# Mass and spin content of a free relativistic particle of arbitrary spins and the group reduction $Sp(4) \supset U(1) \otimes SU(2)$

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**Abstract**

In this paper, we apply Foldy-Wouthuysen transformations to relativistic equation for a free particle of arbitrary spins, which has the symmetry group $O(5)$ associated with it. By noting the isomorphism of $O(5)$ to the symplectic group $Sp(4)$, the mass and spin content of the problem is found using the group reduction $Sp(4) \supset U(1) \otimes SU(2)$ where $U(1)$ is associated with the masses while $SU(2)$ is related to the spins.

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One of the promising approaches to the analysis of relativistic wave equations is connected with Foldy-Wouthuysen (FW) transformations [1]. For relativistic equations of free particle, they diagonalize the Hamiltonian and reduce the associated representations of the Poincare group to the direct sum of irreducible representations. This approach is one of the simplest way to study the mass and spin states of a particular problem.

# 1 FW-Transformation for Free Particle:

In this paper, we apply the FW transformation approach to the free particle of arbitrary spins whose relativistic equation can be obtained from the previous paper [2]

$$H\Psi \equiv (2c\Lambda_{i4}p_i + 2mc^2\Lambda_{45})\Psi = nE\Psi \tag{1}$$

where $\Lambda_{i4}$ and $\Lambda_{45}$ are the matrices associated with the irreducible representations $O(n_1, n_2)$ of the group algebra $O(5)$.

This equation describes a multiplet of particles with several spins and masses which can exist in positive as well as negative energy states. To understand the physical content of this equation, we first diagonalize it using the unitary FW transformations which do not change the associated energy $E$ of the problem. Generalizing the FW approach, we see that the diagonalizing operator exists in the following form

$$U = \exp(i\frac{\Lambda_{i5}p_i}{p}\theta) \tag{2}$$

where $p = \sqrt{p_1^2 + p_2^2 + p_3^2}$ and $\theta$ is an unknown function of $p$. We shall consider equation (1) in the momentum representation.

Using the Campbell-Hausdorff formula [1]

$$\exp(-iA)B\exp(iA) = B - i[A, B] - \frac{1}{2!}[A, [A, B]] - ... \tag{3}$$

and the commutation relation (see eq.(2.11) of ref. [2]) for the $\Lambda-$matrices of the group algebra $O(5)$, we get the transformed Hamiltonian as

$$H' \equiv UHU^\dagger = 2\Lambda_{45}\sqrt{p^2c^2 + m^2c^4} \tag{4}$$

provided $\theta = \arctan(\frac{p}{mc})$. Thus the transformed equation now becomes

$$nE\Psi' = H'\Psi'; \Psi' = U\Psi \tag{5}$$

which reduces to a set of equations

$$E\Psi'_\lambda = \frac{2\lambda}{n}\sqrt{p^2c^2 + m^2c^4}\Psi'_\lambda \tag{6}$$

where $\lambda$ is the eigenvalue of $\Lambda_{45}$ with the eigenfunctions $\Psi'_\lambda$. We note that each component $\Psi'_\lambda$ of $\Psi'$ satisfies the Klein-Gordon equation

$$(E^2 - v_\lambda^2 p^2 - m_\lambda^2 v_\lambda^4)\Psi'_\lambda = 0 \tag{7}$$

$$v_\lambda = \frac{2\lambda}{n}c \text{ and } m_\lambda = \frac{n}{2\lambda}m \tag{8}$$

The eigenvalues $\lambda$ can be found using Gel'fand-Tsetlin inequalities [3] for the quantum numbers $(n_1, n_2)$; $(m_1, m_2); s; \sigma$ associated with the chain of groups $O(5) \supset O(4) \supset O(3) \supset O(2)$ respectively

$$\begin{aligned} n_1 &\geq m_1 \geq n_2; n_2 \geq m_2 \geq -n_2 \\ m_1 &\geq s \geq |m_2|; s \geq \sigma \geq -s \end{aligned} \tag{9}$$

So, the maximum possible range of values for $\sigma$ is $n_1, n_1 - 1 ....... - n_1$. Since the permutation transformation $12345 \rightarrow 54321$ which is an element of group $O(5)$, when applied to $\Lambda_{45}$ gives $\Lambda_{12}$, the associated eigenvalues $\lambda$ and $\sigma$ take the same range of values. Thus the maximum possible range of values for $\lambda$ is $n_1, n_1 - 1 ....... - n_1$. Since $n_1 \leq n/2$,

$$v_\lambda \leq c \text{ and } m_\lambda \geq m \tag{10}$$

The quantity $n_1$ can take integer as well as half integer values, and thus same is true for $\lambda$. Thus, for integer values of $\lambda$, there is a possibility that $\lambda = 0$ for which the particle takes infinite rest-mass with energy $E = 0$. The resulting equation (7) is not hyperbolic. To get rid of this non-physical solution it is sufficient to impose an additional condition on $\Psi'_\lambda$,namely,

$$P_0\Psi'_\lambda = 0, P_0 = \prod_{\lambda \neq 0} \frac{\Lambda_{45}^2 - \lambda^2}{-\lambda^2} \tag{11}$$

where $P_0$ is the projection operator for the space of eigenvectors of the matrices $\Lambda_{45}$ corresponding to zero eigenvalue.

FW-transformed eqs. (7) and (8) of the original eq. (1) for the free particle of arbitrary spin implies that the particle can exist in multiplet of states associated with several spins and several masses correlated with limiting velocities which can exist in positive as well as negative energy states depending on the irreducible representation of $O(5)$ characterized by $(n_1, n_2)$. The states with positive energies are associated with positive masses and positive limiting velocities, while the negative states have negative masses correlated with negative limiting velocities. The state of minimum mass, and thus of the highest spin, is the most probable state a particle can find itself in.

# 2 Mass and Spin Content:

To find out the possible values of spin $s$ for a given mass $m_\lambda$ associated with eigenvalue $\lambda$, we note that the symmetry group $O(5)$ of eq. (1) is isomorphic to the compact symplectic group $Sp(4)$ which has following chain of groups [4]

$$U(4) \supset Sp(4) \supset U_\lambda(1) \otimes SU_S(2) \tag{12}$$

where $SU_S(2)$ is the usual spin group while $U_\lambda(1)$ is the group associated with the generator $\Lambda_{45}$. The basis associated with this chain of groups can be written as

$$\Psi \sim \mid \{h\} \left\langle n^1, n^2 \right\rangle s\sigma\lambda; \gamma\rangle \tag{13}$$

where $\gamma$ distinguishes between repeated values of $s$ and $\lambda$. The group $Sp(4)$ is characterized by the partition $\langle n^1, n^2 \rangle$ which is an integral of motion for the present problem. The quantum numbers $n^1, n^2$ are related with $n_1, n_2$ characterizing the irreps of $O(5)$ as follows

$$n_1 = \frac{1}{2}(n^1 + n^2) \text{ and } n_2 = \frac{1}{2}(n^1 - n^2) \tag{14}$$

The eigenvectors [5]

$$\Psi \sim \mid (n_1 n_2) p_1 p_2 S\sigma \rangle = P^s_{\sigma s} E^{n_1 - p_1}_{2-1} E^{p_1 - s}_{-2-1} E^{p_2}_{-20} \mid HW \rangle \tag{15}$$

with

$$n_1 \geq p_1 \geq n_2 \geq \frac{1}{2} p_2 \geq 0; p_1 \geq s \geq \frac{1}{2} p_2 \tag{16}$$

form a complete basis of the irrep $D^{(n_1, n_2)}$ of the $SO(5)$ algebra in the reduction

$$SO(5) \supset U_\lambda(1) \otimes SU_s(2) \tag{17}$$

where $p_1$ and $p_2$ in (15) replace the values of $\gamma$ and $\lambda$ mentioned above. In eq.(15) and $\mid HW\rangle$ is the highest weight vector of $D^{(n_1,n_2)}$, $E_{ik}$ are the generators of the $SO(5)$ algebra written in while $P^s_{\sigma s}$ is the projection operator for $SU_S(2)$ algebra. For details see ref. [5,6]. it is to be noted that generators $E_{\pm 10}$ and $E_{1-1}$ coincide with the spin operators $S_\pm = S_1 \pm iS_2$ and $S_3$ respectively while $E_{2-2} = \Lambda_{45}$ and $E_{\pm 2\pm 1}, E_{\pm 20}$ are connected with $\Lambda_{i4}$ and $\Lambda_{i5}$. In particular the eigenvalue $\lambda$ of $\Lambda_{45}$ can be determined by [4]

$$\lambda = n_1 + n_2 + s - (2p_1 + p_2) \tag{18}$$

It is to be noted that if $\lambda$ and $s$ are fixed then the number of solutions $p_1, p_2$ satisfying conditions (16) gives the multiplicity associated with the given pair $(\lambda, s)$. The multiplicity $N_s$ associated with any given value of spin $s$ can be calculated using above arguments or by considering the Gel'fand-Tsetlin inequalities (9) which gives that

$$\begin{aligned} N_s &= (n_1 - s + 1)(2n_2 + 1) \text{ for } s \geq n_2 \\ &= (n_1 - n_2 + 1)(2s + 1) \text{ for } s < n_2 \end{aligned} \tag{19}$$

The multiplicities of the spin projection $\sigma$ can be easily calculated using the relation

$$N_\sigma = \sum_{s \geq |\sigma|} N_s \tag{20}$$

Since the multiplicity of $\lambda$ is the same as that of $\sigma$ we get

$$\begin{aligned} N_\lambda &= \frac{1}{2}(n_1 - |\lambda| + 1)(n_1 - |\lambda| + 2)(2n_2 + 1) \text{ for } \lambda \geq n_2 \qquad (21) \\ &= \frac{1}{2}(n_1 - n_2 + 1)\left\{(n_1 - n_2 + 2)(2n_2 + 1) + 2(n_2^2 - \lambda^2)\right\} \text{ for } |\lambda| < n_2 \end{aligned}$$

We now find the maximum value of $\lambda$ for a fixed value of spin $S$. We note that all the numbers $n_1, n_2, s$ and $\lambda$ with

$$n_1 \geq p_1 \geq \max(n_2, s) \tag{22}$$

are integers or half integers. The quantity $p_2$ is an integer which satisfies the condition

$$\min(2s, 2n_2) \geq p_2 \geq 0 \tag{23}$$

By noting from eq.(16) that $\lambda_{\max}$ corresponds to the minimum value of $(2p_1+p_2)$, we obtain using eqs.(18,22,23) that [4]

$$\begin{aligned} \lambda_{\max} &= n_1+n_2-s \text{ for } s \geq n_2 \\ &= n_1-n_2+s \text{ for } s < n_2 \end{aligned} \tag{24}$$

In particular if $s=n_1$, $\lambda$ takes the values $n_2, n_2-1, ..., -n_2$ while if $s=0$ which corresponds to integer values of $n_1$ and $n_2$, $\lambda = n_1-n_2, n_1-n_2-2, ..., -n_1+n_2$. The first relation of the above equation allows us to find out the maximum value of spin $s$ at given value of $\lambda$, [4]

$$\begin{aligned} s_{\max} &= n_1+n_2-|\lambda| \text{ for } |\lambda| \geq n_2 \\ &= n_1 \text{ for } |\lambda| < n_2 \end{aligned} \tag{25}$$

while the second relation of eq.(24) gives the minimum value of spin

$$\begin{aligned} s_{\min} &= -n_1+n_2+|\lambda| \text{ for } |\lambda| \geq n_1-n_2 \\ &= 1/2 \text{ for } |\lambda| < n_1-n_2 \text{ with } n_1, n_2 \text{ being half integer} \\ &= 0 \text{ for } |\lambda| < n_1-n_2 \text{ with even } n_1-n_2-\lambda \\ &= 1 \text{ for } |\lambda| < n_1-n_2 \text{ with odd } n_1-n_2-\lambda \end{aligned} \tag{26}$$

Using relations (24), we get all possible values of masses a particular value of spin state can take, and relations (25,26) give all values of spin a given state associated with mass $m_\lambda$ can take. Thus we conclude by saying that the free particle of arbitrary spin whose relativistic equation is given by (1) describes a multiplet of states associated with several masses (8) and spins given in relations (24-26). In Table 1, we show the values of pair $(\lambda, s)$ for $n_1+n_2 \leq 3$.

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**TABLE 1**

| $n$ | $\{h\}$ | $< n^1, n^2 >$ | $(n_1, n_2)$ | $\lambda$ | $s$ |
|---|---|---|---|---|---|
| 1 | $\{1\}$ | $< 1, 0 >$ | $(1/2, 1/2)$ | $\pm 1/2$ | $1/2$ |
| 2 | $\{2\}$ | $< 2, 0 >$ | $(1, 1)$ | $\pm 1$ | 1 |
| | | | | 0 | 1 |
| | | | | 0 | 0 |
| | $\{11\}$ | $< 1, 1 >$ | $(1, 0)$ | $\pm 1$ | 0 |
| | | | | 0 | 1 |
| | | $< 0, 0 >$ | $(0, 0)$ | 0 | 0 |
| 3 | $\{3\}$ | $< 3, 0 >$ | $(3/2, 3/2)$ | $\pm 3/2$ | $3/2$ |
| | | | | $\pm 1/2$ | $3/2$ |
| | | | | $\pm 1/2$ | $1/2$ |
| | $\{21\}$ | $< 2, 1 >$ | $(3/2, 1/2)$ | $\pm 1/2$ | $3/2$ |
| | | | | $\pm 3/2$ | $1/2$ |
| | $\{111\}$ | $< 1, 0 >$ | $(1/2, 1/2)$ | $\pm 1/2$ | $1/2$ |