WIGNER INDUCED REPRESENTATIONS METHOD AND POINCARÉ PARASUPERALGEBRA

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Abstract

We find irreducible unitary representations of the extended Poincaré parasuperalgebra for timelike, lightlike, and spacelike four-momenta. This parasuperalgebra includes as a particular case the usual Poincaré superalgebra and can serve as the group–theoretical foundation of parasupersymmetric quantum field theory.

1 Introduction

Parasupersymmetric quantum mechanics [1] awaked undoubted interest and stimulated appearance of a lot of papers, see [2] and papers cited therein.

Beckers and Debergh [3] asked for Poincaré invariance of the theory and formulated group–theoretical foundations of so–called parasupersymmetric quantum field theory (QFT). This theory is a natural generalization of SUSY QFT, dealing with parastatistics instead of the usual Fermi or Bose statistics and with the Poincaré parasupergroup (or Poincaré parasuperalgebra, (PPSA)) instead of the Poincaré supergroup (or Poincaré superalgebra (PSA)).

In paper [4] irreducible representations (IRs) of the simplest $N = 1$ PPSA (including the only parasupercharge) were described.

Here, using the Wigner induced representations method, we find IRs of the extended PPSA with *arbitrary* number of parasupercharges. In this way we formulate the group–theoretical foundations of the PSSFT with arbitrary N, and generate a new view–point to the PSA which appears in our approach as a particular realization of the PPSA.

2 The Poincaré parasuperalgebra

The PPSA [3] includes ten generators $P_\nu,\ J_{\nu\sigma}$ of the Poincaré group, satisfying the usual commutation relations

$$\begin{array}{c}[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu),\\ [J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho})\end{array} \tag{1}$$

($J_{\mu\nu} = -J_{\nu\mu}, \quad \mu,\nu = 0,1,2,3$), and N parasupercharges $Q^j_A,\ \bar{Q}^j_A$ ($A = 1,2,\, j = 1,2,...,N$) which satisfy the following double commutation relations

$$[Q^i_A, [Q^j_B, \bar{Q}^k_C]] = -4\delta_{ik}Q^j_B(\sigma_\mu)_{AC}P^\mu, \quad [\bar{Q}^i_A, [Q^j_B, \bar{Q}^k_C]] = 4\delta_{ik}\bar{Q}^j_C(\sigma_\mu)_{BA}P^\mu,$$

$$[Q^i_A, [Q^j_B, Q^k_C]] = [\bar{Q}^i_A, [\bar{Q}^j_B, \bar{Q}^k_C]] = 0. \tag{2}$$

Here σ_ν are the Pauli matrices, $(.)_{AC}$ are the related matrix elements.

Furthermore, parasupercharges commute with generators of the Poincaré group as follows

$$[J_{\mu\nu}, Q^j_A] = -\tfrac{1}{2i}\left(\sigma_{\mu\nu}\right)_{AB} Q^j_B,\ [P_\mu, Q^j_A] = 0, \tag{3}$$

where $\sigma_{\nu\sigma} = -\sigma_{\sigma\nu} = \sigma_\nu\sigma_\sigma$ and commutation relations for $\bar{Q}^i_A$ are obtained from (3) using hermitian conjugation.

The PPSA is a direct (and natural) generalization of the PSA [5] whose masis elements satisfy relations (1)-(3) also. The converse is not true, i.e., basis elements of the PPSA in general does not satisfy PSA.

3 Classification of IRs and representations of class I

It is convenient to introduce the four–vector $B_\mu = W_\mu + X_\mu$ where $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma$ is the Lubanski–Pauli vector, $X_\mu = \left(\sigma_\mu\right)_{AB}\bar{Q}_A Q_B$.

It follows from (1)-(3) that $C_1 = P_\mu P^\mu$ and $C_2 = P_\mu P^\mu B_\nu B^\nu - (B_\mu P^\mu)^2$ are the Casimir operators of the PPSA.

We will search for representations of the algebra (1)– (3) in the momentum representation. As in the case of the ordinary Poincaré algebra [6] we distinguish three main classes of IRs corresponding to: $I.\ \ P_\mu P^\mu = M^2 > 0$, $\ \ II.\ \ P_\mu P^\mu = 0$, $\ \ III.\ \ P_\mu P^\mu = -\eta^2 < 0$.

For representations of class I there exist the additional Casimir operator $C_3 = P_0/|P_0|$. We restrict ourselves to considering IRs corresponding to $\varepsilon = +1$. Let us define "a Wigner little parasuperalgebra " (LPSA) associated with the time–like four–vector $P = (M, 0, 0, 0)$. We set $B_k = W_k + X_k = -MS_k + X_k \equiv Mj_k,\ k = 1, 2, 3$, then $[j_k, Q^i_A] = [j_k, \bar{Q}^i_A] = 0$, and

$$[j_k, j_j] = i\varepsilon_{kjl} j_l \tag{4}$$

$$[Q^i_A, [\bar{Q}^j_A, Q^k_B]] = 4\delta_{ij} M Q^k_B, \qquad [\bar{Q}^i_A, [Q^j_A, \bar{Q}^k_B]] = 4\delta_{ij} M \bar{Q}^k_B, \tag{5}$$

the other double commutators of Q^i_A and $\bar{Q}^j_A$ are equal to zero.

Relations (4) define algebra o(3). To find IRs of algebra (5) we denote $Q^i_A = \sqrt{2M}(S_{4N+1\ 4i+2A-5} + iS_{4N+1\ 4i+2A-4})$, and

$$\begin{array}{l}
[Q^j_A, \bar{Q}^k_B] = 2M\left[S_{4j+2A-5\ 4k+2B-4} - S_{4j+2A-4\ 4k+2B-5}\right.\\
\left.+i\left(S_{4j+2A-5\ 4k+2B-5} + S_{4j+2A-4\ 4k+2B-4}\right)\right],\\
[Q^j_A, Q^k_B] = 2M\left[S_{4j+2A-5\ 4k+2B-4} - S_{4j+2A-5\ 4k+2B-4}\right.\\
\left.+i\left(S_{4j+2A-5\ 4k+2B-5} - S_{4j+2A-4\ 4k+2B-4}\right)\right],
\end{array}$$

and suppose S_{AB} to be hermitian. As a result we come to commutation relations which characterize algebra so(4N+1) $(k, l = 1, 2, \ldots 4N + 1)$:

$$[S_{kl}, S_{mn}] = i(\delta_{km} S_{ln} + \delta_{ln} S_{km} - \delta_{kn} S_{lm} - \delta_{lm} S_{kn}), \tag{6}$$

Thus for $P_\nu P^\nu > 0$ the LPSA reduces to the direct sum of the algebras o(3) and so(4N+1). The explicit form of the corresponding parasupercharges and generators of the Poincaré is

$$\begin{array}{c}
Q^i_1 = \frac{1}{\sqrt{E+M}}[(S_{4N+1\ 4i-3} + iS_{4N+1\ 4i-2})(E + M + \varepsilon p_3)\\
+\varepsilon(S_{4N+1\ 4i-1} + iS_{4N+1\ 4i})(p_1 - ip_2)],\\
Q^i_2 = \frac{1}{\sqrt{E+M}}[\varepsilon(S_{4N+1\ 4i-3} + iS_{4N+1\ 4i-2})(p_1 + ip_2)\\
+(S_{4N+1\ 4i-1} + iS_{4N+1\ 4i})(E + M - \varepsilon p_3)], \quad \bar{Q}_A = Q^+_A,
\end{array} \tag{7}$$

$$P_a = p_a, \quad J_{ab} = x_a p_b - x_b p_a + \varepsilon_{abc} S_c,$$
$$P_0 = \varepsilon E, \quad J_{0a} = x_0 p_a - \frac{i\varepsilon}{2}[\frac{\partial}{\partial p_a}, E]_+ - \varepsilon \frac{\varepsilon_{abc} p_b S_c}{E+M} \tag{8}$$

where $S_a = \frac{1}{2}\Sigma_{i=0}^{n-1}\left(\varepsilon_{a\ b+4i\ c+4i} S_{b+4i\ c+4i} + S_{4(i+1)\ a+4i}\right), E = \sqrt{M^2 + p^2}$, and $p^2 = p_1^2 + p_2^2 + p_3^2$.

In analogous way, choosing the time-like and space-like four-vectors $P = (M, 0, 0, M))$ and $P = (0, 0, 0, \eta)$, we prove, that for representations of classes II and III the LPSA reduces to the direct sum of the algebras so(2N+1)⊕e(2) and so(2N+1,2N)⊕so(1,2). The corresponding basis elements of IRs of the PPSA also can be calculated in an explicit form.

Thus, we have described all possible (up to equivalence) IRs of the PPSA with arbitrary number of parasupercharges. Physical interpretation of these representations can be formulated in analogy with [4].

References

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