ADDITIONAL INVARIANCE OF THE KEMMER–DUFFIN AND RARITA–SCHWINGER EQUATIONS

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Additional (implicit) symmetry of the Kemmer–Duffin, Rarita–Schwinger, and Dirac equations is established. It is shown that the invariance algebra of the Kemmer–Duffin equation is a 34-dimensional Lie algebra containing the algebra of SU(3) as a subalgebra, and that the Rarita–Schwinger equation is invariant under a 64-dimensional Lie algebra including the subalgebra $O(2, 4)$. The explicit form of the operator that reduces the Rarita–Schwinger equation to diagonal form is found and also that of the operator that transforms the Kemmer–Duffin equation into the Tamm–Sakata–Taketani equation. The algebra of the additional invariance of the Dirac and Tamm–Sakata–Taketani equations in the class of differential operators is found.

Introduction

It is well known that some equations of motion in quantum physics have an additional (implicit) symmetry. For example, the Schrödinger equation for the hydrogen atom has an implicit invariance with respect to the group of four-dimensional rotations [1], and the Maxwell equation and Dirac equation (for zero mass) are invariant under the conformal group [2].

In [3, 4] it was established that the Maxwell, Klein–Gordon, and Dirac equations (with zero and nonzero masses) have an additional invariance beyond the Lorentz invariance. The basis elements of this new invariance algebra do not belong, in contrast to the case of Lorentz symmetry, for which the infinitesimal operators are linear first-order differential operators, to the class of differential operators. In this case, the basis elements are integrodifferential (nonlocal) operators in the configuration space. Because of the nonlocality, these operators are not infinitesimal operators of tangent transformations in the sense of Lie, although they do form a finite-dimensional Lie algebra.

In what follows, by an additional invariance of the equations of motion we shall understand any invariance that is not Lorentz invariance.

In the present paper, we investigate the group properties of the free relativistic equations of motion for particles with nonzero mass and spins $s \le {}^3/_2$. We establish theorems on the additional invariance of the Kemmer–Duffin (KD), Tamm–Sakata–Taketani (TST), and Rarita–Schwinger (RS) equations. In addition, we find the invariance algebra of the Dirac and TST equations in the class of differential operators. The theorems are proved by means of a device proposed in [3]. The gist of it is that first the system of first-order differential equations, having been reduced in advance to Hamiltonian form, is reduced by means of a unitary transformation to a different equivalent equation with a diagonal Hamiltonian, and then the additional invariance algebra is established for the transformed equation. Finding basis elements of the additional invariance algebra for the transformed equation and having a unitary operator that diagonalizes the Hamiltonian, we determine the invariance algebra of the original equation.

In recent years, there has been intense study of the group properties of partial differential equations on the basis of the classical Lie methods [5, 6]. These methods differ strongly from ours.

1. Symmetry of the Kemmer–Duffin and Tamm–Sakata–Taketani Equations

A. The KD equation can be written in the form

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$$(\beta_\mu p^\mu - m)\Psi(t, \mathbf{x}) = 0, \quad \mu = 0, 1, 2, 3, \tag{1.1}$$

where $p_\mu = i\partial/\partial x^\mu$, and the matrices β_μ satisfy the algebra

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\mu\nu}. \tag{1.2}$$

The KD equation describes the free motion of a particle with spin 0 or 1. In the first case, the matrices β_μ have five rows, and in the second case, 10 rows.

It is more convenient to write Eq. (1.1) in the Hamiltonian form [7]

$$i\partial\Psi/\partial t = H\Psi(t, x), \quad H = [\beta_0, \beta_a]p_a + \beta_0 m, \tag{1.3}$$

$$\{m(1-\beta_0^2) + (\boldsymbol{\beta}\cdot\mathbf{p})\beta_0^2\}\Psi(t, \mathbf{x}) \equiv mP\Psi = 0. \tag{1.4}$$

The physical meaning of the additional condition (1.4) is that it eliminates the "redundant" components of the wave function Ψ. For spin $s = 0$, the wave function has three redundant components; for spin $s = 1$, four.

The condition of invariance of Eq. (1.1) with respect to a certain set of transformations is equivalent by definition to fulfillment of the conditions

$$\left[i\frac{\partial}{\partial t} - H, Q_A\right]\Psi(t, \mathbf{x}) = 0, \quad [mP, Q_A]\Psi(t, \mathbf{x}) = 0, \tag{1.5}$$

where Q_A are the operators of the transformations, Ψ satisfies Eqs. (1.3) and (1.4), and $\{A\}$ is a set of indices.

The problem of finding the invariance algebra of Eq. (1.1) consists of describing all possible operators Q_A that satisfy conditions (1.5).

We prove

THEOREM 1. The KD equation is invariant under the Lie algebra of the group SU(3). In the case of spin $s = 1$, the KD equation is invariant under a larger, 34-dimensional Lie algebra that contains the SU(3) algebra as a subalgebra. The basis elements of this invariance algebra satisfy the commutation relations (1.10) and (1.14).

Proof. A transition to a representation in which H is diagonal can be made by means of an integral unitary operator of Foldy–Wouthuysen type [8]:

$$\Psi \to \Phi = U\Psi, \quad U = \exp\left\{\frac{\beta_a p_a}{p}\operatorname{arctg}\frac{p}{m}\right\}, \quad p = (p_1^2 + p_2^2 + p_3^2)^{1/2}, \quad a = 1, 2, 3. \tag{1.6}$$

As a result, we obtain the system of integrodifferential equations

$$i\partial\Phi/\partial t = H^\Phi\Phi(t, \mathbf{x}), \quad H^\Phi = UHU^{-1} = \beta_0 E, \quad (1-\beta_0^2)\Phi(t, \mathbf{x}) = 0, \quad E = (p^2 + m^2)^{1/2}, \tag{1.7}$$

and the invariance condition (1.5) reduces to the form

$$\left[i\frac{\partial}{\partial t} - \beta_0 E, Q_A^\Phi\right]\Phi = 0, \quad Q_A^\Phi = UQ_AU^{-1}, \quad [1-\beta_0^2, Q_A^\Phi]\Phi = 0. \tag{1.5'}$$

The condition (1.5′) is satisfied by arbitrary matrices that commute with β_0.

Using the relations (1.2), we can readily see that the condition (1.5′) is satisfied by the matrices

$$S_{ab} = i(\beta_a\beta_b - \beta_b\beta_a), \quad S_{ab} = \varepsilon_{abc}S_c, \quad a, b, c = 1, 2, 3. \tag{1.8}$$

This property is obviously common to all functions of S_{ab}, among which one can choose only eight independent:

$$Q_1^\Phi = -(S_1S_2 + S_2S_1), \quad Q_2^\Phi = S_3, \quad Q_3^\Phi = -i(S_3S_1S_2 - S_1S_2S_3), \quad Q_4^\Phi = -(S_3S_1 + S_1S_2), \quad Q_5^\Phi = -S_2, \quad Q_6^\Phi = -(S_2S_3 + S_3S_2),$$

$$Q_7^\Phi = S_1, \quad Q_8^\Phi = -\frac{i}{\sqrt{3}}(S_3S_1S_2 + S_1S_2S_3 - 2S_2S_3S_1). \tag{1.9}$$

The operators Q_A^Φ, $A = 1, 2, \ldots, 8$, satisfy the commutation relations

$$[Q_M^\Phi, Q_L^\Phi] = if_{MLK}Q_K^\Phi, \quad M, L, K = 1, 2, \ldots, 8, \tag{1.10}$$

where f_{MLK} are the structure constants of the group SU(3).

In the case of spin $s = 0$, the operators (1.10) exhaust all possible (to within equivalence) independent matrices that commute with β_0. For $s = 1$, there are more of these matrices. We construct the complete system of matrices that commute with β_0 as follows. Without loss of generality, we can choose the matrix β_0 in the form

$$\beta_0 = \begin{pmatrix} I^3 & & \\ & -I^3 & \\ & & 0^4 \end{pmatrix}, \tag{1.11}$$

where I^3 and 0^4 are the three-row unit matrix and four-row null matrix and there are zeros in the remaining positions.

The general form of a matrix that commutes with β_0 is given by

$$B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \tag{1.12}$$

where a, b, c are arbitrary square 3×3, 3×3, and 4×4 matrices, respectively. Thus, there are altogether 34 linearly independent matrices that commute with β_0. These 34 matrices include the operators Q_A^Φ, $A = 1, 2, \dots, 8$, from (1.9), and the others can be represented in the form

$$Q_{8+A}^\Phi = \beta_0 Q_A^\Phi, \quad A = 1, 2, \dots, 8, \quad Q_{17}^\Phi = \Gamma_0 = (S_{12} - S_{43})(1 - \beta_0^2),$$

$$Q_{17+a}^\Phi = \Gamma_a = (S_{bc} + S_{4a})(S_{31} - S_{42})(1 - \beta_0^2), \quad S_{4a} = i(\beta_a\beta_4 - \beta_4\beta_a),$$

$$\beta_4 = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma, \quad \{Q_{21}^\Phi, Q_{22}^\Phi, \dots, Q_{32}^\Phi\} = \{\Gamma_\mu\Gamma_\nu; \Gamma_\mu\Gamma_\nu\Gamma_\lambda; \Gamma_0\Gamma_1\Gamma_2\Gamma_3\}, \quad Q_{33}^\Phi = 1, \quad Q_{34}^\Phi = \beta_0, \tag{1.13}$$

$$\mu, \nu, \lambda, \dots = 0, 1, 2, 3, \quad a = 1, 2, 3; \quad (a, b, c) = \text{cyclic perm. of } (1, 2, 3).$$

These operators satisfy the commutation relations

$$[Q_{8+A}^\Phi, Q_{8+B}^\Phi] = if_{ABC} Q_C^\Phi, \quad [Q_{8+A}^\Phi, Q_B^\Phi] = if_{ABC} Q_{8+C}^\Phi; \tag{1.14'}$$

$$[\Gamma_\mu, Q_A^\Phi] = [\Gamma_\mu, Q_{8+A}^\Phi] = 0, \quad (\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu)(1 - \beta_0^2) = 2g_{\mu\nu}(1 - \beta_0^2). \tag{1.14''}$$

The commutation relations (1.10) and (1.14) follow directly from (1.2). The theorem is proved.

To conclude this section, we note that the explicit form of the operators (1.9) and (1.13) in the original Ψ representation is obtained by means of the inverse of the transformation (1.6). In other words, the operators Q_A are obtained from Q_A^Φ, $A = 1, 2, \dots, 34$, by the substitution

$$\mathbf{S} \to \mathbf{S} = U^{-1}\mathbf{S}U = \mathbf{S}\frac{m}{E} - i\frac{\beta \times \mathbf{p}}{E} + \frac{\mathbf{p}(\mathbf{S}\cdot\mathbf{p})}{E(E+m)}. \tag{1.8'}$$

Remark 1. It is well known [9] that Eq. (1.1) in the limiting case $m \to 0$ cannot be used to describe the motion of massless particles. It can be shown however that such a passage to the limit is possible in the Hamiltonian form (1.3)-(1.4) of the KD equation. Theorem 1 remains true.

If we impose on the wave function Ψ the Poincaré-invariant condition of transversality

$$(\mathbf{S}\cdot\mathbf{p})\Psi = 0, \tag{1.15}$$

then Theorem 1 no longer holds.

The system of equations (1.3), (1.4) (with $m = 0$), and (1.15) is equivalent to the Maxwell equations.

Remark 2. For the KD equation, as for the Dirac equation [3], one can find four types of operators that satisfy the commutation relations of the Lie algebra of the Poincaré group for which the condition (1.5) is satisfied. These operators also have an explicit representation:

$$\{Q^1\}: {}^1P_\mu = i\partial/\partial x^\mu, \quad {}^1J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = i(\beta_\mu\beta_\nu - \beta_\nu\beta_\mu); \tag{1.16}$$

$$\{Q^2\}: {}^2P_0 = H, \; {}^2P_a = -i\partial/\partial x_a, \quad {}^2J_{ab} = x_a p_b - x_b p_a + S_{ab}, \quad {}^2J_{0a} = x_0 p_a - {}^1/_2(x_a H + H x_a); \tag{1.17}$$

$$\{Q^3\}: {}^3P_0 = i\partial/\partial t, \; {}^3P_a = -i\partial/\partial x_a, \quad {}^3J_{ab} = \tilde{x}_a p_b - \tilde{x}_b p_a, \quad {}^3J_{0a} = x_0 p_a - \tilde{x}_a p_0; \tag{1.18}$$

$$\{Q^4\}: {}^4P_0 = H, \; {}^4P_a = -i\partial/\partial x_a, \quad {}^4J_{ab} = \tilde{x}_a p_b - \tilde{x}_b p_a, \quad {}^4J_{0a} = x_0 p_a - {}^1/_2(\tilde{x}_a H + H\tilde{x}_a); \tag{1.19}$$

where

$$\tilde{x}_a = x_a - i\frac{\beta_a}{E} + i\frac{(\beta_k p_k) p_a}{E^2(E+m)} + \frac{(\mathbf{p}\times\mathbf{S})_a}{E(E+m)}$$

The operators (1.16) are non-Hermitian in the Hilbert space in which the operators (1.17) are Hermitian. The operators (1.18) and (1.19) are Hermitian and inequivalent to the operators (1.16) and (1.17). This can be readily established by calculating the Casimir operators for the representations (1.16), (1.18), and (1.17), (1.19).

We note further that the operators (1.16)-(1.19) generate completely different laws of transformation of the coordinate and time. Namely, from the explicit form of the operators J_{0a} we obtain directly that in the case (1.17) and (1.19), in contrast to (1.16) and (1.18), the time does not change:

$$x_0' = \exp\{iJ_{0a}\theta_a\} x_0 \exp\{-iJ_{0b}\theta_b\} = x_0. \tag{1.20}$$

B. The TST equation has the form

$$i\partial\Psi^{\mathrm{TST}}/\partial t = H^{\mathrm{TST}}\Psi_{\mathrm{TST}}(t,\mathbf{x}), \quad H^{\mathrm{TST}} = \sigma_2 m - i\sigma_1\frac{(\mathbf{S}\cdot\mathbf{p})^2}{m} + (i\sigma_1+\sigma_2)\frac{p^2}{2m}, \tag{1.21}$$

where Ψ_{TST} is a six-component wave function, S_a are the generators of a representation that is the direct sum of two irreducible representations D(1) of $O(3)$, and σ_1 and σ_2 are six-row Pauli matrices that commute with S_a.

The TST equation describes the motion of a free relativistic particle with spin $s = 0$ and, in contrast to (1.1), does not contain redundant components.

THEOREM 2. The TST equation is invariant under a 16-dimensional Lie algebra that contains the SU(3) algebra as a subalgebra. The basis elements of this algebra satisfy the commutation relations (1.10) and (1.14).

Proof. We first of all establish the connection between the solutions of the KD and TST equations. Usually, the TST equation is obtained from the KD equations by indirect elimination of the redundant components. This procedure is unsuitable for our purposes. We show that the TST equation can be obtained from the KD equations by means of an isometric transformation:

$$\Psi \to \Psi^{\mathrm{TST}} = V\Psi, \quad V = \exp\left\{\frac{\beta_a p_a}{m}\beta_0^2\right\} = 1 + \frac{\beta_a p_a}{m}\beta_0^2, \quad a=1,2,3. \tag{1.22}$$

It is easy to see that Ψ^{TST} satisfies the equations

$$i\partial\Psi^{\mathrm{TST}}/\partial t = VHV^{-1}\Psi^{\mathrm{TST}} = \beta_0\left(m + \frac{\beta_a p_a}{m}\right)\Psi^{\mathrm{TST}}, \quad V(mP)V^{-1}\Psi^{\mathrm{TST}} = m(1-\beta_0^2)\Psi^{\mathrm{TST}} = 0. \tag{1.23}$$

It is well known [7] that the system of equations (1.23) is equivalent to (1.21) since the wave function Ψ^{TST} has only six nonzero components, and one can always set

$$\beta_0 m\Psi^{\mathrm{TST}} = \sigma_2 m\Psi^{\mathrm{TST}}, \quad \beta_0\frac{\beta_a p_a}{m}\Psi^{\mathrm{TST}} = \left[-i\sigma_1\frac{(\mathbf{S}\cdot\mathbf{p})^2}{m} + (\sigma_2+i\sigma_1)\frac{\mathbf{p}^2}{2m}\right]\Psi^{\mathrm{TST}}. \tag{1.24}$$

Since Eqs. (1.3) and (1.4) are invariant with respect to the algebra generated by the operators Q_A, Eq. (1.21) is invariant with respect to the algebra $\{Q_A^{\mathrm{TST}}\}$, $Q_A^{\mathrm{TST}} = VQ_AV^{-1}$. We obtain the explicit form of the operators Q_A^{TST} from (1.9), (1.13), (1.8′), and (1.22):

$$Q_1^{\mathrm{TST}} = -(\check{S}_1\check{S}_2 + \check{S}_2\check{S}_1),\ Q_2^{\mathrm{TST}} = \check{S}_3,\ Q_3^{\mathrm{TST}} = -i(\check{S}_3\check{S}_1\check{S}_2 - \check{S}_1\check{S}_2\check{S}_3),\ Q_4^{\mathrm{TST}} = -(\check{S}_3\check{S}_1 + \check{S}_1\check{S}_3),\ Q_5^{\mathrm{TST}} = -\check{S}_2,\ Q_6^{\mathrm{TST}} = -(\check{S}_2\check{S}_3 + \check{S}_3\check{S}_2),\ Q_7^{\mathrm{TST}} = -\check{S}_1,\ Q_8^{\mathrm{TST}} = -\frac{i}{\sqrt{3}}(\check{S}_3\check{S}_1\check{S}_2 + \check{S}_1\check{S}_2\check{S}_3 - 2\check{S}_2\check{S}_3\check{S}_1),$$

$$Q_{8+A}^{\mathrm{TST}} = \frac{H^{\mathrm{TST}}}{E}Q_A^{\mathrm{TST}}, \quad \check{\mathbf{S}} = \mathbf{S}\frac{m}{E} + \frac{\mathbf{p}(\mathbf{S}\cdot\mathbf{p})}{E(E+m)} + \frac{i}{mE}\{\sigma_3(\mathbf{S}\times\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) + \frac{i}{2}(1+\sigma_3)[\mathbf{p}(\mathbf{S}\cdot\mathbf{p}) - \mathbf{S}p^2]\}, \tag{1.25}$$

$$Q_{17}^{\mathrm{TST}} = H^{\mathrm{TST}}/E, \quad Q_{18}^{\mathrm{TST}} = 1. \tag{1.26}$$

The operators (1.25) satisfy the same commutation relations (1.9) and (1.14′) as the operators Q_A^{Φ}, Q_{A+8}^{Φ}. The operators (1.26) commute with (1.25).

The invariance algebra (1.25)-(1.26) of the TST equation is of course smaller than the algebra (1.9), (1.14) of the KD equation. This is because the TST wave function has fewer components than the KD's, and therefore the operators VQ_{17}, $Q_{18}, \ldots, Q_{32}V^{-1}$ are not defined on solutions of the TST equation. The theorem is proved.

Remark 3. The relativistic equations without redundant components for particles with spin s = 1 obtained in [11] are also invariant with respect to the transformations that satisfy the algebra (1.10), (1.14). This is proved in the same way as above, because these equations can be reduced to diagonal form.

2. Symmetry of the Rarita–Schwinger Equation

The RS equation for a particle with spin $s = {}^3/_2$ can be written in the form

$$(\gamma_\mu p^\mu - m)\Psi^\nu(t, \mathbf{x}) = 0, \quad \gamma_\nu \Psi^\nu(t, \mathbf{x}) = 0, \quad \mu, \nu = 0, 1, 2, 3, \tag{2.1}$$

where γ_μ are 4×4 Dirac matrices. The RS wave function has 16 components Ψ^ν_α, $\alpha = 1, 2, 3, 4$.

We write the system of equations (2.1) in the Hamiltonian form

$$i\partial\Psi/\partial t = H\Psi(t, \mathbf{x}), \quad \gamma_\nu\Psi^\nu(t, \mathbf{x}) = 0, \quad H = \begin{pmatrix} \hat{H} & 0 & 0 & 0 \\ 0 & \hat{H} & 0 & 0 \\ 0 & 0 & \hat{H} & 0 \\ 0 & 0 & 0 & \hat{H} \end{pmatrix}, \quad \Psi = \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \quad \hat{H} = \gamma_0\gamma_a p_a + \gamma_0 m. \tag{2.2}$$

The following manifestly covariant representation of the Lie algebra of the Poincaré group is realized on the solutions of Eqs. (2.2):

$$P_0 = H, \quad P_a = p_a = -i\partial/\partial x_a, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \tag{2.3}$$

where the spin matrices $S_{\mu\nu}$ are generators of the representation $D({}^1/_2, {}^1/_2) \times [D({}^1/_2, 0) \oplus D(0, {}^1/_2)]$ of the group $O(1, 3)$, and therefore can be represented in the form

$$S_{\mu\nu} = j_{\mu\nu} + \tau_{\mu\nu}, \quad [j_{\mu\nu}, \tau_{\mu'\nu'}] = 0, \quad \tau_{\mu\nu} = \frac{i}{2}\gamma_\mu\gamma_\nu, \quad j_{ab} = j_c^1 + j_c^2, \quad j_{0a} = i(j_a^1 - j_a^2), \quad [j_a^1, j_b^2] = 0, \tag{2.4}$$

where j_a^1, j_b^2 are the generators of the representation $D(\frac{1}{2})$ of $O(3)$. We now show that the following theorem holds

THEOREM 3. The RS equation is invariant under a 64-dimensional Lie algebra that contains the Lie algebra of the group $O(2, 4)$ as a subalgebra. The basis elements of this algebra are all possible independent products of the operators (2.12).

Proof. As in the preceding section, to prove the theorem we go over to a representation in which the Hamiltonian H is diagonal and the wave function has only $2(2s + 1)$ nonzero components. The transition to such a representation for the RS equation is discussed in [12], but there the explicit form of the transformation operator is not found.

We have obtained such an operator in the form

$$W = \exp\left\{i\gamma_0 \frac{j_{0a}p_a}{p} \operatorname{arth}\frac{p}{E}\right\} \exp\left\{\frac{\gamma_a p_a}{p} \operatorname{arctg}\frac{p}{m}\right\}. \tag{2.5}$$

This operator not only diagonalizes the Hamiltonian H (2.2) but also reduces the remaining generators (2.3) to the canonical Foldy–Shirokov form.

Equations (2.2) after the transformation W take the form

$$i\partial\Phi/\partial t = H^\Phi\Phi(t, \mathbf{x}), \quad H^\Phi = WHW^{-1} = \Gamma_0^{(16)}E, \quad S_{ab}^2\Phi = {}^3/_2({}^3/_2 + 1)\Phi; \quad \Phi = W\Psi; \quad E = (p^2 + m^2)^{1/2}, \tag{2.6}$$

where the 16-row matrix $\Gamma_0^{(16)}$ can always be chosen in the form

$$\Gamma_0^{(16)} = \begin{pmatrix} \hat{I} & 0 & 0 & 0 \\ 0 & \hat{I} & 0 & 0 \\ 0 & 0 & -\hat{I} & 0 \\ 0 & 0 & 0 & -\hat{I} \end{pmatrix}, \tag{2.7}$$

and $\hat{I}$ and 0 are four-row unit and null matrices.

It is clear from (2.6) that the additional invariance of the RS equations is generated by the same matrices B_N that satisfy the conditions

$$[B_N, \Gamma_0^{(16)}]=0, \quad [B_N, S_{ab}^2]=0. \tag{2.8}$$

Without loss of generality, the matrix S^2_{ab} can be taken in the diagonal form

$$S^2_{ab} = \frac{3}{4}\begin{pmatrix} 5\hat{I} & 0 & 0 & 0 \\ 0 & \hat{I} & 0 & 0 \\ 0 & 0 & 5\hat{I} & 0 \\ 0 & 0 & 0 & \hat{I} \end{pmatrix}, \tag{2.9}$$

It can be seen from (2.7) and (2.9) that the most general form of a matrix that commutes with $\Gamma_0^{(16)}$ and S^2_{ab} is given by

$$A = \begin{pmatrix} l & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & h \end{pmatrix}, \tag{2.10}$$

where l, f, g, h are arbitrary square four-row matrices. Therefore, the matrix A can be represented as a linear combination of 64 linearly independent matrices B_N that commute with $\Gamma_0^{(16)}$ and S^2_{ab}:

$$A = \sum_{N=1}^{64} a_N B_N, \tag{2.11}$$

with arbitrary coefficients a_N.

A system of basis matrices B_N can be constructed explicitly. Namely, we choose six 16×16 matrices:

$$\Gamma_0 = \frac{1}{\sqrt{3}}(S_{23}S_{31}+S_{31}S_{23}-i\varepsilon_{abc}j_{0a}\tau_{bc}), \quad \Gamma_1=2i\tau_{23}(1-2j_{23}^2)(j_{ab}^2-1), \quad \Gamma_2=2i\tau_{31}(1-2j_{31}^2)(j_{ab}^2-1),$$
$$\Gamma_3=2i[\tau_{12}(1-j_{12}^2)+2j_{12}\tau_{12}](j_{ab}^2-1), \quad L_1=\Gamma_0^{(16)}, \quad L_2={}^2/_3 S_{ab}^2-{}^3/_2, \tag{2.12}$$

which satisfy the condition (2.8).

Using the relation (2.4) and making fairly lengthy calculations, we can establish that the operators (2.12) satisfy

$$\Gamma_\mu\Gamma_\nu+\Gamma_\nu\Gamma_\mu=2g_{\mu\nu}, \quad [L_1, L_2]=[\Gamma_\mu, L_1]=[\Gamma_\mu, L_2]=0, \quad L_1^2=L_2^2=1. \tag{2.13}$$

If we now take all possible independent products of the operators (2.13), we obtain exactly 64 elements, which form the basis system of matrices satisfying (2.8). In particular, the set of all possible independent products of the matrices Γ_μ forms, as follows from (2.13), the Clifford algebra C_4, whose elements are basis elements of the Lie algebra of $O(2, 4)$.

To complete the exposition, we give the explicit form of the matrices Γ_μ, L_1, L_2 in the Ψ representation, where $\Psi = W^{-1}\Phi$. By means of the inverse transformation W^{-1}, we obtain

$$\hat{\Gamma}_\mu=W^{-1}\Gamma_\mu W, \tag{2.14}$$

$$\hat{\Gamma}_0 = \frac{1}{\sqrt{3}}(\hat{S}_{23}\hat{S}_{31}+\hat{S}_{31}\hat{S}_{23}-i\varepsilon_{abc}\hat{j}_{0a}\hat{\tau}_{bc}), \quad \hat{\Gamma}_1=2i\hat{\tau}_{23}(1-2\hat{j}_{23}^2)(j_{ab}^2-1),$$
$$\hat{\Gamma}_2=2i\hat{\tau}_{31}(1-2\hat{j}_{31}^2)(\hat{j}_{ab}^2-1), \quad \hat{\Gamma}_3=2i\hat{\tau}_{12}(1-\hat{j}_{12}^2+2\hat{j}_{12}\hat{\tau}_{12})(j_{ab}^2-1), \quad \hat{L}_1=H/E, \quad \hat{L}_2={}^2/_3\hat{S}_{ab}^2-{}^3/_2, \tag{2.15}$$

where

$$\hat{\tau}_{ab}=\tau_{ab}\frac{m}{E}+i\frac{\gamma_a p_b-\gamma_b p_a}{m}+\frac{p_c(p_a\cdot\tau_a)}{E(E+m)}, \quad \hat{j}_{ab}=j_{ab}\frac{m}{E}-\frac{H}{Em}(j_{0a}p_b-j_{0b}p_a)+\frac{p_c(p_b j_{0b})}{E(E+m)},$$
$$\hat{j}_{0a}=j_{0a}+\frac{p_a(p_b\cdot j_b)-j_a p_b^2}{E(E+m)}-\frac{j_{ab}p_b}{Em}H, \quad \hat{S}_{ab}=\hat{j}_{ab}+\hat{\tau}_{ab}, \quad (a, b, c)=\text{cyclic perm. of } (1, 2, 3). \tag{2.16}$$

In conclusion we note that the assertions made above about additional invariance also hold for the Bargmann–Wigner, Dirac–Fierz–Pauli, and Bhabha equations, which describe particles with spin 1 and $^3/_2$. The additional symmetry of relativistic equations for particles with spin $s > {}^3/_2$ can also be investigated by means of the methods used in the present paper.

3. Invariance Algebra of the Dirac and TST Equations in the Class of Differential Operators

In the Introduction it was noted that the Dirac equation is implicitly invariant under the algebra $O(4)$ as well as Poincaré invariant. The algebra $O(4)$ is defined by integrodifferential operators and is in a certain sense the maximal algebra of additional invariance of the Dirac equation [3]. In connection with this result, it is natural to clarify the following question: does there exist an algebra of implicit invariance of the Dirac and TST equations in the class of differential operators?

In what follows, we shall prove theorems that provide a positive answer to this question.

THEOREM 4. The Dirac equation is invariant with respect to the algebra of $O(4)$ with basis elements given by differential operators.

Proof. We subject the Dirac equation

$$(\gamma_\mu p^\mu - m)\Psi = 0 \tag{3.1}$$

to the transformation

$$\Psi \to \Phi = V\Psi,\quad (m-\gamma_\mu p^\mu) \to V(m-\gamma_\mu p^\mu)V^{-1} = m - (P_\mu P_\mu)^{1/2}\gamma_5;$$
$$V = \exp\left(\frac{S_{5\mu}p_\mu}{\sqrt{p_\mu p_\mu}}\cdot\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(1+\frac{2S_{5\mu}p^\mu}{(p_\mu p^\mu)^{1/2}}\right),\quad S_{5\mu} = \frac{i}{2}\gamma_5\gamma_\mu,\quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{3.2}$$

The invariance condition takes the form

$$[m-(p_\mu p_\mu)^{1/2}\gamma_5, Q']\Phi(t,x) = 0. \tag{3.3}$$

Equation (3.3) is satisfied by arbitrary matrices that commute with γ_5. Any such matrix can be represented as a linear combination of the quantities

$$S_{ab} = \frac{i}{2}\gamma_a\gamma_b,\quad S_{4a} = \frac{1}{2}\gamma_0\gamma_a. \tag{3.4}$$

The matrices (3.4) realize, as is well known, the direct sum $D(1/2, 0)\oplus D(0, 1/2)$ of two irreducible representations of the $O(4)$ algebra. By the transformation that is the inverse of (3.2), we obtain the basis elements of the algebra of the additional invariance of Eq. (3.1):

$$\hat{S}_{ab} = V^{-1}S_{ab}V = S_{ab} - \frac{i}{m}(1+\varphi_5)(\gamma_a p_b - \gamma_b p_a),\quad S_{4a} = S_{4a} - \frac{1}{m}(1+\gamma_5)(\gamma_0 p_a - \gamma_a p_0). \tag{3.5}$$

It should be noted that this algebra is not equivalent to the Lie algebra of the group of three-dimensional rotations defined by the generators $J_{ab} = x_a p_b - x_b p_a + S_{ab}$ of the Poincaré group. The theorem is proved.

Remark 4. The operators $\hat{S}_{ab}$ are non-Hermitian with respect to the ordinary scalar product

$$(\Psi_1, \Psi_2) = \int d^3x \Psi_1^+(x)\Psi_2(x), \tag{3.6}$$

but they are Hermitian in the indefinite scalar product

$$(\Psi_1, \Psi_2) = \int d^3x \Psi_1^+\left[\gamma_0 + (1-\gamma_4)\frac{2(\mathbf{S}\cdot\mathbf{p})}{m}\right]\Psi_2. \tag{3.7}$$

In the scalar product (3.7), the Dirac Hamiltonian (3.1) is also Hermitian.

THEOREM 5. The TST equation is invariant with respect to the algebra of SU(3) with basis elements given by differential operators.

Proof. We subject the TST equation (1.21) to the transformation

$$\Psi^{\mathrm{TST}} \to \Psi'^{\mathrm{TST}} = W\Psi^{\mathrm{TST}},\quad H^{\mathrm{TST}} \to WH^{\mathrm{TST}}W^{-1} = \sigma_2 m + (\sigma_2 + i\sigma_1)\frac{p^2}{2m} = H'^{\mathrm{TST}},$$
$$W = 1 + \sigma_2\frac{(S\cdot p)}{m} + (1+\sigma_3)\frac{(S\cdot p)^2}{2m^2}. \tag{3.8}$$

The operators H'^{TST} (3.8) commute with the spin matrices S_a. From this we conclude that the operator $i\partial/\partial t - H'^{\mathrm{TST}}$ commutes with the set

$$Q_1'^{TST}=-(S_1'S_2'+S_2'S_1'),\quad Q_2'^{TST}=S_3',\quad Q_3'^{TST}=-i(S_3'S_1'S_2'-S_1'S_2'S_3'),\quad Q_4'^{TST}=-(S_3'S_1'+S_1'S_3'),$$
$$Q_5'^{TST}=-S_2',\quad Q_6'^{TST}=-(S_2'S_3'+S_3'S_2'),\quad Q_7'^{TST}=S_1',\quad Q_8'^{TST}=-\frac{i}{\sqrt{3}}(S_3'S_1'S_2'+S_1'S_2'S_3'-2S_2'S_3'S_1'),\tag{3.9}$$

where

$$S_a'=S_a+i\left\{\sigma_2\frac{\varepsilon_{abc}S_bp_c}{m}+(1+\sigma_3)\frac{[\varepsilon_{abc}S_bp_c,(\mathbf{S}\cdot\mathbf{p})]_+}{2m^2}\right\}\left\{1-\sigma_2\frac{(\mathbf{S}\cdot\mathbf{p})}{m}+(1-\sigma_3)\frac{(\mathbf{S}\cdot\mathbf{p})^2}{2m^2}\right\}.$$

This means that the operators (3.9) satisfy the invariance condition of the TST equation. By direct verification one can show that the operators (3.9) satisfy the commutation relations (1.10) of the algebra SU(3). These basis elements of the invariance algebra of the TST equation are Hermitian with respect to the indefinite scalar product

$$(\Psi_1,\Psi_2)=\int d^3x\Psi_1^+(t,\mathbf{x})W^+\sigma_2W\Psi_2(t,\mathbf{x})=\int d^3x\Psi_1^+\left\{\sigma_2+2\frac{\mathbf{S}\cdot\mathbf{p}}{m}+2\sigma_2\frac{(\mathbf{S}\cdot\mathbf{p})^2}{m^2}+(1-\sigma_3)\frac{(\mathbf{S}\cdot\mathbf{p})^3}{m^3}\right\}\Psi_2.\tag{3.10}$$

The theorem is proved.

The above results can be used to find integrals of the motion of particles interacting with an external field. For example, for a particle with spin $s=\frac{1}{2}$ in a uniform magnetic field $\mathbf{H}$ an integral of the motion is the operator $Q=\varepsilon_{abc}\hat{S}_{bc}(\pi)H_c$, where $\hat{S}_{ab}(\pi)$ is obtained from (3.5) by the substitution $p_a\to\pi_a=p_a-eA_a$.

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