\tilde{G}(\tau,\tau) = \int\limits_{\tau}^{+\infty} G_1(\tau,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds + \sum_{\tau \le \tilde{\tau}_j} G_1(\tau,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau).$$

Since $\tilde{G}(\tau,\tau) - \tilde{G}(\tau-0,\tau) = I$, we have by (19), for $t < \tau$,

$$\tilde{G}(t,\tau) = U_1(t,\tau)\Bigg(\int\limits_{\tau}^{+\infty} G_1(\tau,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds +$$

$$+ \sum_{\tau \le \tilde{\tau}_j} G_1(\tau,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau)\Bigg) - U_1(t,\tau)P_1(\tau) +$$

$$+ \int\limits_{-\infty}^{\tau} G_1(t,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds + \sum_{\tau > \tilde{\tau}_j} G_1(t,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau) =$$

$$= G_1(t,\tau) + \int\limits_{-\infty}^{\infty} G_1(t,s)(\tilde{A}(s) - A_1(s))\tilde{G}(s,\tau)ds + \sum_{j} G_1(t,\tilde{\tau}_j)(\tilde{B}_j - B_j)\tilde{G}(\tilde{\tau}_j,\tau).$$

The case $t \ge \tau$ is considered analogously.
Lemma 3 is proved.
By (17), we obtain the estimate

$$\|\tilde{G}(t,\tau) - G_1(t,\tau)\| \le \delta_0 M_2 e^{-\beta_2|t-\tau|}, \quad t,\tau \in \mathbb{R}, \tag{20}$$

with some $\beta_2 \le \beta_1$ and $M_2 \ge 1$.

Lemma 4. *Let systems (3), (4) and (12), (13) satisfy assumptions of Lemma 2 with sufficiently small $\delta_0 > 0$. Then the corresponding Green functions of these systems satisfy the inequality*

$$\|\tilde{G}(t,\tau) - G(t,\tau)\| \le \delta_0 \tilde{M}_2 e^{-\beta_2|t-s|}, \tag{21}$$

for all t and s such that $|t-\tau_j| > \delta_0$, $|s-\tau_j| > \delta_0$ for all $j \in \mathbb{Z}$.

Proof. We have

$$\|G(t,\tau)-\tilde G(t,\tau)\| \le \|G(t,\tau)-G(\alpha(t),\alpha(\tau))\| + \|G(\alpha(t),\alpha(\tau))-\tilde G(t,\tau)\|.$$

Let $t > s$ (the case $t < s$ is considered analogously). Then

$$\begin{aligned}
\|G(t,s)-G(\alpha(t),\alpha(s))\| &= \|U(t,s)P(s)-U(\alpha(t),\alpha(s))P(\alpha(s))\| \le \\
&\le \|U(t,s)P(s)-U(\alpha(t),s)P(s)\| + \\
&\quad + \|U(\alpha(t),s)P(s)-U(\alpha(t),\alpha(s))P(\alpha(s))\| \le \\
&\le \|U(t,s)P(s)-U(\alpha(t),t)U(t,s)P(s)\| + \\
&\quad + \|U(\alpha(t),s)P(s)-U(\alpha(t),s)U(s,\alpha(s))P(\alpha(s))\| \le \\
&\le \|I-U(\alpha(t),t)\|\|U(t,s)P(s)\| + \\
&\quad + \|U(\alpha(t),s)P(s)\|\|I-U(s,\alpha(s))\| \le \\
&\le Me^{-\gamma(t-s)}\|I-U(\alpha(t),t)\| + Me^{-\gamma(\alpha(t)-s)}\|I-U(s,\alpha(s)\|.
\end{aligned}$$

Here, for definiteness $s \ge \alpha(s)$, $t \le \alpha(t)$. If $t \in (\tau_j+\delta_0, \tau_{j+1}-\delta_0)$, then $\alpha(t) \in (\tau_j, \tau_{j+1})$. Therefore, by continuity there exists a positive constant C_1 independent of t such that $\|I - -U(\alpha(t),t)\| \le C_1\delta_0$. As a result, we obtain $\|G(t,s)-G(\alpha(t),\alpha(s))\| \le \delta_0 M_3 e^{-\beta|t-s|}$ with some positive constant M_3 independent of $t,s \in \mathbb{R}$ and δ_0. Now, taking into account (20) we obtain (21).

Lemma 4 is proved.

Corollary 1. *Assume that system (3), (4) satisfies conditions (H_1) – (H_3) and is exponentially dichotomous with constants β and M. Then for any $\varepsilon > 0$, $t,s \in \mathbb{R}$, $|t-\tau_j| > \varepsilon$, $|s-\tau_j| > \varepsilon$, $j \in \mathbb{Z}$, there exists a relatively dense set of ε-almost periods r such that*

$$\|G(t+r,s+r)-G(t,s)\| \le \varepsilon C_1 \exp\left(-\frac{\beta}{2}|t-s|\right), \tag{22}$$

where C_1 is a positive constant independent on ε.

Proof. If $u(t) = U(t,s)u_0$, $u(s) = u_0$, is a solution of the impulsive equation (3), (4), then $u_1(t) = U(t+r,s+r)u_0$ is a solution of the equation

$$\frac{du}{dt} = A(t+r)u, \quad t \ne \tau_j, \tag{23}$$

$$\Delta u|_{t=\tau_{j+q}} = u(\tau_{j+q}+0) - u(\tau_{j+q}) = B_{j+q}u(\tau_{j+q}), \quad j \in \mathbb{Z}. \tag{24}$$

By Lemma 1, there exists a positive integer q such that $\tau_{j+q} \in (s+r, t+r)$ if $\tau_j \in (s,t)$. Now we apply Lemma 4.

4. Almost periodic solutions of linear inhomogeneous system. We prove existence of almost periodic solutions in a linear inhomogeneous system.

Lemma 5. *Let a linear homogeneous system satisfy assumptions (H_1) – (H_3) and be exponentially dichotomous on the axis. If the function $f(t)$ is W-almost periodic and the sequence $\{g_j\}$ is almost periodic, then the linear inhomogeneous system (6), (7) has unique solution, which is bounded on $\mathbb{R}$ and W-almost periodic.*

Proof. The unique solution, bounded on $\mathbb{R}$, of the system (6), (7) is defined by formula (8). We show that it is W-almost periodic.

Take an ε-almost period r for the right-hand side of the equation. Then

$$u_0(t+r)-u_0(t)=\int_{-\infty}^{+\infty} G(t+r,s)f(s)ds+\sum_j G(t+r,\tau_j)g_j-\int_{-\infty}^{+\infty} G(t,s)f(s)ds-$$

$$-\sum_j G(t,\tau_j)g_j = \int_{-\infty}^{+\infty}(G(t+r,s+r)-G(t,s))f(s+h)ds+$$

$$+\int_{-\infty}^{+\infty} G(t,s)(f(s+r)-f(s))ds+\sum_j(G(t+r,\tau_{j+q})-G(t,\tau_j))g_{j+q}+$$

$$+\sum_j G(t,\tau_j)(g_{j+q}-g_j).$$

We estimate the first integral,

$$\int_{-\infty}^{\infty}\|(G(t+r,s+r)-G(t,s))f(s+r)\|ds\le$$

$$\le\sum_{k\in\mathbb{Z}}\int_{\tau_k+\varepsilon}^{\tau_{k+1}-\varepsilon}\|(G(t+r,s+r)-G(t,s))f(s+r)\|ds+$$

$$+\sum_{k\in\mathbb{Z}}\int_{\tau_k-\varepsilon}^{\tau_k+\varepsilon}\|(G(t+r,s+r)-G(t,s))f(s+r)\|ds\le$$

$$\le\int_{-\infty}^{\infty}\varepsilon C_1e^{-\frac{\beta}{2}|t-s|}\|f(s)\|ds+\sum_{k\in\mathbb{Z}}\int_{\tau_k-\varepsilon}^{\tau_k+\varepsilon}\|G(t+r,s+r)f(s+r)\|ds+$$

$$+\sum_{k\in\mathbb{Z}}\int_{\tau_k-\varepsilon}^{\tau_k+\varepsilon}\|G(t,s)f(s+r)\|ds.$$

By Lemma 1, $|\tau_{j+q}-\tau_j-r|<\varepsilon$, therefore $\tau_j+r+\varepsilon>\tau_{j+q}$ (we assume that $r>0$ for definiteness). The difference $G(t,\tau_j)-G(t+r,\tau_{j+q})$ is estimated as follows. Let $t-\tau_j\ge\varepsilon$.

Then

$$\|G(t,\tau_j) - G(t+r,\tau_{j+q})\| = \|U(t,\tau_j)(I-P(\tau_j)) - U(t+r,\tau_{j+q})(I-P(\tau_{j+q}))\| \le$$
$$\le \|U(t,\tau_j)(I-P(\tau_j)) - U(t,\tau_j+\varepsilon)(I-P(\tau_j+\varepsilon))\| +$$
$$+ \|U(t,\tau_j+\varepsilon)(I-P(\tau_j+\varepsilon)) - U(t+r,\tau_j+\varepsilon+r)(I-P(\tau_j+\varepsilon+r))\| +$$
$$+ \|U(t+r,\tau_j+\varepsilon+r)(I-P(\tau_j+\varepsilon+r)) - U(t+r,\tau_{j+q})(I-P(\tau_{j+q}))\|. \qquad (25)$$

The first and the third differences are small because of continuity of the function $U(t,s)$ at intervals between the points of impulses,

$$\|U(t,\tau_j)(I-P(\tau_j)) - U(t,\tau_j+\varepsilon)(I-P(\tau_j+\varepsilon))\| \le$$
$$\le \|U(t,\tau_j+\varepsilon)(I-P(\tau_j+\varepsilon))(U(\tau_j+\varepsilon,\tau_j)-I)\| \le$$
$$\le \varepsilon C_2 e^{-\beta(t-\tau_j-\varepsilon)},$$

$$\|U(t+r,\tau_j+\varepsilon+r)(I-P(\tau_j+\varepsilon+r)) - U(t+r,\tau_{j+q})(I-P(\tau_{j+q}))\| =$$
$$= \|U(t+r,\tau_j+\varepsilon+r)(I-P(\tau_j+\varepsilon+r))(U(\tau_j+\varepsilon+r,\tau_{j+q})-I)\| \le$$
$$\le \varepsilon C_2 e^{-\gamma(t-\tau_j-\varepsilon)}.$$

The second difference in (25) is small because of (22).

5. Proof of Theorem 1. Denote by $\mathfrak{M}$ the space of all W-almost periodic functions with discontinuous at points of the same sequence $\{\tau_j\}$. The norm in the space $\mathfrak{M}$ is defined as $\|\varphi\|_0 = \sup_{t\in\mathbb{R}} \|\varphi(t)\|$, $\varphi \in \mathfrak{M}$. We define an operator T on $\mathfrak{M}$ as follows: if $\varphi(t) \in \mathfrak{M}$, then

$$(T\varphi)(t) = \int_{-\infty}^{\infty} G(t,s) f(s,\varphi(s))ds + \sum_j G(t,\tau_j) g(\varphi(\tau_j)).$$

First, we prove that $T(D_h) \subseteq D_h$ for some $h > 0$ where

$$D_h = \{\varphi \in \mathfrak{M}, \|\varphi\|_0 \le h\}.$$

Indeed, if $\|\varphi\|_0 \le h$, then

$$\|T\varphi\| \le \int_{-\infty}^{\infty} \|G(t,s)\|\|f(s,\varphi(s))\|ds + \sum_j \|G(t,\tau_j)\|\|g(\varphi(\tau_j))\| \le$$
$$\le \int_{-\infty}^{\infty} Me^{-\beta|t-s|}Hds + \sum_j Me^{-\beta|t-\tau_j|}H \le$$
$$\le 2MH\left(\frac{1}{\beta} + \frac{1}{1-e^{-\beta\theta}}\right) \le h.$$

By Lemma 37 [2, p. 214], if $\varphi(t)$ is an W-almost periodic function and $\inf_i \tau_i^1 = \theta > 0$, then $\{\varphi(\tau_i)\}$ is an almost periodic sequence. Using the method of finding common almost periods, it is possible to show that the sequence $\{g_i(\varphi(\tau_i))\}$ is almost periodic.

Let r be an ε-almost period of the function $\varphi(t)$. Analogously to the proof of Lemma 5 we show that for $t \in \mathbb{R}$, $|t - \tau_j| > \varepsilon$, $j \in \mathbb{Z}$, the following inequality holds:

$$\|(T\varphi)(t+r) - (T\varphi)(t)\| = \left\| \int\limits_{-\infty}^{+\infty} G(t+r,s)f(s)ds + \sum_j G(t+r,\tau_j)g_j - \right.$$

$$\left. - \int\limits_{-\infty}^{+\infty} G(t,s)f(s)ds - \sum_j G(t,\tau_j)g_j \right\| \leq \Gamma(\varepsilon)\varepsilon,$$

where $\Gamma(\varepsilon)$ is some bounded function of ε. Hence, we proved that $T(D_h) \subseteq D_h$.

If $\varphi, \psi \in D_h$, then

$$\|(T\varphi)(t) - (T\psi)(t)\| \leq \int\limits_{-\infty}^{\infty} \|G(t,s)\| \|f(s,\varphi(s)) - f(s,\phi(s))\| ds +$$

$$+ \sum_k \|G(t,\tau_k)\| \|g_k(\varphi(\tau_k)) - g_k(\phi(\tau_k))\| \leq$$

$$\leq 2MH \left(\frac{1}{\beta} + \frac{1}{1 - e^{-\beta\theta}} \right) \|\varphi - \psi\|_0.$$

For sufficiently small $N > 0$, the operator T is a contraction in the domain D_h, and so there exists a unique W-almost periodic solution of system (1), (2).

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