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PARASUPERSYMMETRIES AND NON-LIE CONSTANTS OF MOTION FOR TWO-PARTICLE EQUATIONS

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We search for hidden symmetries of two-particle equations with oscillator-equivalent potential, proposed by Moshinsky with collaborators. We proved that these equations admit hidden symmetries and parasupersymmetries which enable one to easily find the Hamiltonian spectra using algebraic methods and to construct exact Foldy–Wouthuysen transformations. Moreover, we demonstrate that these equations are reducible and generate Hamiltonians for pararelativistic or Kemmer oscillators. We also establish equivalence relations between different approaches to Kemmer oscillator and propose new one- and two-particle equations with oscillator-equivalent potentials.

1. Introduction

Non-Lie symmetries[1,2] have many useful applications in mathematical physics. Among them are investigations of conservation laws which cannot be found in the classical Lie approach,[2] searching for coordinate systems in which solutions in separated variables for partial differential equations exist,[3,4] etc. Moreover, non-Lie symmetries of relativistic wave equations generate algebraic structures which are typical for super- and parasupersymmetric quantum mechanics.[2,5,6]

Parasupersymmetry (PSUSY) which was introduced by Rubakov and Spiridonov[7] attracts attention of many physicists and mathematicians. Physically, PSUSY is a symmetry between particles obeying parastatistics of different order. Mathematically, PSUSY can be interpreted as a non-Lie symmetry of motion equations, which have a more complicated algebraic structure than the usual Lie algebras. More precisely, these symmetries obey polynomial relations defining structures which are called parasuperalgebras.

PSUSY has already found useful applications in the analysis of spectral properties of various Hamiltonians.[8] In particular it is realized on solutions of quantum mechanical equations for particles interacting with magnetic field.[9] PSUSY has already been used in relativistic quantum mechanics.[10] Moreover, backgrounds of parasupersymmetric quantum field theory were formulated in Refs. 11 and 12.

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Of course it is interesting to investigate physical and mathematical models which admit this new type of symmetry. It happens that a number of relatively recently proposed equations (connected with two-body[13–16] and one-body[18–20] problems) have a PSUSY nature and, moreover, there exist interesting connections between them.

In the present paper, we investigate symmetries and parasupersymmetries of relativistic two-particle equations with oscillator-equivalent potentials proposed by Moshinsky with collaborators.[13–16] These equations appear as two-body extensions of Dirac oscillator,[17] and, like the last, have a very rich symmetry structure.

We also analyze connections between two-body equations,[13–16] pararelativistic quantum oscillator equations[18,19] and the Kemmer oscillator equation.[20] Moreover, we propose a new Kemmer oscillator equation and two-particle equations for arbitrary spin particle with oscillator-equivalent potential.

We show that two-particle equations with oscillator-like potential have non-Lie constants of motion and hidden parasupersymmetries. We use them to find Hamiltonian spectra without solving equations of motion and to construct exact Foldy–Wouthuysen (FW) transformations. In this way we continue searching for symmetries of two-particle equations,[21–23] and parasupersymmetries of relativistic wave equations, refer to Refs. 18 and 24.

In Sec. 2, we search for the usual Lie symmetries of the two-particle equations,[13–16] in Sec. 3, we reduce these equations to the three noncoupled systems for ten-, five- and one-component wave functions. In Secs. 4 and 5 we search for non-Lie symmetries and hidden parasupersymmetries of the considered equations. These symmetries are used in Secs. 6, 7 and 9 to construct exact FW transformation and to generate the Hamiltonian spectra by algebraic methods.

In Sec. 8 we analyze connections between the two-body Dirac oscillator[13–16] and the other equations with oscillator-like potentials, refer to Refs. 18–20. We propose a new version of the Kemmer oscillator with a good energy spectrum. Finally, in Sec. 10, we summarize the obtained results and discuss a new two-body equation with oscillator-like potential for a system "boson + fermion."

2. Lie Symmetries of Two-Particle Equations

Consider the two-particle equations of Moshinsky *et al.*[13–16] in c.m. frame

$$L_1\psi \equiv \left[(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2)\left(\mathbf{p} - i\,\frac{\omega}{2}\,\mathbf{x}\beta_1\beta_2\right) + m(\beta_1+\beta_2) - E'\right]\psi = 0 \qquad (2.1)$$

and[16]

$$L_2\psi \equiv \left[(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2)\left(\mathbf{p} - i\,\frac{\omega}{2}\,\mathbf{x}\beta_1\beta_2\gamma_{51}\gamma_{52}\right) + m(\beta_1+\beta_2) - E'\right]\psi = 0\,, \qquad (2.2)$$

where $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, β_1, β_2, γ_{51}, γ_{52} are the 16×16 matrices

$$\begin{aligned} \boldsymbol{\alpha}_1 &= \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \boldsymbol{\alpha}_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} 0 & \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2 & 0 \end{pmatrix}, \\ \beta_1 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \\ \gamma_{51} &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \gamma_{52} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \end{aligned} \tag{2.3}$$

$\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are Pauli matrices related to the first and second particle, $A\otimes B$ is the direct (Kronecker) product.

Equation (2.1) corresponds to a single Poincaré-invariant equation in arbitrary frame of reference which is suggested by the approach for two free Dirac particles and by Barut analysis of the Lagrangian of quantum electrodynamics.[25,15] Equation (2.2) represents an alternative approach to the two-body problem,[26,16] which starts with two independent Dirac equations whose potentials preserve compatibility of these equations.

Our interest in Eqs. (2.1) and (2.2) is connected with their parasupersymmetric nature. Indeed, as it will be shown in the following, (2.1) and (2.2) present examples of realization of PSUSY on the set of solutions of physically significant equations.

Making the similarity transformation

$$\psi \to \psi' = \beta_2\psi\,, \qquad L_\eta \to L'_\eta = \beta_2 L_\eta \beta_2\,, \qquad \eta = 1,2\,,$$

we reduce (2.1), (2.2) to the form

$$L'_1\psi' \equiv \left\{[\hat\beta_0,\hat\beta_a]\left(p_a + i\,\frac{\omega}{2}\,x_a\eta\right) + \hat\beta_0 m - E\right\}\psi' = 0\,, \tag{2.1$'$}$$

$$L'_2\psi' \equiv \left\{[\hat\beta_0,\hat\beta_a]\left(p_a - i\,\frac{\omega}{2}\,x_a\xi\right) + \hat\beta_0 m - E\right\}\psi' = 0\,, \tag{2.2$'$}$$

where

$$\begin{gathered} \eta = 1 - 2\hat\beta_0^2\,, \qquad \xi = (1-2\hat\beta_0^2)(1-2\hat\beta_5^2)\,, \qquad E = \frac{1}{2}\,E'\,, \\ \hat\beta_\mu = \frac{1}{2}\,(\gamma_\mu^{(1)} + \gamma_\mu^{(2)})\,, \quad \mu = 0,1,2,3,5\,, \\ \gamma_0^{(i)} = \beta_i\,, \qquad \gamma_a^{(i)} = \beta_i\alpha_{ai}\,, \qquad \gamma_5^{(i)} = \gamma_{5i}\,, \quad i=1,2\,. \end{gathered} \tag{2.4}$$

Equations (2.1′), and (2.2′) are more convenient for symmetry analysis than (2.1) and (2.2). Indeed, the matrices $\hat\beta_\mu$ of (2.4) satisfy the Kemmer–Duffin–Petiau (KDP) algebra

$$\hat\beta_\mu\hat\beta_\nu\hat\beta_\lambda + \hat\beta_\lambda\hat\beta_\nu\hat\beta_\mu = g_{\mu\nu}\hat\beta_\lambda + g_{\nu\lambda}\hat\beta_\mu \tag{2.5}$$

which enables to use the known results[27] connected with complete sets of irreducible KDP matrices.

We say a differential operator Q is *a symmetry* of (2.1′) (or (2.2′)) if it is defined on 16-component function $\Psi' = \Psi'(x)$ and commutes with L'_α, i.e. transforms solutions into solutions. To describe Lie symmetries of (2.1′) (or (2.2′)) it is sufficient to find a complete set of first-order differential operators $Q = b_a p_a + C$ (b_a are functions of x, C is a matrix depending on x) which commute with L'_1 (or L'_2). Such operators form a Lie algebra and are generators of symmetry groups of the considered equations.

Using the classical Lie algorithm (see, e.g. Refs. 2 and 3) it is possible to prove the following assertion.

Proposition. Equations (2.1′) and (2.2′) are invariant under a six-parametrical Lie group whose generators are

$$\begin{aligned} &J_a = \varepsilon_{abc}(x_b p_c + i\hat{\beta}_b\hat{\beta}_c)\,,\\ &Q_1 = \big(1+\gamma_\mu^{(1)}\gamma^{(2)\mu}\big)\big(1+2\gamma_\mu^{(1)}\gamma^{(2)\mu}\big)\,,\\ &Q_2 = -\big(3+2\gamma_\mu^{(1)}\gamma^{(2)\mu}\big)\gamma_\mu^{(1)}\gamma^{(2)\mu}\,, \qquad Q_3 = 1 - Q_1 - Q_2\,, \end{aligned} \tag{2.6}$$

where covariant summation is imposed over the repeated indices $\mu = 0, 1, 2, 3$.

We do not present straightforward but cumbersome proof.

In accordance with (2.6) the number of Lie symmetries is rather restricted.[a] The operators J_a are generators of the rotations group O(3). As to Q_1, Q_2 and Q_3 these symmetries exist due to the well-known fact that the (16×16)-dimensional representation of the KDP algebra is reducible and includes 10×10, 5×5 and 1×1 (trivial) irreducible representations. It means that Eq. (2.1′), (or (2.2′)) can be reduced to three uncoupled subsystems for ten-, five- and one-component functions; the operators Q_1, Q_2 and Q_3 are nothing but orthoprojectors on subspaces of these functions, see the following section.

3. Reduction of Eqs. (2.1′) and (2.2′)

Let us reduce $\hat{\beta}$-matrices (2.4) and (2.3) to a direct sum of irreducible KDP matrices. Using the unitary transformation $\hat{\beta}_\mu \to \hat{\beta}_\mu = U\hat{\beta}_\mu U^\dagger$ where

$$\begin{aligned} U = \frac{(1-i)}{2}\big(&e_{1,1}+e_{1,13}+e_{2,2}+e_{2,14}+e_{3,3}+e_{3,15}-e_{10,8}+e_{10,12}\\ &-e_{11,4}-e_{11,16}+e_{13,15}-e_{13,9}+e_{14,6}-e_{14,10}+e_{15,7}-e_{15,11}\big)\\ &+\frac{(1+i)}{2}\big(-e_{4,5}-e_{4,9}-e_{5,6}-e_{5,10}-e_{6,7}-e_{6,11}-e_{7,1}+e_{7,13}\\ &-e_{8,2}+e_{8,14}-e_{9,3}+e_{9,15}-e_{12,4}+e_{12,16}+e_{16,8}+e_{16,12}\big)\,, \end{aligned} \tag{3.1}$$

[a]It was demonstrated by Moshinsky *et al.*[14] that the operator L of (2.1) can be interpreted as Poincaré-invariant mass operator in a particular (rest) frame of reference.

($e_{k,l}$ are unit matrix elements placed at the intersection of the kth row and lth column), we obtain

$$\hat{\beta}_5 = \begin{pmatrix} \hat{\beta}_5^{(10)} & \cdot \\ \cdot & \beta_5^{(6)} \end{pmatrix}, \qquad \hat{\beta}_\mu = \begin{pmatrix} \beta_\mu^{(10)} & \cdot & \cdot \\ \cdot & \beta_\mu^{(5)} & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}, \quad \mu = 0, 1, 2, 3\,, \tag{3.2}$$

where $\beta_\mu^{(10)}$, $\beta_5^{(10)}$, $\beta_5^{(6)}$ and $\beta_\mu^{(5)}$ are the 10×10, 6×6 and 5×5 KDP matrices, the dots denote the zero matrices of corresponding dimension. Moreover,

$$\begin{aligned}
\beta_0^{(10)} &= i(e_{1,7} + e_{2,8} + e_{3,9} - e_{7,1} - e_{8,2} - e_{9,3})\,,\\
\beta_1^{(10)} &= -i(e_{1,10} - e_{5,9} + e_{6,8} + e_{8,6} - e_{9,5} + e_{10,1})\,,\\
\beta_2^{(10)} &= -i(e_{2,10} + e_{4,9} - e_{6,7} - e_{7,6} + e_{9,4} + e_{10,2})\,,\\
\beta_3^{(10)} &= -i(e_{3,10} - e_{4,8} + e_{5,7} + e_{7,5} - e_{8,4} + e_{10,3})\,,\\
\beta_5^{(10)} &= i(e_{1,4} + e_{2,5} + e_{3,6} - e_{4,1} - e_{5,2} - e_{6,3})\,,\\
\beta_0^{(5)} &= -i(e_{1,2} - e_{2,1})\,, \qquad \beta_1^{(5)} = i(e_{1,3} + e_{3,1})\,,\\
\beta_2^{(5)} &= i(e_{1,4} + e_{4,1})\,, \qquad \beta_3^{(5)} = i(e_{1,5} + e_{5,1})\,,\\
\beta_5^{(5)} &= i(e_{1,6} + e_{6,1})\,.
\end{aligned} \tag{3.3}$$

Denoting $\Psi'' = U\Psi' =$ column $(\Psi_{(10)}, \Psi_{(5)}, \Psi_{(1)})$ where $\Psi_{(10)}$, $\Psi_{(5)}$ and $\Psi_{(1)}$ are ten-, five- and one-component functions, we obtain from (2.1′) and (3.2)

$$\begin{aligned}
(H_1 - E)\psi_{(10)} &\equiv \left\{ \left[\beta_0^{(10)}, \beta_a^{(10)}\right] \left(p_a + i\,\frac{\omega}{2}\,x_a \eta^{(10)}\right) + \beta_0^{(10)} m - E \right\} \psi_{(10)}\\
&= 0\,,
\end{aligned} \tag{3.4a}$$

$$\begin{aligned}
(H_0 - E)\psi_{(5)} &\equiv \left\{ \left[\beta_0^{(5)}, \beta_a^{(5)}\right] \left(p_a + i\,\frac{\omega}{2}\,x_a \eta^{(5)}\right) + \beta_0^{(5)} m - E \right\} \psi_{(5)}\\
&= 0\,,
\end{aligned} \tag{3.4b}$$

$$E\psi_{(1)} = 0\,, \tag{3.4c}$$

where $\eta^{(10)}$ and $\eta^{(5)}$ are matrices (2.4) and (3.3). As to (2.2′) it reduces to the ten-component equation

$$(H_1 - E)\psi_{(10)} \equiv \left\{ \left[\beta_0^{(10)}, \beta_a^{(10)}\right] \left(p_a - i\,\frac{\omega}{2}\,x_a \xi^{(10)}\right) + \beta_0^{(10)} m - E \right\} \psi_{(10)} = 0 \tag{3.5}$$

and to Eqs. (3.4b) and (3.4c) for five- and one-component functions.

We see that the considered two-particle equations reduce to the KDP equations in the Schrödinger form with a specific interaction potential linear in x. However, (2.1) or (2.2) are not equivalent to a direct sum of the KDP equations, inasmuch as the last include the additional conditions,[28]

$$\tilde{L}\psi \equiv (-H_\eta \beta_0 + m)\psi = 0\,, \quad \eta = 0, 1 \tag{3.6}$$

(where H_η and β_0 are the corresponding Hamiltonian and KDP matrix), which delete nonphysical components of the wave function. Conditions (3.6) does not follow from either (2.1) or (2.2); this results in existence of nonphysical solutions corresponding to zero energies (i.e. in the "cockroach nest"[29]). One more source of these solutions is Eq. (3.4c).

A systematic way for separation of nonphysical solutions is to use the FW transformation, refer to Sec. 9.

Equations (3.4) with subsidiary condition (3.6) have been called pararelativistic oscillator.[18]

4. Non-Lie Symmetries

Besides invariance with respect to the group O(3), Eqs. (2.1′) and (2.2′) are invariant under the space inversion transformation

$$\psi'(\mathbf{x}) \to \eta\psi'(-\mathbf{x})\,, \tag{4.1}$$

where η is the matrix defined in (2.4). From this it follows[23] that these equations automatically admit a specific non-Lie symmetry which we have called the Dirac type constant of motion. Indeed, it is not difficult to verify by direct calculation that the operator[23]

$$Q_4 = \eta\big(2(\mathbf{S}\cdot\mathbf{J})^2 - 2\mathbf{S}\cdot\mathbf{J} - \mathbf{J}^2\big) \tag{4.2}$$

(where $S = i\hat{\beta}\times\hat{\beta}$, J is a vector whose components J_a are defined in (2.6)) commutes with L' of (2.1′) and (2.2′).

We note that the operator (4.2) does not belong to an enveloping algebra generated by symmetries (2.6) and so is an essentially non-Lie symmetry.

To find one more constant of motion for (2.1′) we use the fact that the operators

$$a_a^+ = \frac{1}{\sqrt{2}}\left(p_a + i\,\frac{\omega}{2}\,x_a\eta\right), \qquad a_a^- = \frac{1}{\sqrt{2}}\left(p_a - i\,\frac{\omega}{2}\,x_a\eta\right) \tag{4.3}$$

satisfy the usual relations for bosonic creation and annihilation operators, and so

$$[\hat{N}, a_a^+] = \frac{\omega}{2}\,\eta a_a^+\,, \qquad [\hat{N}, a_a^-] = -\frac{\omega}{2}\,\eta a_a^-\,, \tag{4.4}$$

where

$$\hat{N} = \frac{1}{2}\sum_a (a_a^+ a_a + a_a a_a^+) = \frac{1}{2}\left(p^2 + \frac{\omega^2}{4}\,x^2\right). \tag{4.5}$$

Using (4.4) and bearing in mind that $[B_0, \eta] = \{\beta_a, \eta\} = 0$, we find immediately, that the operator

$$Q_5 = 2\hat{N} + \frac{1}{2}\,\omega\eta = p^2 + \frac{1}{4}\,\omega^2 x^2 + \frac{1}{2}\,\omega\eta \tag{4.6}$$

commutes with L_1' and so is a symmetry of (2.1′).

Finally, using (2.5) we find the following cubic relation for H of (2.1′)

$$H^3 = H(Q_5 + m^2 - \omega) + \omega Q_6\,, \tag{4.7}$$

where

$$\begin{gathered} Q_6 = -imS_{5a}L_a + i\varepsilon_{abc}S_{4a}S_{5b}\left(p_c + \frac{i}{2}\,\omega x_c\eta\right), \\ S_{\mu\eta} = i[\hat{\beta}\mu, \hat{\beta}\eta]\,, \qquad S_{4\mu} = i\hat{\beta}_\mu\,, \\ \hat{\beta}_5 = \frac{i}{4!}\,\epsilon_{\nu\lambda\rho\sigma}\hat{\beta}_\nu\hat{\beta}_\lambda\hat{\beta}_\rho\hat{\beta}_\sigma\,, \qquad \mu,\ \eta = 1,2,3,5\,. \end{gathered} \tag{4.8}$$

It follows from (4.7) that Q_6 commutes with H and so is one more symmetry of (2.1′).

Repeating the above reasoning for Eq. (2.2′) we recognize, that in addition to (4.2) there exist one more symmetry

$$Q_7 = \boldsymbol{p}^2 + \frac{1}{4}\,\omega^2\boldsymbol{x}^2 - \frac{1}{2}\,\omega\hat{\eta}\,, \tag{4.9}$$

where

$$\hat{\eta} = 3 - 2\hat{\beta}_0^2 - 4\hat{\beta}_5^2(1 - \hat{\beta}_0^2)\,. \tag{4.10}$$

It is not difficult to verify that the symmetries (4.2), (4.6), (4.8) and (4.9) commute with generators (2.6).

Thus in addition to Lie symmetries (2.6) Eq. (2.1′) has three non-Lie constants of motion (4.2), (4.6) and (4.8), moreover, these operators form a basis of nine-dimensional Lie algebra, satisfying the following relations

$$\begin{aligned} &[J_a, J_b] = i\epsilon_{abc}J_c\,, \\ &[Q_A, J_b] = [Q_A, Q_B] = 0\,, \quad A,\ B = 1,2,\ldots,6\,. \end{aligned} \tag{4.11}$$

Lie and non-Lie symmetries of Eq. (2.2′), are given in (2.6) and (4.2), (4.9); they satisfy relations (4.11) with $A,\ B = 1,2,3,4,7$.

5. Hidden Parasupersymmetries

Let us investigate hidden symmetries of (2.1′) and (2.2′), which appear to have structures typical for parasupersymmetric quantum mechanics (PSSQM).

The Rubakov–Spiridonov[7] version of PSSQM is characterized by the parasuperalgebra involving odd elements (parasupercharges) $\hat{Q}_A$ and an even element (Hamiltonian) $H_{\rm PSS}$, which satisfy the following relations[30]

$$\begin{aligned}&\{\hat{Q}_A,\{\hat{Q}_B,\hat{Q}_C\}-2\delta_{BC}H_{\rm PSS}\}+\{\hat{Q}_B,\{\hat{Q}_C,\hat{Q}_A\}-2\delta_{CA}H_{\rm PSS}\}\\&\qquad+\{\hat{Q}_C,\{\hat{Q}_A,\hat{Q}_B\}-2\delta_{AB}H_{\rm PSS}\}=0\,,\\&[\hat{Q}_A,H_{\rm PSS}]=0\,,\quad A,\ B,\ C=1,2\,.\end{aligned}\tag{5.1}$$

This theory corresponds to so-called Ξ-type quantum mechanical systems.[31]

An independent version of PSSQM was formulated by Beckers and Debergh.[32] It is associated with V-type three-level quantum mechanical systems[31] and characterized by the following parasuperalgebra

$$\begin{aligned}&[H_{\rm PSS},\hat{Q}_A]=0\,,\\&\big[[\hat{Q}_A,\hat{Q}_B],\hat{Q}_C\big]=4(\delta_{AC}\hat{Q}_B-\delta_{BC}\hat{Q}_A)H_{\rm PSS}\,,\end{aligned}\tag{5.2}$$

which involves double commutators instead of double anticommutators in (5.1).

It happens the Hamiltonian H of (2.1′) can be represented in the form

$$H=\hat{Q}_1+\hat{\beta}_0 m\,,\tag{5.3}$$

where $\hat{Q}_1$ is a parasupercharge

$$\hat{Q}_1=[\hat{\beta}_0,\hat{\beta}_a]\left(p_a+i\,\frac{\omega}{2}\,x_a\eta\right).\tag{5.4}$$

Indeed, $\hat{Q}_1$ and $\hat{Q}_2=i\eta\hat{Q}_1$ satisfy relations (5.1) and (5.2) together with $H_{\rm PSS}=Q_5+\omega(1-\hat{\beta}_5{}^2)$, where Q_5 is given in (4.6).[18] In other words, the Hamiltonian of (2.1′) is connected with both types of PSUSY, i.e. of Rubakov–Spiridonov and Beckers–Debergh ones.

The representation (5.3) demonstrates hidden PSUSY of Eq. (2.1′). Such a representation has already been recognized for the Dirac equation (either for free[33] or interacting[34] particle, the corresponding β_0 is the Dirac matrix and $\hat{Q}_1$ is a supercharge), and for the KDP equation.[18] An important property of the representation (5.3) is that it enables to construct the FW transformation using relations (5.2), refer to Sec. 9.

Equation (2.2′), also possesses a hidden PSUSY. The corresponding Hamiltonian admits the representation (5.3), moreover, parasupercharges and parasuper-Hamiltonian have the form

$$\hat{Q}_1=[\hat{\beta}_0,\hat{\beta}_a]\left(p_a-i\,\frac{\omega}{2}\,x_a\xi\right),\qquad \hat{Q}_2=i[\hat{\beta}_0,\hat{Q}_1]\,,\qquad H_{\rm PSS}=\frac{1}{4}\,Q_7\,,\tag{5.5}$$

where ξ and Q_7 are given in (2.4) and (4.9). Operators (5.5) realize a representation of the parasuperalgebra (5.2).

We note that the Hamiltonian H of (2.2′) is also a parasupercharge, inasmuch as

$$H^3 = (Q_7 + m^2)H\,, \qquad [H, Q_7 + m^2] = 0\,, \qquad H\xi + \xi H = 0\,. \tag{5.6}$$

It follows from (5.6) that the operators

$$\hat{Q}_1 = H\,, \qquad \hat{Q}_2 = i\xi H\,, \qquad H_{\mathrm{PSS}} = Q_7 + m^2$$

satisfy the algebra (5.2), and so the squared eigenvalues E^2 of (2.2′) have typical PSUSY degeneration.[32]

We note that all the results of this section are valid for the reduced equations (3.4) and (3.5), inasmuch as they are based on relations (2.5) satisfied by matrices $\hat{\beta}_\mu$ and $\beta_\mu^{(10)}$, $\beta_\mu^{(5)}$ as well. For applications of PSUSY and the KDP matrices in relativistic quantum mechanics refer to Ref. 10.

6. Hamiltonian Eigenvalues for Parastates

In this and the following section we use symmetries and hidden parasupersymmetries of (2.1′) and (2.2′) to find eigenvalues of the Hamiltonians by purely algebraic methods, without solving the corresponding equations.

In accordance with Sec. 3, Eqs. (2.1′) and (2.2′) reduce to uncoupled subsystems (3.4) and (3.5). Here we consider the simplest nontrivial subsystem, i.e. (3.4b).

A system of two spin-1/2 particles described by (2.1′) or (2.2′) have two spin states corresponding to the total spin values $s = 0$ (parastate) or $s = 1$ (orthostate). Equation (3.4b) describes spin zero,[18] or parastates. Indeed, in absence of interaction ($\omega = 0$) the Hamiltonian H of (3.4b) reduces to the KDP Hamiltonian for spinless particles.[28]

To find possible eigenvalues E we use the fact that in accordance with (4.8) for 5×5 KDP matrices $\beta_5^{(5)} \equiv 0$, and relation (4.7) reduce to the form

$$H_0^3 = H_0(Q_5 + m^2 - \omega)\,. \tag{6.1}$$

Inasmuch as $[H_0, Q_5] = 0$, relation (6.1) leads to the corresponding relation for eigenvalues E of H_0 and q of Q_5

$$E(E^2 - q - m^2 + \omega) = 0\,. \tag{6.2}$$

In accordance with (4.6)

$$q = (2N + 3 + \varepsilon)\,\frac{\omega}{2}\,, \tag{6.3}$$

where $N = 2n + j$, $n = 0, 1, 2, \ldots$; $j = 0, 1, \ldots, N$; $\varepsilon = \pm 1$ are eigenvalues of η (this known result[18] can be obtained by algebraic methods using (4.4)–(4.6)). Thus,

$$E = \mu\sqrt{(2N + 1 + \varepsilon)\,\frac{\omega}{2} + m^2}\,, \tag{6.4}$$

where $\mu = \pm 1$ (μ is the energy sign), or

$$E = 0\,. \tag{6.5}$$

Inasmuch as the matrix η does not commute with H_0, the values of μ and ε are not independent. Using the FW transformation (refer to Sec. 9) it is possible to show that $\varepsilon\mu = -\mu$ and so nonzero values of E are

$$E = \pm\sqrt{N\omega + m^2}\,. \tag{6.6}$$

Thus, we find algebraically known eigenvalues[16] of E for parastates.

7. Hamiltonian Eigenvalues for Orthostates

Consider Eq. (3.4a) describing spin-one states. In this section we omit indices (10) for 10×10 matrices and ten-component wave function.

To find eigenvalues of H_1 we use relation (4.7). Moreover, it can be verified directly, that $H = H_1$ satisfies one more algebraic relation

$$Q_6(Q_6 - H_1) = \frac{m^2}{2}\,(J^2 + Q_4)\,, \tag{7.1}$$

where $\mathbf{J} = (J_1, J_2, J_3)$, $J^2 = J_1^2 + J_2^2 + J_3^2$, and Q_4 are the symmetry (4.2). Combining (4.7) with (7.1) and taking into account commutativity of all the operators involved to these relations, we obtain

$$H^2(H^2 - Q_5 - m^2)(H^2 - Q_5 - m^2 + \omega) = \frac{m^2\omega^2}{2}\,(J^2 + Q_4)\,. \tag{7.2}$$

Let us change in (7.2) commuting operators H_1^2, Q_5, J^2 and Q_4 by their eigenvalues E^2, $(2N + 3 + \varepsilon)\omega/2$, $j(j + 1)$ and[22,23] $\nu j(j + 1)$, where

$$j = 0, 1, 2, \ldots, \qquad \nu = \pm 1\,. \tag{7.3}$$

As a result we obtain

$$E^2\left[E^2 - m^2 - (2N+1+\varepsilon)\,\frac{\omega}{2}\right]\left[E^2 - m^2 - (2N+3+\varepsilon)\,\frac{\omega}{2}\right] = \frac{\omega^2 m^2}{2}\,j(j+1)(\nu+1)\,. \tag{7.4}$$

For $\nu = -1$ we have three possibilities

$$E = 0\,, \tag{7.5a}$$

$$E = \mu\sqrt{m^2 + (2N + 1 + \varepsilon)\,\frac{\omega}{2}}\,, \quad \mu = \pm 1\,, \tag{7.5b}$$

$$E = \mu\sqrt{m^2 + (2N + 3 + \varepsilon)\,\frac{\omega}{2}}\,. \tag{7.5c}$$

Like the case of parastates we conclude that values of μ and ε are not independent. Using nonrelativistic approximation it is possible to show that in (7.5b) $\mu\varepsilon = \mu$, and in (7.5c) $\mu\varepsilon = -\mu$, and so nonzero energies are defined by the relation

$$E = \pm\sqrt{m^2 + (N+1)\omega}\,. \tag{7.6}$$

For $\nu = 1$ (7.4) reduces to third-order algebraic equation for E^2

$$E^2[E^2 - m^2 - (N+1)\omega][E^2 - m^2 - (N+2)\omega] = m^2\omega^2 j(j+1)\,. \tag{7.7}$$

Formulae (6.6), (7.6) and (7.7) are in good accordance with the results of Moshinsky *et al.*[13–16] Using hidden symmetries of (2.1′) and (2.2′) we obtain these results in a straightforward and easy way.

The eigenvalues problem (2.2′) also can be solved algebraically using (5.6). Replacing in (5.6) operators H_1, Q_7 by their eigenvalues E^2 and $q = (2N + 3 - \varepsilon)\omega/2$ (where ε are eigenvalues of the matrix $\hat{\eta}$ (4.10); in accordance with (9.10) $\varepsilon\mu = -\mu$), we come to the relations

$$E = 0 \quad \text{or} \quad E^2 = m^2 + (N+2)\omega\,, \tag{7.8}$$

i.e. obtain the recognized result.[16]

8. Connections with Kemmer–Duffin–Petiau Oscillator

The Duffin–Kemmer–Petiau (Kemmer) oscillator equation[20] for stationary states have the form

$$L\psi = \left[-\beta_0\tilde{E} + \beta_a\left(p_a + i\,\frac{\omega}{2}\,x_a\eta\right) + \tilde{m}\right]\psi = 0\,, \tag{8.1}$$

where $\eta = 1 - 2\beta_0^2$, β_0, β_a are 10×10 or 5×5 KDP matrices. We set in the original equation[20] $c = h = 1$ and replace $\omega \to \omega/2m$.

We note that for the case when β_μ are 5×5 Kemmer matrices equation (8.1) is equivalent to system (3.4b) and (3.6). For a 10×10 realization of β_μ Eqs. (3.4a) and (3.6) are equivalent to (8.1) supplemented by the additional condition $Q_4\psi = j(j+1)\psi$ where Q_4 is given in (4.2).[35]

We recognize that (8.1) is closely related to the two-particle equations (2.1). Inasmuch as Eq. (8.1) was solved exactly, it is interesting to compare energies $\tilde{E}$ generated by (8.1)[20]

$$S = 0\,, \qquad \tilde{E}^2 = \tilde{m}^2 + N\omega\,, \tag{8.2}$$

$$S = 1\,, \qquad \tilde{E}^2 = \tilde{m}^2 + (N+1)\omega\,, \tag{8.3}$$

or

$$S = 1\,, \quad \tilde{m}^2[\tilde{E}^2 - \tilde{m}^2 - (N+1)\omega][\tilde{E}^2 - \tilde{m}^2 - (N+2)\omega] = \tilde{E}^2\tilde{\omega}^2 j(j+1) \tag{8.4}$$

with (6.6), (7.6) and (7.7).

We point out that formulae (8.2), (8.3) and (8.4) are reduced to (6.6), (7.6) and (7.7) correspondingly, if we change

$$\tilde{E} = im\,, \qquad \tilde{m} = -iE\,. \tag{8.5}$$

This observation is in accordance with the fact that the change (8.5) together with the similarity transformation

$$\psi \to \psi' = \exp\left(i\beta_0\,\frac{\pi}{2}\right)\psi\,, \qquad L \to L' = \exp\left(i\beta_0\,\frac{\pi}{2}\right)L\exp\left(-i\beta_0\,\frac{\pi}{2}\right) \tag{8.6}$$

reduce (8.1) to (2.1′). The transformation (8.6) enables to reformulate all the results related to symmetries and parasupersymmetries of Eq. (2.1′) (refer to Secs. 2, 4 and 5) for the case of the Kemmer oscillator equation (8.1).

In analogy with the above it is not difficult to find the Kemmer oscillator equation corresponding to (2.2′). Using the change and transformation inverse to (8.5) and (8.6) we obtain from (3.5)

$$\left[\beta_0\tilde{E} - \beta_a\left(p_a + i\,\frac{\omega}{2}\,x_a(2\beta_5^2 - 1)\right) - \tilde{m}\right]\psi = 0\,, \tag{8.7}$$

where β_μ are 10×10 KDP matrices.

Like (8.1), Eq. (8.7) is exactly solvable; moreover it generates oscillator-like energy spectrum (7.8). In contrast to (3.5) this equation does not have nonphysical solutions corresponding to zero energies, and admits symmetries (2.6), (4.2), (4.9) and hidden parasupersymmetries. We will return to (8.7) in the following section where FW transformations are considered.

9. FW Transformations

Hidden parasupersymmetries discussed in Sec. 5 enable to construct exact FW transformations for the corresponding equations.

We start with Eq. (3.4b) and represent the Hamiltonian H_0 in the form (5.3) and (5.4). The second parasupercharge can be chosen as follows

$$\hat{Q}_2 = i\beta_a\left(p_a + i\,\frac{\omega}{2}\,x_a\eta\right) \equiv [\beta_0, \hat{Q}_1]\,. \tag{9.1}$$

The operators $\hat{Q}_1$ (5.4) and $\hat{Q}_2$ satisfy relations (5.2) where

$$H_{\rm PSS} = \frac{1}{4}\left(p^2 + \frac{x^2\omega^2}{4} + \frac{(\eta-2)\omega}{2}\right).$$

Moreover,

$$[\hat{Q}_1, \hat{Q}_2] = -4i\beta_0 H_{\rm PSS}\,. \tag{9.2}$$

Using (5.2), (9.1) and (9.2) we easily prove the following identity

$$
\begin{aligned}
&\exp(i\hat{Q}_2\theta)\beta_0\exp(-i\hat{Q}_2\theta) \\
&\quad = \beta_0 + i[\hat{Q}_2, \beta_0]\theta - \frac{1}{2!}\big[\hat{Q}_2, [\hat{Q}_2, \beta_0]\big]\theta^2 + \cdots \\
&\quad = \beta_0 + \hat{Q}_1\theta - \frac{4}{2!}\,\beta_0 H_{\mathrm{PSS}}\theta^2 - \frac{4}{3!}\,\hat{Q}_1\theta^3 H_{\mathrm{PSS}} + \cdots \\
&\quad = \beta_0\cos\big(2\sqrt{H_{\mathrm{PSS}}\theta}\big) + \frac{\hat{Q}_1}{2\sqrt{H_{\mathrm{PSS}}}}\sin\big(2\sqrt{H_{\mathrm{PSS}}\theta}\big)
\end{aligned}
\tag{9.3}
$$

for any θ commuting with β_0 and $\hat{Q}_2$.

Choosing $\theta = \frac{1}{2\sqrt{H_{\mathrm{PSS}}}}\operatorname{arctg}\frac{2\sqrt{H_{\mathrm{PSS}}}}{m}$, we come to the FW operator

$$
\begin{aligned}
U^{\mathrm{FW}} &= \exp\left(\frac{\nu - i\hat{Q}_2}{2\sqrt{H_{\mathrm{PSS}}}}\operatorname{arctg}\frac{2\sqrt{H_{\mathrm{PSS}}}}{m}\right) \\
&\equiv 1 + \frac{\beta_a\big(p_a + i\,\frac{\omega}{2}\,x_a\eta\big)}{\hat{E}} + \frac{\big[\beta_a\big(p_a + i\,\frac{\omega}{2}\,x_a\eta\big)\big]^2}{\hat{E}(\hat{E}+m)}\,,
\end{aligned}
\tag{9.4}
$$

where

$$
\hat{E} = \sqrt{p^2 + m^2 + \frac{1}{4}\,\omega^2 x^2 - (2-\eta)\,\frac{\omega}{2}}\,. \tag{9.5}
$$

This operator diagonalizes the energy sign operator, so that

$$
U^{\mathrm{FW}} H_0 (U^{\mathrm{FW}})^\dagger = \beta_0\hat{E} = H_0'\,. \tag{9.6}
$$

Eigenvalues of H_0' have to coincide with eigenvalues of H_0 which are given in (6.4) and (6.5). Inasmuch as $\beta_0\eta = -\beta_0$ and eigenvalues of β_0 are equal to 0 or to $\mu = \pm 1$, it follows from (9.6) that (6.4) and (6.5) reduce to (6.6) and (6.5).

Zero energies correspond to nonphysical solutions of (3.4b). In the FW representation these solutions correspond to zero eigenvalues of β_0, so requiring

$$
\beta_0^2\psi_{\mathrm{FW}} = \psi_{\mathrm{FW}} \tag{9.7}
$$

we can select physical solutions corresponding to nonzero energies (6.6).

In an analogous way, it is possible to construct FW transformation for Eq. (3.5) describing spin-one states. The corresponding Hamiltonian admits the representation (5.3) and (5.5), moreover, the parasupercharges (5.5) satisfy relations (5.2) and (9.2) with $H_{\mathrm{PSS}} = Q_7/4$. Thus the FW transformation operator has the form

$$U^{\rm FW} = \exp\left(\frac{-i\hat{Q}_2}{2\hat{p}}\,\mathrm{arctg}\,\frac{2\hat{p}}{m}\right)$$

$$\equiv 1 + \frac{\beta_a\left(p_a - i\,\frac{\omega}{2}\,x_a\xi\right)}{\hat{E}_1} + \frac{\left[\beta_a\left(p_a - i\,\frac{\omega}{2}\,x_a\xi\right)\right]^2}{\hat{E}(\hat{E}_1+m)}\,, \tag{9.8}$$

where

$$\hat{p} = \sqrt{p^2 + \frac{1}{4}\,\omega^2x^2 - \hat{\eta}\,\frac{\omega}{2}}\,, \qquad \hat{E}_1 = \sqrt{p^2 + m^2 + \frac{1}{4}\,\omega^2x^2 - \hat{\eta}\,\frac{\omega}{2}} \tag{9.9}$$

and ξ, $\hat{\eta}$ are matrices defined in (2.4) and (4.10).

Using (9.8) and (5.3) we obtain

$$U^{\rm FW}H_1(U^{\rm FW})^\dagger = \beta_0\hat{E}_1 = H_1' \equiv \beta_0\sqrt{p^2 + m^2 + \frac{1}{4}\,\omega^2x^2 - \frac{\omega}{2}}\,. \tag{9.10}$$

Moreover the corresponding eigenvalues are given by relations (7.8), and the physical states are selected by relations (9.7).

The hidden parasupersymmetries (5.3), (5.4), (9.1) of Eq. (3.4a) does not satisfies relations (9.2) and so cannot be used to construct exact FW transformation. An approximate FW transformation for (3.4a) was found earlier.[19]

To construct FW transformation for (8.7) we multiply it from the left by $\{1 - \beta_a\big(p_a + i\frac{\omega}{2}x_a(2\beta_5^2-1)\big)/m - (1-\beta_0^2)\frac{\tilde{E}}{m}\}$ and $(1-\beta_0^2)$. As a result we come to the equivalent system of Eqs. (3.5) and (3.6) with $E \to \tilde{E}$. Then, using the transformation

$$(H_1 - E) \to U^{\rm FW}(H_1 - E)(U^{\rm FW})^\dagger\,,$$

$$\tilde{L} \to U^{\rm FW}\tilde{L}(U^{\rm FW})^\dagger\,, \qquad \psi \to \psi' = U^{\rm FW}\psi\,,$$

where $U^{\rm FW}$ is given in (9.8) we obtain

$$E\psi' \equiv \beta_0\sqrt{p^2 + \frac{1}{4}\,\omega^2x^2 - \hat{\eta}\,\frac{\omega}{2} + m^2\psi'}\,, \tag{9.11}$$

$$[(1-2\beta_0^2)m + \beta_0H_1]\psi' = 0\,, \tag{9.12}$$

where H_1 and $\hat{\eta}$ are defined in (3.5) and (4.10). Multiplying Eq. (9.12) by β_0^2 and $1-\beta_0^2$ we come to the conclusion that it is equivalent to the condition

$$(1-\beta_0^2)\psi' = 0\,, \tag{9.13}$$

deleting nonphysical components. In accordance with (9.11) and (9.13) eigenvalues $\tilde{E}$ satisfy (7.8).

10. Discussions

Thus, we investigate symmetries (as Lie as hidden ones) of two-particle equations with oscillator-like potentials. These symmetries appear to be rather nontrivial and include non-Lie constants of motion (4.2), (4.6), (4.8) and (4.9) which enable easily to find eigenvalues of the corresponding Hamiltonians using algebraic methods.

The other important feature of the considered equations is existence of hidden parasupersymmetries which enable to construct exact FW transformation. In other words, we recognize that these equations present an example of physically profound theory which admits this new type of symmetry.

The pararelativistic, or Kemmer oscillator equations[18–20] also generate hidden PSUSY. These equations predict oscillator-like energy spectra and, moreover, do not lead to difficulties with "cockroach nest"[29] which are typical for considered two-particle equations. A new version of Kemmer oscillator is discussed in Sec. 8, where we consider relations between different approaches to bosonic oscillators.

The non-Lie approach used in this paper presents effective tools for investigation of various systems of partial differential equations, refer to monographs.[1,2] In particular, the hidden symmetry analogous to (4.2) exists for any O(3)-invariant equation which admits the parity transformation.[2] The PSUSY aspects of the present paper also can be generalized for multiparticle equations and equations of motion for particles of higher spins. We plan to consider such generalizations elsewhere.

Let us notice that the mathematical models considered in the above have very interesting physical contents. Indeed, Eqs. (2.1) and (2.2) are two-particle equations for systems of particles of spin 1/2.[13–16] In the nonrelativistic approximation equation (2.2) is reduced to the 3D harmonic oscillator (refer to Sec. 9), and Eq. (2.1) is equivalent to the 3D harmonic oscillator with a spin-orbit coupling (this result follows from the analysis present in Sec. 8 and results of paper[19]). These oscillators was fundamental tools for the development of the nuclear shell model and are used as effective quark-confining potentials (refer, e.g. to Ref. 37). Taking into account that Eqs. (2.1) and (2.2) are exactly solvable we can wait that they can be used for calculations of relativistic (and two-particle) corrections for the mentioned nonrelativistic models. Morever, they can serve as suitable analytic bases for more realistic relativistic interactions.

In conclusion we note that Eq. (2.1′) can be treated as a particular case of a more general equation with oscillator-like potential

$$(H-E)\psi = 0\,, \qquad H = -iS_{0a}\left(p_a + i\,\frac{\omega}{2}\,x_a\eta\right) + iS_{04}m\,, \tag{10.1}$$

where S_{0a}, S_{04} are arbitrary matrices satisfying the following commutation relations (which characterize the algebra O(1,4))

$$\begin{aligned} [S_{\mu\nu}, S_{\rho\sigma}] &= i(g_{\mu\sigma}S_{\nu\rho} + g_{\nu\rho}S_{\mu\sigma} - g_{\mu\rho}S_{\nu\sigma} - g_{\nu\sigma}S_{\mu\rho})\,, \\ g_{\mu\nu} &= \mathrm{diag}(1,-1,-1,-1,-1)\,, \end{aligned} \tag{10.2}$$

η is a space reflection matrix which can be chosen in the form

$$\eta = \sum_{\nu} (-1)^{\nu} \Lambda_{\nu} \,, \qquad \Lambda_{\nu} = \prod_{\nu' \neq \nu} \frac{S_{04} - \nu'}{\nu - \nu'} \,.$$

Here ν, ν' run over all the possible eigenvalues of the matrix S_{04}.

Indeed, choosing

$$S_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \,, \qquad S_{4\mu} = -i\gamma_\mu \,, \qquad \mu,\ \eta = 0, 1, 2, 3, 5 \,. \tag{10.3}$$

where γ_μ are 4×4 Dirac matrices, we reduce (10.1) to the Dirac oscillator.[17] If

$$S_{\mu\nu} = \frac{i}{4} \left[\gamma_\mu^{(1)}, \gamma_\nu^{(1)}\right] + \frac{i}{4} \left[\gamma_\mu^{(2)}, \gamma_\nu^{(2)}\right] , \qquad S_{4\mu} = -\frac{i}{2} \left(\gamma_\mu^{(1)}, \gamma_\mu^{(2)}\right) ,$$

(where $\left\{\gamma_\mu^{(1)}\right\}$ and $\left\{\gamma_\mu^{(2)}\right\}$ are commuting sets of Dirac matrices) then (10.1) reduce to (2.1′). If

$$S_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] + i[\beta_\mu, \beta_\nu] \,, \quad \eta = \gamma_0 (1 - 2\beta_0^2) \,, \quad S_{4\mu} = -(\gamma_\mu + \beta_\mu) \,, \tag{10.4}$$

where $\{\gamma_\mu\}$ and $\{\beta_\mu\}$ are commuting sets of Dirac and KDP matrices, then, in analogy with arguments of Moshinsky *et al.*,[14,15] we can interpret (10.1) as a motion equation in the c.m. frame for a system of Dirac and KDP particles.

We emphasize that any equation of the type (10.1) admits nine symmetries (2.6), (4.2), (4.6) and (4.8) (with the corresponding matrices $S_{\mu\nu}$ and η); for the case of Dirac oscillator (refer to (10.1) and (10.3)) (4.8) reduce to the form

$$\begin{gathered} \hat{Q}_6 = 2mQ_D + 2H_D \,, \qquad Q_D = \gamma_0 (2\mathbf{S} \cdot \mathbf{L} + 1) \,, \\ H_D = \gamma_0 \gamma \cdot \left(\mathbf{p} + i \, \frac{\omega}{2} \, \mathbf{x} \eta \right) - m\gamma_0 \,, \end{gathered} \tag{10.5}$$

where Q_D is the Dirac constant of motion,[36] H_D is the Hamiltonian of the Dirac oscillator.[17]

In accordance with the results of paper[23] an analog of the symmetry (4.2) exists for any Eq. (10.1); for (10.1) and (10.4) we have

$$\begin{gathered} Q_4 = \eta \left\{ \frac{4}{3} [q^3 + q^2 - (7\mathbf{J}^2 + \mathbf{S}^2) q + (4\mathbf{S}^2 - 6)\mathbf{J}^2] + 3 \right\} , \\ q = 2\mathbf{S} \cdot \mathbf{J} - \frac{3}{2} \,, \qquad S_a = \frac{1}{2} \, \varepsilon_{abc} S_{bc} \,. \end{gathered}$$

Thus, the Lie and non-Lie symmetries discussed in Secs. 2 and 4 are valid for rather extended class of partial differential equations.

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