Analysis of relativistic particles of arbitrary spin through different chains of groups

M. Moshinsky *[a], A. G. Nikitin[a,c], A. Sharma[a], and Yu. F. Smirnov[b]

[a] Instituto de Física, UNAM, Apartado Postal 20-364, 01000 México, D. F. México
[b] Instituto de Ciencias Nucleares, UNAM. Apdo. Postal 70-543, 04510 México, D. F. México
[c] Institute of Mathematics, National Academy of Sciences of the Ukraine

Abstract

In a previous paper we analyzed the equation of particles of arbitrary spins characterized by the chain of groups SU(4)$\supset$ [S $\hat{\rm U}(2) \otimes$ S $\check{\rm U}(2)$]. In the present paper we find that the symmetry group of the problem rather than SU(4) is its smaller subgroup Sp(4) which is a unitary symplectic one. As Sp(4) is isomorphic to O(5) we can replace it by the latter and write our equation in terms of the generators of O(5)$\supset$ O(4)$\supset$ O(3) $\supset$ O(2) groups. These are more convenient as there are no multiplicity indices and the matrix elements of the generators can be given explicitly for an arbitrary irrep (n_1n_2) of O(5). The analysis is applied variationally to particles in an harmonic oscillator potential corresponding to the irreps $(\frac{1}{2}\frac{1}{2}), (11), (10)$ of O(5).

1 Introduction

In a previous paper[1] that will be denoted below as I, we discussed the particular problem of a free particle with arbitrary spin in a constant magnetic field. In the present publication we shall consider the situation in an external field involving a scalar potential $V(r)$. Our aim will be to get the appropriate equation in terms of the generators of a O(6) $\supset$ O(5) $\supset$ O(4) $\supset$ O(3), chain of groups rather than the U(4) $\supset$ $\hat{\rm U}$ (2) $\otimes$ $\check{\rm U}(2)$ used in I. The advantage will be that the first chain is canonical *i.e.* the representations of O$(p-1)$ contained in a given representation of O(p) will appear either once or not at all, and thus the matrix representation

*Member of El Colegio Nacional

of the generators of O(5) (which will turn out to be our fundamental group) can be obtained in an explicit analytical form.

Once we have the Hamiltonian operator of our equation appropriately formulated, we shall indicate a complete basis formed from the standard harmonic oscillator states in configuration space combined with the spin part, with the help of which this Hamiltonian can be transformed into an hermitian matrix of infinite number of components. Restricting ourselves to a given maximum number of quanta in our oscillator, the matrix becomes finite, thus giving us the possibility of diagonalizing it to get the energy eigenvalues for different representation of O(5), which in turn contain a finite number of possibilities for spin. We shall discuss explicitly the energy spectra when $V(r) = \frac{1}{2}m\Omega r^2$.

2 The Hamiltonian expressed in the O(6)⊃O(5)⊃ O(4)⊃ O(3) chain of groups

We use *c.g.s.* units in which we shall indicate momenta and positions by

$$\mathbf{r}', \mathbf{p}' \tag{1}$$

where we use this notations to reserves $\mathbf{r}, \mathbf{p}$ for more appropriate units. The Dirac equation for a spin $\frac{1}{2}$ particle in an external field can be written as

$$[c\boldsymbol{\alpha} \cdot \mathbf{p}' + mc^2\beta + V(r')]\psi = E'\psi \tag{2}$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{3}$$

and $\boldsymbol{\sigma}$ is the vector of the 2×2 Pauli spin matrices.

Our next point is to note that the 4×4 matrices $\boldsymbol{\alpha}, \beta$ in (2.3) can be converted into direct products of 2×2 ones by introducing the definitions[1]

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, s_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, s_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

$$\check{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, t_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, t_2 = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, t_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$\boldsymbol{\alpha} = 4\mathbf{s} \otimes t_1, \qquad \beta = 2\hat{I} \otimes t_3, \tag{5}$$

Following the procedure discussed in (2.6I) we can write (2.2) in the form

$$\left\{4c\sum_{i=1}^{3}(s_i \otimes t_i)p'_i + 2mc^2(\hat{I} \otimes t_3) + V(r')\right\}\psi = E'\psi \tag{6}$$

where $s_i, t_i, i = 1,2,3$ are the standard matrices for ordinary and sign spins given (2.4I, 2.5I) and $\otimes$ stands for the direct product. The E' indicates the energy in *c.g.s.* units. When we want to go to a problem of larger spins[1] we introduce an index $u = 1, 2 \ldots n$ for all the variables appearing in (2.6), sum the corresponding Hamiltonians and then make all $\mathbf{r}'_u, \mathbf{p}'_u$ equal to a single $\mathbf{r}', \mathbf{p}'$ thus getting an equation of the form

$$\sum_{u=1}^{n} \left\{ 4c \sum_{i=1}^{3} (s_{iu} \otimes t_{iu}) p'_i + 2mc^2 (\hat{I} \otimes t_{3u}) + nV(r') \right\} \psi = nE'\psi. \qquad (7)$$

whose spin can range from $(n/2), (n/2) - l, \ldots (1/2)$ or 0.

Now as in (3.9I) we define

$$S_i \equiv \sum_{u=1}^{n} (s_{iu} \otimes \check{I}), \qquad R_{ij} \equiv \sum_{u=1}^{n} (s_{iu} \otimes t_{ju}), \qquad T_i \equiv \sum_{u=1}^{n} (\hat{I} \otimes t_{iu}) \quad (8)$$

with $S_i, T_i, = 1,2,3$ being respectively the components of the total ordinary and sign spins, which together with the nine R_{ij}'s are the 15 generators of SU(4) group as shown by their commutation relations[2,3]

$$\begin{aligned} \Big[S_i, S_j\Big] &= i\epsilon_{ijk} S_k, \ \Big[T_i, T_j\Big] = i\epsilon_{ijk} T_k, \ \Big[S_i, T_j\Big] = 0, \\ \Big[S_i, R_{jk}\Big] &= i\epsilon_{ijl} R_{lk}, \ \Big[T_i, R_{jk}\Big] = i\epsilon_{ikl} R_{jl}, \\ \Big[R_{ij}, R_{kl}\Big] &= \frac{1}{4} i\epsilon_{ikm} S_m \delta_{jl} + \frac{1}{4} i\delta_{ik}\ \epsilon_{jln} T_n. \end{aligned} \qquad (9)$$

Using the definitions (2.8) we can rewrite Eq. (2.7) as

$$[4c \sum_{i=1}^{3} R_{i1} p'_i + 2mc^2 T_3 + nV(r')]\psi = nE'\psi \qquad (10)$$

This last equation can not, in general, be solved exactly and thus we need a convenient complete basis in which to express the operator in the square bracket in (2.10) as a numerical matrix.

The first requirement concerns a basis for the ordinary and sign spins. In I we characterized them by the chain of groups U(4)$\supset$ [S Û (2)$\otimes$ S Ǔ (2)], familiar in nuclear physics[4] when we combine the ordinary spin with the isospin to get **supermultiplets**. The states can then be expressed by the kets $|\{h\}\gamma s\sigma t\tau\rangle$, where $\{h\} = [h_1 h_2 h_3 h_4]$ is the partition of n corresponding to the representation of U(4), while $s(s+1), t(t+1)$ are the eigenvalues of Casimir operators associated with the SU(2) of ordinary and sign spins, and σ, τ characterize the corresponding orthogonal O(2) subgroups of SU(2).

We notice that the ket in the previous paragraph has an extra index γ, that serves to distinguish representations of SÛ(2) $\otimes$ SǓ(2) that appear more than once in a given representation of SU(4). This feature complicates greatly the *general* representation of R_{ij} in the basis of the ket given above. Thus we decided to follow another chain using the fact SU(4) is isomorphic to the orthogonal group O(6).

The generators of O(6) can be characterized by the antisymmetric operators $\wedge_{mm'} = -\wedge_{m'm}$ with $m, m' = 1, 2, 3, 4, 5, 6$ and thus there are 15 of them that satisfy the commutation rules[5]

$$[\wedge_{mm'},\ \wedge_{nn'}] = i[\delta_{m'n} \wedge_{n'm} + \delta_{mn'} \wedge_{nm'} + \delta_{mn} \wedge_{m'n'} + \delta_{m'n'} \wedge_{mn}] \tag{11}$$

Comparing them with the commutation rules (3.11I) we easily see that $\wedge_{mm'}$ with $m < m'$ (to avoid the repetition due to the antisymmetry) are correlated with $S_i, R_{ij}, T_j, i, j = 1, 2, 3$ in the following way:

$$\begin{aligned} \tfrac{1}{2}\epsilon_{ijk}\wedge_{jk} &= S_i \\ \wedge_{i4} &= 2R_{i1} \\ \wedge_{i5} &= 2R_{i2} \\ \wedge_{i6} &= 2R_{i3} \\ \wedge_{45} &= T_3 \\ \wedge_{46} &= -T_2 \\ \wedge_{56} &= T_1 \end{aligned} \tag{12}$$

where i, j, k take the values 1,2,3 and repeated indices are summed over these values.

Now O(6) has the following chain of subgroups O(6)⊃O(5)⊃ O(4)⊃O(3)⊃O(2) whose generators in terms of the operators (2.9) are given by

$$\begin{array}{cccc} 15 & S_i, R_{ij}, T_i & i, j = 1, 2, 3 & O(6) \\ 10 & S_i, R_{i1}, R_{i2}, T_3 & i = 1, 2, 3 & O(5) \\ 6 & S_i, R_{i1} & i = 1, 2, 3 & O(4) \\ 3 & S_i & i = 1, 2, 3 & O(3) \\ 1 & S_3 & & O(2) \end{array} \tag{13}$$

where on the left hand side we give the number of generators. Using (3.11 I), we easily check that the generators of each subgroup close under commutation.

We note now that in Eq. (2.10) only R_{i1} and T_3 appear so we can restrict, ourselves to O(5) as the symmetry group. Nevertheless we consider the representations starting from O(6) as we would like to characterize our kets also by the $\{h\}$ which is the partition of n that was mentioned above and characterizes the irreducible representation of U(4).

To achieve the purpose of the last paragraph we note that

$$\wedge_{12} = S_3, \qquad \wedge_{45} = T_3, \qquad \wedge_{36} = 2R_{33} \tag{14}$$

commute among themselves as seen from the relations (2.9) or (2.11). They could then be considered as three weight generators[6] of the

O(6) group, while the 12 that remain can be divided into groups of 6 each, corresponding to the raising and lowering generators of the group mentioned. If we consider the state of highest weight[7], which is an eigenstate of the operators (2.14), the corresponding eigenvalues can be denoted by

$$q_1, q_2, q_3 \tag{15}$$

Turning now our attention to U(4) group, its generators are of the form

$$C_r^{r'}, r < r'; \qquad C_r^{r'}, r = r'; \qquad C_r^{r'}\ r > r' \tag{16}$$

where $r, r' = 1, 2, 3, 4$ and we indicate the raising, weight and lowering generators. Again if we have a state of highest weight it would be an eigenstate of $C_1^1, C_2^2, C_3^3, C_4^4$ and their eigenvalues

$$h_1, h_2, h_3, h_4 \tag{17}$$

characterize the irreducible representations of U(4). In table II of reference 3 we give, in spherical component form, the S_i, T_i, R_{ij} as linear functions of $C_r^{r'}$ and, in particular, for the weight generators, where the index 0 is equivalent to the index 3 of cartesian components, we have that (2.15) are related to (2.17) by[3]

$$q_1 = \frac{1}{2}(h_1{+}h_2{-}h_3{-}h_4), q_2 = \frac{1}{2}(h_1{-}h_2{+}h_3{-}h_4), q_3 = \frac{1}{2}(h_1{-}h_2{-}h_3{+}h_4), \tag{18}$$

As $h_1 + h_2 + h_3 + h_4 = n$ with $h_1 \geq h_2 \geq h_3 \geq h_4$ we see that q_1, q_2, q_3 can be integer or semi integer numbers depending on wether n is even or odd and furthermore $q_1 \geq q_2 \geq q_3$ with q_1, q_2 being positive while, in some cases, q_3 could also take negative values.

Having established the relation between the irreducible representations (irreps) of O(6) and U(4), we turn our attention to O(5) which, as we indicated before, is a smaller symmetry group for the Hamiltonian in Eq.(2.10) as, using (2.12), it can be written in the form

$$[2c \sum_{i=1}^{3} \wedge_{i4} p'_i + 2mc^2 \wedge_{45} + nV(r')]\psi = nE'\psi. \tag{19}$$

3 Matrix elements of the generators $\wedge_{45}, \wedge_{i4}$, $i = 1, 2, 3$ in a basis of irreps in the chain O(5)⊃ O(4), O(3) ⊃ O(2)

As is well known[8,9] the irreps of O(2 k+1) and O(2k) are characterized by partitions involving only k numbers that can be integer or seminteger and non-negative, except for the last one in the even case which sometimes can be negative.

Rather than discussing the general theory analyzed in references [7,8], we shall restrict our analysis to the chain of orthogonal groups that appear in the title of this section, where the irreps will be denoted as follows:

$$\begin{array}{l} O(5)\ ;\ n_1, n_2 \\ O(4)\ ;\ m_1, m_2 \\ O(3)\ ;\ s \\ O(2)\ ;\ \sigma \end{array} \tag{1}$$

As O(5) is a subgroups of O(6), n_1, n_2 are restricted by the in equalities[8]

$$q_1 \geq n_1 \geq q_2 \geq n_2 \geq |q_3|. \tag{2}$$

Turning now our attention to O(4), m_1, m_2 are restricted by the inequalities[8,9]

$$n_1 \geq m_1 \geq n_2 \geq |m_2|. \tag{3}$$

For O(3) we have the single number s restricted by

$$m_1 \geq s \geq |m_2|. \tag{4}$$

Finally σ of O(2) is restricted by $|\sigma| \leq s$ which implies that is given by

$$\sigma = s, s-1, \ldots, -s+1, -s \tag{5}$$

as all the values indicated can only change by one unit at a time within the limits indicated in the inequalities. We note then that the integer or seminteger character of the representation (q_1, q_2, q_3) of O(6) propagates to all of its subgroups.

The kets for the spin part of O(5) $\supset$ O(4)$\supset$ O(3) $\supset$ O(2) chain of groups, can be denoted by

$$\left| \begin{array}{c} n_1 n_2 \\ m_1 m_2 \\ s \\ \sigma \end{array} \right\rangle \tag{6}$$

and the matrix elements of $\wedge_{45}, \wedge_{34}$ with respect to them have been calculated in references [10,11]. Before giving them explicitly here, we note that $\wedge_{i4}$ is a Racah tensor of order 1 with respect to the O(3) group and, in particular, $\wedge_{34}$ corresponds to the component 0 of this tensor so we have by the Wigner–Eckart theorem that[9]

$$\left\langle \begin{array}{c} n_1 n_2 \\ m_1' m_2' \\ s' \\ \sigma' \end{array} \right| \wedge_{34} \left| \begin{array}{c} n_1 n_2 \\ m_1 m_2 \\ s \\ \sigma \end{array} \right\rangle = \langle s\sigma, 10 | s'\sigma' \rangle \left\langle \begin{array}{c} n_1 n_2 \\ m_1' m_2' \\ s' \end{array} \right\| \wedge_4 \left\| \begin{array}{c} n_1 n_2 \\ m_1 m_2 \\ s \end{array} \right\rangle, \tag{7}$$

where $\langle | \rangle$ is a standard O(3) Clebsch–Gordan coefficient. Thus for $\wedge_{14}, \wedge_{24}$ we need only the reduced matrix element on the right hand side of (3.7), and its explicit value, together with that of $\wedge_{45}$, is given below [10,11]

$$\left\langle \begin{matrix} n_1 n_2 \\ m_1' m_2' \\ s \end{matrix} \right| \wedge_{45} \left| \begin{matrix} n_1 n_2 \\ m_1 m_2 \\ s \end{matrix} \right\rangle =$$

$$-\frac{i}{2}\sqrt{\frac{(m_1-s+1)(m_1+s+2)(n_1-m_1)(n_1+m_1+3)(m_1-n_2+1)(m_1+n_2+2)}{(m_1+m_2+1)(m_1+m_2+2)(m_1-m_2+1)(m_1-m_2+2)}}\delta$$

$$-\frac{i}{2}\sqrt{\frac{(s-m_2)(s+m_2+1)(n_2-m_2)(n_2+m_2+1)(n_1-m_2+1)(n_1+m_2+2)}{(m_1+m_2+2)(m_1+m_2+1)(m_1-m_2)(m_1-m_2+1)}}\delta_{m_1',m}$$

$$+\frac{i}{2}\sqrt{\frac{(s+m_1+1)(m_1-s)(n_1-m_1+1)(n_1+m_1+2)(m_1-n_2)(m_1+n_2+1)}{(m_1+m_2)(m_1+m_2+1)(m_1-m_2)(m_1-m_2+1)}}\delta_{m_1',m}$$

$$+\frac{i}{2}\sqrt{\frac{(s-m_2+1)(s+m_2)(n_2-m_2+1)(n_2+m_2)(n_1-m_2+2)(m_2+n_1+1)}{(m_1+m_2)(m_1+m_2+1)(m_1-m_2+2)(m_1-m_2+1)}}\delta_{m_1',m}$$

$$\left\langle \begin{matrix} n_1 n_2 \\ m_1 m_2 \\ s' \end{matrix} \right\| \wedge_4 \left\| \begin{matrix} n_1 n_2 \\ m_1 m_2 \\ s \end{matrix} \right\rangle = -i\sqrt{\frac{(m_1-s)(m_1+s+2)(s-m_2+1)(s+m_2+1)}{(2s+3)(s+1)}}\delta_{s',s+1}$$

$$+\frac{(m_1+1)m_2}{\sqrt{s(s+1)}}\delta_{s',s} + i\sqrt{\frac{(m_1-s+1)(m_1+s+1)(s-m_2)(s+m_2)}{(2s-1)s}}\delta_{s',s-1}$$

Note now that in Eq. (2.19) only $\wedge_{i4}, \wedge_{45}$ appear, which are generators of O(5), and thus (n_1, n_2), that give the irrep of O(5), are integrals of motion for the Hamiltonian operator in the square bracket of (2.19).

4 The complete set of variational states and the matrix elements of our Hamiltonian with respect to them.

So far we have not mentioned that part of our state that is a function of $\mathbf{r}'$ in our configuration space. As, in general, the Eq. (2.19) does not admit an exact solution we choose the simplest set of states, that of the harmonic oscillator, to carry our analysis variationally. As the frequency ω of the oscillator is our only parameter, we can introduce it in the Hamiltonian[11] and thus consider only states of frequency 1 given by the ket

$$|Nl\mu\rangle = R_{Nl}(r')Y_{l\mu}(\theta,\varphi) \qquad (1)$$

with Y being a spherical harmonic where l is the orbital angular momentum, while R is the radial part of the ket characterized by the number of quanta N.

As the total angular momentum

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \quad \mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{2}$$

is obviously an integral of motion, we can write our full ket as

$$\left| N \begin{pmatrix} l, & \begin{matrix} n_1 n_2 \\ m_1 m_2 \\ s \end{matrix} \end{pmatrix} jm \right\rangle = \sum_{\sigma\mu} \langle l\mu, s\sigma | jm \rangle |Nl\mu\rangle \left| \begin{matrix} n_1 n_2 \\ m_1 m_2 \\ s \\ \sigma \end{matrix} \right\rangle \tag{3}$$

where $\langle | \rangle$ is a Clebsch Gordon coefficient.

We now have to apply standard Racah algebra[11] to the state (4.3) and we get the following result

$$\left\langle N \begin{pmatrix} l', & \begin{matrix} n_1 n_2 \\ m_1' m_2' \\ s' \end{matrix} \end{pmatrix} jm \right| 2c \sum_{i=1}^{3} \wedge_{i4} p_i + 2mc^2 \wedge_{45} + nV(r') \left| N \begin{pmatrix} l, & \begin{matrix} n_1 n_2 \\ m_1 m_2 \\ s \end{matrix} \end{pmatrix} jm \right\rangle =$$

$$= 2(-1)^{l'+s-j} W(ll'ss';j)[(2l'+1)(2s'+1)]^{1/2}$$

$$\langle N'l' \| p' \| Nl \rangle \left\langle \begin{matrix} n_1 n_2 \\ m_1' m_2' \\ s' \end{matrix} \right\| \wedge_4 \left\| \begin{matrix} n_1 n_2 \\ m_1 m_2 \\ s \end{matrix} \right\rangle$$

$$+ 2\delta_{N'N} \delta_{l'l} \left\langle \begin{matrix} n_1 n_2 \\ m_1' m_2' \\ s' \end{matrix} \right| \wedge_{45} \left| \begin{matrix} n_1 n_2 \\ m_1 m_2 \\ s \end{matrix} \right\rangle$$

$$+ n\delta_{l'l} \delta_{m'm_1} \delta_{m_2' m_2} \delta_{s's} \langle N'l \| V(r') \| Nl \rangle \tag{4}$$

where we have assumed that $V(r)$ is only function of the magnitude of $\mathbf{r}$.

In (4.4) the W is a Racah coefficient and the reduced matrix elements of $\wedge_{i4}, \wedge_{45}$ are given in (3.8,3.9). To get then the full matrix representation of our Hamiltonian we only to need determine the reduced matrix elements of p' and $V(r')$ which will be discussed in the next section where we shall introduce more appropriate units.

5 Example: The energy spectra of a relativistic harmonic oscillator corresponding to a definite irrep of O(5)

The equation is now (2.19) where

$$V(r') = \tfrac{1}{2} m \Omega^2 r'^2 \tag{1}$$

with m the mass and Ω the frequency of the oscillator, all in *cgs* units.

We shall introduce dimensionless units by dividing (2.19) by $\hbar\Omega$ and defining

$$E = (\hbar\Omega)^{-1}(E' - mc^2) \qquad (2)$$

We now note that $\mathbf{r}', \mathbf{p}'$ are in *c.g.s* units associated with a frequency ω of the variational states. Instead of having ω in the state (4.1) we introduce it in the Hamiltonian by replacing $\mathbf{r}', \mathbf{p}'$ by dimensionless expressions through the relation

$$\mathbf{r}' = (\hbar/m\omega)^{1/2}\mathbf{r}, \mathbf{p}' = (m\omega\hbar)^{-1/2}\mathbf{p} \qquad (3)$$

The equation (2.19) can then be written as

$$\left\{2a\epsilon\sum_{2=1}^{3}(\wedge_{i4}p_i) + a^2(2\wedge_{45} - n) + \frac{nr^2}{2\epsilon^2}\right\}\psi = nE\psi \qquad (4)$$

where we substracted the total rest energy and

$$a \equiv \left(\frac{mc^2}{\hbar\Omega}\right)^{1/2}, \qquad \epsilon = (\omega/\Omega)^{1/2} \qquad (5)$$

and ϵ will be now the variational parameter against which we plot the energy E.

The matrix representation of the left hand side of Eq. (5.4) can be obtained in a way similar to (4.4) replacing $\mathbf{r}', \mathbf{p}'$ by $\mathbf{r}, \mathbf{p}$ so we need only to introduce in (4.4) the well know [12,13] reduced matrix elements

$$\begin{aligned}
&\langle N'l'\|p\|Nl\rangle \\
&= \frac{i}{\sqrt{2}}\left\{\left[(N+l+3)^{1/2}\delta_{N'N+1} + (N-l)^{1/2}\delta_{N'N-1}\right]\sqrt{\frac{(l+1)}{(2l+3)}}\delta_{l'l+1}\right. \\
&\left. + \left[(N-l+2)^{1/2}\delta_{N'N+1} + (N+l+1)^{1/2}\delta_{N'N-1}\right]\sqrt{\frac{l}{(2l-1)}}\delta_{l',l-1}\right\} \\
&\langle N'l'\|r^2\|Nl\rangle = -\frac{1}{2}\sqrt{(N-l)(N+l+1)}\delta_{N'N-2} \\
&+(N+\tfrac{3}{2})\delta_{N'N} - (1/2)\sqrt{(N-l+2)(N+l+3)}\delta_{N'N+2} \qquad (6)
\end{aligned}$$

From (3.8), (3.9), (4.4) and (5.6), (5.7) we can then write the matrix representation of the operator of the left hand side of (5.4), in which the irrep (n_1n_2) of O(5) is an integral of motion together with the total angular momentum j. This implies that in (4.4), $(n_1n_2), j$ are fixed and the same in bra and ket.

In the following subsections we shall discuss simple particular cases of this matrix taking the lowest value of j allowed *i.e.* $j = 0$ if n is even or $j = \frac{1}{2}$ if n is odd.

In the title of the examples given below we have both the partitions $\{h\}$ irrep of U(4), and (n_1n_2) irrep of O(5) related by (2.15), as well as the values of j mentioned in the previous phrase.

a) The case $\{h\} = 1, (n_1 n_2) = (\frac{1}{2}\frac{1}{2}), j = 1/2$

This corresponds to the ordinary Dirac equation in an oscillator potential and from the inequalities (3.3) to (3.5) we see that O(5) part (3.6) takes the form

$$\left| \begin{matrix} \frac{1}{2}\frac{1}{2} \\ \frac{1}{2}m_2 \\ \frac{1}{2} \\ \sigma \end{matrix} \right\rangle \tag{7}$$

As $j = \frac{1}{2}, l$ could only be 0 or 1 and this is given automatically if N is even or odd.

Thus the kets (4.3) could be written in the short hand notation

$$|N^{m_2}\rangle, \quad m_2 = \pm\tfrac{1}{2}, \quad N = 0, 1, 2, \ldots . \tag{8}$$

Ordering the kets by increasing values of N and, instead of $m_2 = \pm\frac{1}{2}$, just denoting it as $m_2 = \pm$ the set of states, (where we suppress the ket notation), can be indicated by

$$0^+0^-1^+1^- \ldots 14^+14^-15^+ \ldots \tag{9}$$

where we end with an N^+ to get a better fit to the positive energy levels. In Fig. 1(a,b) we give as a function of the variational parameter ϵ, the values of the positive and negative energies separate and together, for $a = 10$ or 1. For $a = 10$ we have the non-relativistic limit and we see that the positive energies are equally separated and in fact are close to the value $E = (N + \frac{3}{2})$ at least for the levels up to $N = 10$.

b) $\{h\} = \{2\}, (n_1, n_2) = (1, 1); j = 0$

The present case concerns $n = 2$ in which the spin s can take values 0 and 1. Since we consider here the lowest possible value of total spin $j = 0, l$ can take only two values viz 0 and 1. Thus even value of N occurs with $l = 0, s = 0$, while odd N corresponds to the combination $l = 1, s = 1$. In the chain of groups O(5)$\supset$ O(4) $\supset$ O(3) $\supset$O(2), the symmetric partition $\{2\}$ can have the following ket

$$\left| \begin{matrix} 1\,1 \\ 1\,m_2 \\ s \\ \sigma \end{matrix} \right\rangle \qquad \text{with} \qquad m_2 = \pm 1, 0 \text{ and } s = 0, 1. \tag{10}$$

$s = 1$ can have all possible values of $m_2, viz, \pm 1$ or 0 while $s = 0$ can correspond to only $m_2 = 0$. Since all quantum numbers except s and m_2 are fixed in the above ket (with σ not occurring explicitly in the matrix) we can denote the states in short hand notation as

$$|N^{sm_2}\rangle \tag{11}$$

which according to above arguments would give rise to following chain of states in the matrix.

$$0^{00}1^{11}1^{10}1^{1-1}2^{00}3^{11}3^{10}3^{1-1}\dots \tag{12}$$

We restrict the hamiltonian matrix in this chain of states to 14^{00} whose diagonalized eigenvalues (E) are plotted against the variational parameter ϵ in Figs. 2a and 2b for $a = 10$ and $a = 1$ respectively. The general behavior of the curves can be easily explained by considering eq. (5.4). When $\epsilon \to 0$ the kinetic energy as well as the rest-mass terms become negligible compared to the potential energy which goes to $+\infty$, thus $E \to \infty$. But as $\epsilon \to \infty$ potential energy term goes to zero and the equation reduces to that of free particle whose energy spectra is given by[14]

$$E_m = \left(\frac{n-2m}{n}\right)\sqrt{p^2+1} - a^2; \qquad m = 0, 1, 2, \dots n \tag{13}$$

Thus for the present case with $n = 2$, there appears sets of curves corresponding to positive energies forming bound states, a negative continuum and a set which converges to $-a^2$ as $\epsilon \to \infty$. The chain of states (5.12) gives rise to positive bound states occurring for odd values of N, thus the ground state does not exist in this case of $j = 0$. Fig. 2a gives the result for non-relativistic case $a = 10$ for which positive bound states occurring at $\epsilon = 1.02$ follow the non-relativistic result $E = N + 3/2$. It is to be noted that a gap of a^2 (or mc^2) exists between the three sets of levels starting from where bound states are formed. Fig. 2b gives the results for fully relativistic region $a = 1$ wherein the bound states are formed at higher values of $\epsilon(\epsilon = 2.28)$, which no longer follow the non-relativistic result noted above.

c) $\{h\} = \{11\}, (n_1, n_2) = (1, 0); j = 0$

The antisymmetric partition $\{11\}$ in the canonical chain of orthogonal groups correspond to following ket

$$\left| \begin{matrix} 1\,0 \\ m_1\,0 \\ s \\ \sigma \end{matrix} \right\rangle \qquad \text{with} \qquad m_1 = 0, 1 \text{ and } s = 0, 1. \tag{14}$$

which can be written for convenience as

$$|N^{sm_1}\rangle \tag{15}$$

In the present case the chain of states can be written as

$$0^{01}0^{00}1^{11}2^{01}2^{00}\dots \tag{16}$$

which we restrict to 15^{11} while calculating the matrix whose diagonalized results are shown in Figs. 3a and 3b corresponding to $a = 10$ and

$a = 1$ respectively. The behavior of the curves is similar to the previous case of $\{2\}$ except that the present case gives rise to bound states with even values of N. The minimum for E for non-relativistic case $a = 10$ occurs at $\epsilon = 1.02$ as in the $\{2\}$ but for $a = 1$ it occurs at $\epsilon = 2.62$.

Table 1 gives the results for bound states for non-relativistic case $a = 10$, which appear at $\epsilon = 1.02$ for all partitions discussed. Table 2 shows the bound states for fully relativistic region of $a = 1$. For $\{1\}$, they are formed at $\epsilon = 2.5$ while for the other cases of $j = 0, n = 2$, they appear at $\epsilon = 2.62$.

6 Conclusions

We have derived a wave equation for a relativistic particle with arbitrary spin using the generators of the O(5)$\supset$ O(4)$\supset$ O(3) chain of groups. We did not discuss the Lorentz invariance of our equation as its initial formulation is in terms of the α's rather than the γ's matrices. We shall use the latter in a future publication showing not only that the equations are Poincaré invariant, but also that they lead through our simple supermultiple formulation to the Bhabha equation proposed long ago[15].

References

[1] M. Moshinsky and Yu. F. Smirnov *J. Phys. A: Math. Gen.* **29**, 6027 (1996).

[2] M. Moshinsky and J. G. Nagel, *Phys. Lett.* **5**, 173 (1963).

[3] M. Moshinsky, *J. Math. Phys.* **7**, 691 (1966)

[4] E. P. Wigner, *Phys. Rev.* **51**, 106 (1937).

[5] M. Moshinsky, *Group Theory and the many body problem* (Gordon and Breach, New York, 1968) pp. 36.

[6] M.Moshinsky, loc. cit. p.14

[7] I. M. Gelfand and M. L.Zetlin, *Dok. Akad. Nauk. USSR* **71**, 147 (1950) (In Russian).

[8] S.C. Pang and K. T. Hecht , *J. Math. Phys* **8**, 1233 (1967) .

[9] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley and Sons, New York 1957) p. 85-88.

[10] G. F.Filippov, V. I. Ovcharenko, Yu F. Smirnov *Microscopic theory of collective excitations of atomic nuclei.* Kiev, Nauka Dumka 1981 (In Russian) pp. 252-254

[11] Nikitin

[12] M. Moshinsky and Yu. F. Smirnov, *The Harmonic Oscillator in Modern Physics*, (Harwood Academic Press, The Netherlands 1996) Eq. (10.35) in pp. 35.

[13] M. Moshinsky and C. Quesne, *Ann. Phys. (N.Y.)* **148**, 462 (1983).

[14] M. Moshinsky, *Rev. Mex. Fis.* **43**, 511 (1997)

[15] H. J. Bhabha, *Rev. Mod. Phys.* **17**, 200 (1945).

Figure Captions

Fig. 1. This figure gives plots for the energy E versus the variational parameter ϵ for the case $n = 1$ with partition $\{1\}$ for which the lowest possible value of the total spin $j = 1/2$ is taken. Fig. 1a shows the results for $a = 10$ corresponding to the non-relativistic case, which contains two parts wherein positive energy curves are drawn separately in (i) and then along with those corresponding to the negative energies (ii). The bound states with minimum value of positive E occur at $\epsilon = 1.02$, and agree fairly well with the non-relativistic result (5.11). Fig. 1b shows the energy plots for the case $a = 1$ which corresponds to the fully relativistic region. The curves in general show behaviour similar to the above case except that positive bound states occur for higher values of $\epsilon = 2.5$, which no longer follow the non-relativistic result.

Fig. 2. This figure shows the curves of E versus ϵ for $n = 2$ with partition $\{2\}$ for which lowest spin $j = 0$ is taken. This case gives rise to three sets of curves corresponding to the positive energies, negative continuum, and the energies which converge to $-a^2$ as $\epsilon \to \infty$. Figs. 2a and 2b show the results for $a = 10$ and $a = 1$ respectively with each figure containing two parts wherein positive energies are drawn separately in (i) and then along with other sets in part (ii) thus showing all the energy curves together. The positive energy curves for $a = 10$ show a behaviour similar to that in partition $\{1\}$, and the bound states occurring at $\epsilon = 1.02$, follow the non-relativistic rule. In this case, the positive bound states occur for odd values of N. For $a = 1$, the bound states occur at $\epsilon = 2.28$, which no longer follow the non-relativistic rule.

Fig.3. This figure gives the result for partition $\{11\}$ with $j = 0$. Figs. 3a and 3b show the curves for $a = 10$ and $a = 1$ respectively. The general behaviour of the curves is exactly similar to those of partition $\{2\}$ except that in the present case the positive bound states occur for even values of N. For $a = 1$, the bound states occur at $\epsilon = 2.62$ instead of $\epsilon = 2.28$ as in the previous case of $\{2\}$.