On extended supersymmetries in quantum mechanics

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1 Introduction

Twenty five years ago a very constructive idea and concept of supersymmetry (SUSY) was born in particle physics. This concept provided new and effective ways for solution of fundamental problems of physics and caused the generation of new branches of mathematics. But till now we do not have a satisfactory answer for the question of principle: does this fine kind of symmetry be realized in nature?

A positive answer for this question can be formulated on quantum mechanical level. Indeed, there exist such realistic physical problems which can be described with good accuracy in frames of ordinary quantum mechanics and generate exact SUSY. They are the Coulomb problem for the Dirac particle, interaction of a spin 1/2 particle with a constant and homogeneous magnetic field and others [1-3]. But at the best of my knowledge, the problems with an extended SUSY (i.e., generating more then two supercharges) were not indicated yet.

In the following I will present some results of systematic search for extended SUSY in quantum mechanics. We will see that this kind of symmetry is generated by a number of realistic problems.

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2 Dirac particle interacting with a magnetic field

Consider the Dirac equation for a charged particle interacting with an external (time-independent) magnetic field

$$L\Psi \equiv (\gamma^\mu \pi_\mu - m)\,\Psi = 0 \tag{1}$$

where $\pi_0 = p_0 = i\frac{\partial}{\partial x_0}, \quad \pi_a = p_a - eA_a(\mathbf{x}), \quad p_a = -\frac{\partial}{\partial \mathbf{x_a}}$.

We say a linear first order differential operator Q be a symmetry operator (SO) for equation (1) provided [5]

$$[L, Q]\Psi = 0 \tag{2}$$

where Ψ is a solution of equation (1).

For general methods of solution of equation (2) for arbitrary order SOs refer to [5].

Let us search for special SOs for the Dirac equation, which satisfy superalgebra the Witten superalgebra

$$\{Q_a, Q_b\} = 2\delta_{ab}H, \quad [Q_a, H] = 0, \tag{3}$$

where $[.,.]$ and $\{.,.\}$ are commutator and anticommutator respectively, $a, b = 1, 2, ...$

It is convenient to transform (1) to the following equivalent form

$$(\pi_\mu \pi^\mu - m^2 - 2e\mathbf{S}\cdot\mathbf{H})\,\hat{\Phi} = 0, \quad (1 + i\gamma_5)\,\hat{\Phi} = 0 \tag{4}$$

where $\mathbf{S} = \frac{i}{2}\gamma \times \gamma, \quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \quad \mathbf{H} = i\pi \times \pi$. The corresponding transformation has the form [5]

$$\hat{\Phi} = V^+\hat{\Phi}, \quad \Psi = V^-\hat{\Phi}, \quad V^\pm = 1 \pm (1 + i\gamma_5)\gamma_\mu\pi^\mu/m. \tag{5}$$

For the diagonal γ_5, $i\gamma_5 = diag(1, 1, -1, -1)$, the function $\hat{\Phi}$ has two non-zero components only which we denote by Φ. Moreover,

$$(p_0^2 - m^2)\,\Phi = \hat{H}\Phi, \quad \hat{H} = \pi^2 + e\sigma\cdot\mathbf{H}. \tag{6}$$

Consider the eight-dimension group formed by the unity transformation and reflections of spatial variables $\mathbf{x} = (x_1, x_2, x_3)$:

$$
\begin{gathered}
r_1\mathbf{x} = (-x_1, x_2, x_3), \quad r_2\mathbf{x} = (x_1, -x_2, x_3), \quad r_3\mathbf{x} = (x_1, x_2, -x_3), \quad r\mathbf{x} = -\mathbf{x} \\
r_{12}\mathbf{x} = (-x_1, -x_2, x_3), \quad r_{23}\mathbf{x} = (x_1, -x_2, -x_3), \quad r_{31}\mathbf{x} = (-x_1, x_2, -x_3).
\end{gathered} \tag{7}
$$

We say that the vector-potential $\mathbf{A}(\mathbf{x})$ has the proper parity w.r.t. one of reflections (7) if it satisfies one of the following relations correspondingly (for fixed a, b)

$$
\mathbf{A}(r_a\mathbf{x}) = r_a\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{ab}\mathbf{x}) = r_{ab}\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r\mathbf{x}) = -\mathbf{A}(\mathbf{x}). \tag{8}
$$

$\mathbf{A}(\mathbf{x})$ has unproper parities if the r.h.s. of (8) have the opposite signs.

If $\mathbf{A}(\mathbf{x})$ satisfies two or more relations (8) simultaneously then problem (6) generates extended SUSY. Let us present the corresponding supercharges explicitly:

$$
\left\{ \begin{array}{c}
\mathbf{A}(r_1\mathbf{x}) = r_1\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_2\mathbf{x}) = r_2\mathbf{A}(\mathbf{x}), \\
Q_1 = iR_1\sigma\cdot\pi, \quad Q_2 = iR_2\sigma\cdot\pi, \quad Q_3 = \sigma\cdot\pi.
\end{array} \right. \tag{9a}
$$

$$
\left\{ \begin{array}{c}
\mathbf{A}(r_{12}\mathbf{x}) = r_{12}\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{23}\mathbf{x}) = r_{23}\mathbf{A}(\mathbf{x}), \\
Q_1 = R_{23}\sigma\cdot\pi, \quad Q_2 = R_{31}\sigma\cdot\pi, \quad Q_3 = R_{12}\sigma\cdot\pi.
\end{array} \right. \tag{9b}
$$

$$
\left\{ \begin{array}{c}
\mathbf{A}(r_1\mathbf{x}) = r_1\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_2\mathbf{x}) = r_2\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_3\mathbf{x}) = r_3\mathbf{A}(\mathbf{x}), \\
Q_1 = iR_1\sigma\cdot\pi, \quad Q_2 = iR_2\sigma\cdot\pi, \quad Q_3 = iR_3\sigma\cdot\pi, \quad Q_4 = \sigma\cdot\pi;
\end{array} \right. \tag{10}
$$

Here R_a, R_{ab} denote space reflection transformations:

$$
R_a\Phi(\mathbf{x}) = \sigma_a\Phi(r_a\mathbf{x}), \quad R_{ab}\Phi(\mathbf{x}) = \sigma_a\sigma_b\Phi(r_{ab}\mathbf{x}).
$$

For cases (9) we have $N = 3$ extended SUSY, while case (10) corresponds to $N = 4$ extended SUSY.

Supposing that $\mathbf{A}(\mathbf{x})$ has unproper parities w.r.t. two or more reflections (7), or that $\mathbf{A}(\mathbf{x})$ has combined parities (e.g., proper w.r.t. r_1 and unproper w.r.t. r_2), we come to a wide class of problems admitting extended SUSY. Here we consider only one example.

Let $\mathbf{A}$ has unproper parities w.r.t. the reflection of any component of x,

$$
\mathbf{A}(r_1\mathbf{x}) = -r_1\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_2\mathbf{x}) = -r_2\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_3\mathbf{x}) = -r_3\mathbf{A}(\mathbf{x}),
$$

then equation (1) admits $N=4$ SUSY generated by the following operators

$$Q_1 = i\sigma_2 c R_1 \sigma\cdot\pi, \quad Q_2 = i\sigma_2 c R_2 \sigma\cdot\pi, \quad Q_3 = i\sigma_2 c R_3 \sigma\cdot\pi, \quad Q_0 = \sigma\cdot\pi, \tag{11}$$

where c is the complex conjugation operator.

Operators (11) satisfy the following relations

$$\begin{gathered} \{Q_\mu, Q_\nu\}_+ = 2g_{\mu\nu}\hat{H}, \quad \left[Q_\mu, \hat{H}\right] = 0 \\ \mu,\nu = 0,1,2,3, \quad g_{00} = -g_{11} = -g_{22} = -g_{33} = 1; \qquad g_{\mu\nu} = 0, \quad \mu \neq \nu. \end{gathered} \tag{12}$$

In contrast with (3), the structure constants of superalgebra (12) are defined via components of the metric tensor $g_{\mu\nu}$.

3 Dirac particle in the Coulomb field

The corresponding Dirac equation again has form (1), where, however,

$$\pi_0 = p_0 - \tfrac{\alpha}{x}, \quad \pi_a = p_a, \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{13}$$

In analogy with (4)-(6) we come to the following two-component equation :

$$\left(p_0^2 - m^2\right)\phi = \left(\mathbf{p}^2 + i\alpha\tfrac{\sigma\cdot\mathbf{x}}{x^3} - \tfrac{\alpha^2}{x\ 2} + \tfrac{2\alpha p_0}{x}\right)\Phi.$$

Changing p_0 by an eigenvalue E and choosing the new variables $\rho = E\mathbf{x}$ we obtain

$$\tfrac{m^2}{E^2}\Phi = \hat{H}\Phi, \quad -\hat{H} = \mathbf{p}^2 + i\alpha\tfrac{\sigma\cdot\mathbf{x}}{x^3} + \left(\tfrac{\alpha}{x} - 1\right)^2. \tag{14}$$

The "Hamiltonian" $\hat{H}$ can be represented as a square of any of the following supercharges

$$Q_0 = i\sigma_2 c\left(\sigma\cdot\mathbf{p} - 1 + \tfrac{\alpha}{x}\right), \quad Q_a = \sigma_a r_a\left(\sigma\cdot\mathbf{p} - 1 + \tfrac{\alpha}{x}\right), \quad a = 1,2,3, \tag{15}$$

which satisfy relations (12). $N=2$ SUSY for the problem (1), (12) was established long time ago, refer, e.g., to [2,3]. We prove that this problem generates N=4 extended SUSY.

4 The Dirac equation with a scalar potential

Let us investigate supersymmetries of the following equation

$$(\gamma_\mu \pi_\mu - m - \varphi)\,\psi = 0 \tag{16}$$

where φ is a scalar potential which we suppose to be time independent.

Equation (16) has many useful applications in particle and nuclear physics [1,6]. The corresponding eigenvalue problem for the operator $p_0 = i\frac{\partial}{\partial x_0}$ reduces to the following two-component equation (compare with (14))

$$p_0^2 \Phi = \hat{H}\Phi, \quad \hat{H} = \mathbf{p}^2 - \sigma \cdot \mathbf{E} + m^2 + 2m\varphi + \varphi^2 \tag{17}$$

where $\mathbf{E} = \nabla\varphi$.

Let $\varphi(\mathbf{x}$ has definite parities w.r.t. the reflections (7). Then equation (16) is supersymmetric. More precisely, let $\varphi(x)$ satisfies one of the following relations for fixed a, b

$$\varphi(r_a\mathbf{x}) = \varphi(\mathbf{x}), \tag{18}$$

$$\varphi(r\mathbf{x}) = \varphi(\mathbf{x}), \tag{19}$$

$$\varphi(r_{ab}\mathbf{x}) = \varphi(\mathbf{x}), \tag{20}$$

$$\varphi(r_a\mathbf{x}) = \varphi(\mathbf{x}), \quad \varphi(r_b\mathbf{x}) = \varphi(\mathbf{x}), \tag{21}$$

or

$$\varphi(r_a\mathbf{x}) = \varphi(\mathbf{x}), \quad a = 1, 2, 3; \tag{22}$$

$$\varphi(r_{ab}\mathbf{x}) = \varphi(\mathbf{x}), \quad \varphi(r_{bc}\mathbf{x}) = \varphi(\mathbf{x}). \tag{23}$$

Then equation (17) admits $N = 2$ SUSY for the cases (18), (19) and (20), the related supercharges have the form (24), (25) and (26) respectively:

$$Q_0 = i\sigma_2 c(i\sigma \cdot \pi + m + \varphi), \quad Q_1 = R_{12} Q_0 \tag{24}$$

$$Q_0 = i\sigma_2 c(i\sigma \cdot \pi + m + \varphi), \quad Q_1 = R_1(i\sigma \cdot \pi + m + \varphi), \tag{25}$$

$$Q_0 = i\sigma_2 c(i\sigma \cdot \pi + m + \varphi), \quad Q_1 = R_{23} Q_0. \tag{26}$$

For the cases (21), (22) and (23) equation (17) admits $N = 4$ SUSY generated by the supercharges (27), (28) and (29) correspondingly:

$$\begin{gathered} Q_0 = i\sigma_2 c(i\sigma \cdot \pi + m + \varphi), \quad Q_1 = R_{12} Q_0, \\ Q_2 = iR_a Q_0, \quad Q_3 = R_b Q_0, \end{gathered} \tag{27}$$

$$Q_0 = i\sigma_2 c(i\sigma\cdot\pi + m + \varphi), \quad Q_a = R_a(i\sigma\cdot\pi + m + \varphi), a = 1,2,3, \tag{28}$$

$$Q_0 = i\sigma_2 c(i\sigma\cdot\pi + m + \varphi), \quad Q_a = R_{bc}Q_0, \quad a = 1,2,3,, a \neq b, b \neq c. \tag{29}$$

Relations (18)-(23) are valid for an extended class of potentials including φ being an arbitrary function of x. It is possible to show that for $\varphi = \frac{C}{x}$ the problem (16) admits a SUSY formulation with a shape invariant potential, and so the corresponding energy spectra can be calculated algebraically.

5 Conclusions and comments

Thus we present a number of relativistic physical problems generating extended SUSY. The very existence of such problems gives a proof that this find kind of symmetry does be realized in nature. Moreover, the found extended SUSYs enables to make *a priori* predictions about specific four-fold degeneration of the corresponding energy spectra.

An extended SUSY is generated also by another equations of quantum mechanics. For example, all results of Section 2 can be reformulated for the Schrődinger-Pauli equation.

Without any doubts, the number of problems admitting an extended SUSY can be added by analysis of another types of external fields for the Dirac equation. We plan to study such problems elthewhere.

In conclusion we notice that the relatively new kind of symmetry called parasupersymmetry [7] also is admitted by realistic physical problems, which are connected with a motion of spin-one particles in an external fields. A short survey of such problems is present in [8].

References

[1] E.E. Gendenstein and I,V,Krive, Usp. Fiz. Nauk **146**, 553,1985.

[2] B. Thaller, The Dirac Equation (Springer-Verlag, 1992)

[3] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **211**, 268, 1995.

[4] E. Witten, Nucl. Phys. **188**, 513, 1981.

[5] W.I. Fushchich and A.G. Nikitin, Symmetries of Equations of Quantum Mechanics, Allerton Press Inc., N.Y., 1994.

[6] R. Tagen, Ann. Phys. **197**, 439, 1990.

[7] V.A. Rubakov and V.P. Spiridonov, Mod. Phys. Lett. **A 3**, 1337, 1988; J. Beckers and N. Debergh, Nucl. Phys. **B 340**, 767, 1990.

[8] J. Beckers, N. Debergh, and A.G. Nikitin, I, II, Fortschr. Phys. **43**, 67, 81, 1995.