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LETTER TO THE EDITOR

More on symmetries of the Schrödinger equation

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Abstract. Besides the usual and well known Lie symmetries of the Schrödinger equation, we study higher-order symmetry operators and point out their impact on classes of solvable potentials when time-independent interactions take place for one-dimensional (space) systems. We consider the case of *n*th-order symmetry operators and more particularly the third-order context.

One-dimensional non-relativistic systems have already been studied through the Schrödinger equation

$$\mathrm{i}\partial_t\Psi(x, t) = [-\tfrac{1}{2}\partial_x^2 + U(x)]\Psi(x, t) \qquad \hbar = 1 \quad m = 1 \quad \partial_t \equiv \frac{\partial}{\partial t} \quad \partial_x \equiv \frac{\partial}{\partial x} \tag{1}$$

where we have introduced time-independent potentials $U(x)$. From the symmetry point of view, such an equation has been visited by many authors [1–5] and the admissible (solvable) cases have been listed. Let us just recall that the free case ($U = 0$) and the harmonic oscillator case ($U = \frac{1}{2}\omega^2x^2$) are isomorphic and admit six symmetries [1, 2] while others admit four or two symmetries according to some specificities we do not collect here [3, 4]. In connection with these results, Miller [5] has also given the first systematic treatment showing how separation of variables relates the sets of symmetry operators associated with specific interaction terms, i.e. through the corresponding invariance Lie algebras.

In fact, as it is evident from equation (1), the above studies only consider the *first*- and *second*-order symmetry operators leaving invariant the Schrödinger equation. Let us also point out that a first-order derivation in time is directly connected to a second-order derivation in space when, as it will be the case in the following, we limit ourselves to symmetry operators acting on solutions of (1).

In an elegant and compact way, such developments can be summarized in the following problem. Let us consider (1) written on the form

$$L\Psi(x, t) \equiv [\mathrm{i}\partial_t - \tfrac{1}{2}(p^2 + V(x))]\Psi(x, t) = 0 \qquad p = -\mathrm{i}\partial_x \tag{2}$$

and let us search for symmetry operators Q satisfying the invariance condition

$$[L, Q] = 0 \tag{3}$$

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when these Q-operators have the following structure:

$$Q_1 = (h_0 \cdot p)_0 + (h_1 \cdot p)_1 \tag{4}$$

if *first*-order operators only are considered, *or*

$$Q_2 = (h_0 \cdot p)_0 + (h_1 \cdot p)_1 + (h_2 \cdot p)_2 \tag{5}$$

if the *second* order is required. Here we have introduced the notation

$$(h_n \cdot p)_n \equiv \{(h_n \cdot p)_{n-1}, p\} \qquad (h_0 \cdot p)_0 = h_0 \tag{6}$$

where $\{A, B\}$ is the anticommutator of A and B $(=AB+BA)$ and where $h_n (n = 0, 1, 2, 3, \ldots)$ are *arbitrary* functions of x *and* t.

It is easy to recover the old results [1-4] corresponding to the solution of (3) with $Q_2 \equiv (5)$. They lead to the most general potential [6]

$$V(x) = \tfrac{1}{2}\alpha x^2 + \beta x + \gamma + \frac{\delta}{(\varepsilon + \mu x)^2} \qquad \mu \neq 0 \tag{7}$$

which can be included in an invariant Schrödinger equation (admitting six symmetries when $\delta = 0$ or four symmetries when $\delta \neq 0$).

Here we now propose to extend these developments to arbitrary nth-order symmetries with

$$Q_n = \sum_{k=0}^{n} (h_k \cdot p)_k \tag{8}$$

and to search for the corresponding classes of solvable potentials admitting such nth-order symmetry operators.

For any order n, we notice the following operator identity containing n anticommutators:

$$\{\ldots\{\{h_n, p\}, p\}, \ldots, p\} = (-\mathrm{i})^n \sum_{k=0}^{n} \frac{n!\,2^{n-k}}{(n-k)!\,k!} (\partial_x^k h_n) \partial_x^{n-k}. \tag{9}$$

It will be very useful in the following in order to get the systems of differential equations which are characteristic of the general Q_n-problem (2), (3), (8). Let us briefly sketch the steps leading to these equations. For each k-term in the operator (8) we can evaluate the following commutators:

$$[\mathrm{i}\partial_t, (h_k \cdot p)_k] = \mathrm{i}(\dot{h}_k \cdot p)_k \tag{10}$$

and

$$[-\tfrac{1}{2}p^2, (h_k \cdot p)_k] = \tfrac{1}{2}\mathrm{i}(h_k' \cdot p)_{k+1} \tag{11}$$

due to the x- and t-dependences in the arbitrary functions h_k $(k = 0, 1, \ldots, n)$. Notice that dots refer to time derivatives and primes to spatial ones. From (10) and (11) we immediately get, *for the free case* $(V = 0)$, the set of commutators

$$[L_0, (h_k \cdot p)_k] = \mathrm{i}(\dot{h}_k \cdot p)_k + \tfrac{1}{2}\mathrm{i}(h_k' \cdot p)_{k+1} \tag{12}$$

where evidently

$$L_0 \equiv \mathrm{i}\partial_t - \tfrac{1}{2}p^2 = \mathrm{i}\partial_t + \tfrac{1}{2}\partial_x^2. \tag{13}$$

For considering the Q_n-problem *with interaction*, we need the additional commutators $[V(x), (h_k \cdot p)_k]$, $\forall k = 0, 1, \ldots, n$. These tedious cases can be more easily handled by considering separately even and odd k's. By using explicitly the identity (9), we have obtained for $k \geqslant 1$

$$[V(x), (h_{2k} \cdot p)_{2k}] = -\mathrm{i} \sum_{m=0}^{k-1} (-1)^{m+k} \frac{2(2k)!}{(2k-2m-1)!(2m+1)!} \times (h_{2k} \partial_x^{2k-2m-1} V(x) \cdot p)_{2m+1} \tag{14}$$

and for $k \geqslant 0$

$$[V(x), (h_{2k+1} \cdot p)_{2k+1}] = -\mathrm{i} \sum_{m=0}^{k} (-1)^{m+k+1} \frac{2(2k+1)!}{(2k-2m+1)!(2m)!} \times (h_{2k+1} \partial_x^{2k-2m+1} V(x) \cdot p)_{2m}. \tag{15}$$

By superposing the results (10), (11), (14) and (15), it is easy to show that the Q_n-problem (3) associated with the Schrödinger equation (2) is characterized by the following sets of commutators

$$[L, (h_0 \cdot p)_0] = \mathrm{i}(\dot{h}_0 \cdot p)_0 + \frac{1}{2}\mathrm{i}(h_0' \cdot p)_1 \tag{16a}$$

$$[L, (h_{2k} \cdot p)_{2k}] = \mathrm{i}(\dot{h}_{2k} \cdot p)_{2k} + \frac{1}{2}\mathrm{i}(h_{2k}' \cdot p)_{2k+1} + \frac{1}{2}\mathrm{i} \sum_{m=0}^{k-1} (-1)^{m+k} \frac{2(2k)!}{(2k-2m-1)!(2m+1)!} (h_{2k} \partial_x^{2k-2m-1} V(x) \cdot p)_{2m+1} \tag{16b}$$

and

$$[L, (h_{2k+1} \cdot p)_{2k+1}] = \mathrm{i}(\dot{h}_{2k+1} \cdot p)_{2k+1} + \frac{1}{2}\mathrm{i}(h_{2k+1}' \cdot p)_{2k+2} + \frac{1}{2}\mathrm{i} \sum_{m=0}^{k} (-1)^{m+k+1} \frac{2(2k+1)!}{(2k-2m+1)!(2m)!} (h_{2k+1} \partial_x^{2k-2m+1} V(x) \cdot p)_{2m} \tag{16c}$$

respectively for $k=0$, $k \geqslant 1$ and $k \geqslant 0$.

These general results can now be exploited. Let us here only point out and discuss the first new *non-trivial* context, i.e. the *third*-order symmetry operators. In such a problem we are asking for symmetries of the Schrödinger equation (2) according to the condition

$$[L, Q_3] = 0 \tag{17}$$

with

$$Q_3 = \sum_{k=0}^{3} (h_k \cdot p)_k = (h_0 \cdot p)_0 + (h_1 \cdot p)_1 + (h_2 \cdot p)_2 + (h_3 \cdot p)_3. \tag{18}$$

By using the above identity (9) we immediately notice its explicit form

$$Q_3 = 8\mathrm{i}h_3\partial_x^3 + [12\mathrm{i}(\partial_x h_3) - 4h_2]\partial_x^2 + [6\mathrm{i}(\partial_x^2 h_3) - 4(\partial_x h_2) - 2\mathrm{i}h_1]\partial_x$$

$$+ [\mathrm{i}(\partial_x^3 h_3) - (\partial_x^2 h_2) - \mathrm{i}\partial_x h_1 + h_0]. \tag{19}$$

Then, through equations (16), we easily get the following system of five differential equations:

$$h_3' = 0 \tag{20a}$$

$$2\dot{h}_3 + h_2' = 0 \tag{20b}$$

$$2\dot{h}_2 + h_1' - 6h_3 V' = 0 \tag{20c}$$

$$2\dot{h}_1 + h_0' - 4h_2 V' = 0 \tag{20d}$$

$$\dot{h}_0 - h_1 V' + h_3 V''' = 0. \tag{20e}$$

Evidently, if h_3 is identically zero, we recover the Boyer system [4, 6] associated with the second-order symmetry problem.

Equations (20*a*, *b*, *c*) lead to three arbitrary unknown functions (hereafter called *a*, *b*, *c*) depending only on the time variable and the equations (20*d*, *e*) to an inhomogeneous differential equation of the fourth order in terms of the potential $V(x)$ given by

$$aV'''' - (2\ddot{a}x^2 + 6aV + c - 2\dot{b}x)V'' - 6(2\ddot{a}x + aV' - \dot{b})V' - 12\ddot{a}V$$

$$= 2(\ddddot{a}x^2 - 2\dddot{b}x + \ddot{c}). \tag{21}$$

It admits as a particular solution the explicit form

$$V_1(x) = \tfrac{1}{2}\alpha x^2 + \beta x + \gamma \tag{22}$$

where α, β and γ are arbitrary constants. It also ensures that this potential (22) and more generally the potential (7) admits third-order symmetries we are discussing.

The general resolution of (21) then asks for the three subcases $\alpha = 0$, $\alpha = \omega^2 > 0$ and $\alpha = -\omega^2 < 0$ as it was the case in the second order problem [4, 6] where we also know [6] that these are isomorphic contexts with respect to Lie structures subtended by the corresponding symmetry operators. Consequently, let us point out some results and comments by considering here the $\alpha = 0$-case only reducing the difficulty in a significant manner.

The resulting potential evidently becomes

$$V_1^{(0)}(x) = \beta x + \gamma \tag{23}$$

and it is easy to get from (21) and (23) that

$$\ddddot{a}(t) = 0 \qquad \dddot{b}(t) = 6\beta\ddot{a}(t)$$

$$\ddot{c}(t) = -6\gamma\ddot{a}(t) - 3\beta^2 a(t) + 3\beta\dot{b}(t).$$

The complete determination of the functions $a(t)$, $b(t)$, $c(t)$ and $h_0(x, t)$ requires ten arbitrary constants leading to *ten* symmetry operators. With respect to the well known results concerning the second-order problem [4, 6] (remember that the context of the potential (23) is isomorphic to the *free* case admitting six symmetries), we thus get *four* new symmetries which evidently correspond to third-order symmetry operators.

Let us mention their explicit forms easily obtained for the free case as follows

$$X_1 = x^2 + \mathrm{i}t - t^2\partial_x^2 + 2\mathrm{i}xt\partial_x \tag{24a}$$

$$X_2 = 2\mathrm{i}x\partial_x - 2t\partial_x^2 + \mathrm{i} \tag{24b}$$

$$X_3 = -\partial_x^2 \tag{24c}$$

$$X_4 = -(\mathrm{i}t\partial_x + x) \tag{25a}$$

$$X_5 = -\mathrm{i}\partial_x \tag{25b}$$

$$X_6 = 1 \tag{25c}$$

$$X_7 = \mathrm{i}t^3\partial_x^3 + 3t^2x\partial_x^2 + 3t^2\partial_x - 3\mathrm{i}tx^2\partial_x - 3tx - x^3 \tag{26a}$$

$$X_8 = \mathrm{i}t^2\partial_x^3 + 2tx\partial_x^2 + 2t\partial_x - \mathrm{i}x^2\partial_x - \mathrm{i}x \tag{26b}$$

$$X_9 = \mathrm{i}t\partial_x^3 + x\partial_x^2 + \partial_x \tag{26c}$$

$$X_{10} = \mathrm{i}\partial_x^3. \tag{26d}$$

We recognize the generators $\{X_1, \ldots, X_6\}$ of the semi-direct sum of the *second-order* operators of the conformal algebra $\{X_1, X_2, X_3\}$ with the *first-order* operators of the Heisenberg algebra $\{X_4, X_5, X_6\}$, i.e. the six symmetries belonging to the largest kinematical invariance algebra for one-dimensional systems [1, 2]. The other operators $\{X_7, \ldots, X_{10}\}$ are the *third-order* symmetry generators which do not form a closed set, neither between themselves, nor with the above six operators. Consequently they correspond to the so-called 'non-classical Lie' symmetries as referred and described in particular by Fushchich and Nikitin [7] which have also been recently studied in the supersymmetric context [8]. Let us just notice further possible determinations of *closed* Lie algebras among these ten generators but with *only six* of them. It can be verified that the following largest structures close:

$$\{X_{10}, X_2, X_3, X_4, X_5, X_6\} \qquad \text{and} \qquad \{X_1, X_2, X_7, X_4, X_5, X_6\}.$$

Both structures are once again semi-direct sums between the Heisenberg algebra and other non-semisimple three-dimensional ones which are isomorphic to each other as it can be easily shown.

Let us now add to this letter two main comments, one concerning the constructive information on classes of potentials admitting *third-order* symmetries and another one on a supplementary result on *nth-order* symmetries.

(a) On classes of admissible potentials and third-order symmetries. If third-order symmetries are studied we thus get the inhomogeneous differential equation (21) on the admissible potentials. As already noticed, the functions (7) are particular solutions and, besides the general resolution of (21), we want also to mention that there exists a lot of *new* solutions extending in that way the classes of admissible potentials. Let us here comment on the inverse way, i.e. let us show that unusual but admissible potentials have third-order symmetries. As particular potentials which lead to *exact* solutions as discussed by Bagrov and Gitman [9], let us collect the five following ones:

$$V(x) = \frac{2c^2}{\cos^2 cx} \qquad V(x) = 2c^2\tan^2 cx$$

$$V(x) = 2c^2(\tanh^2 cx - 1) \qquad V(x) = 2c^2(\operatorname{cotanh}^2 cx - 1) \tag{27}$$

$$V(x) = \frac{c^2}{\sinh^2 cx} \pm c^2 \frac{\cosh cx}{\sinh^2 cx}.$$

We can show that all these potentials admit the symmetries associated with the generators X'_5, X'_6, X'_9 and X'_{10} where primes refer here to their explicit form corresponding to (25) and (26) but in this interacting context. Only X'_5, X'_6 and X'_{10} form a closed Lie algebra. We thus conclude that the requirement of symmetry under the third order is less restrictive than under the first and second orders. It permits the extension of classes of admissible potentials.

As a last remark, let us come back on particular potentials (also considered by Bagrov and Gitman [9]) appearing as special cases of (7), i.e.

$$V(x)=\beta x+\gamma \qquad V(x)=\frac{2\nu^2}{(\nu x+\eta)} \quad \nu\neq 0$$
$$V(x)=\frac{\mu}{(\nu x+\eta)^2} \qquad \mu\neq 2\nu^2 \quad \mu\neq 0 \quad \nu\neq 0. \tag{28}$$

The first potential admits *ten* symmetries, the second one admits *eight* symmetries (which contain the four third-order ones) and the third potential admits only four symmetries (but none of the third-order type).

(b) On nth-order symmetries. By looking at the ten symmetries subtended by the four first orders of differentiation as characterized by equations (24-26), we can distinguish among these ten operators that the zeroth order gives *one* symmetry ($X_6\equiv 1$), the first order *two* symmetries (X_4 and X_5), the second order *three* symmetries (X_1, X_2 and X_3) and the third order *four* symmetries (X_7, X_8, X_9 and X_{10}). Then the total number of operators included in this Q_3-problem is equal to $1+2+3+4=\frac{1}{2}(3+1)(3+2)$. These rules appear as interesting properties which can be generalized in the following proposition concerning the free Q_n-problem.

Proposition. For the one-dimensional *free* Schrödinger problem, the number of *n*th-order symmetry operators is always equal to $n+1$ and the total number of operators included in the Q_n-problem is

$$N=\tfrac{1}{2}(n+1)(n+2). \tag{29}$$

Let us exploit the results contained in the commutators (12) by noticing that the definitions (6) lead immediately to a basis on the derivatives. By considering successively the zeroth, $(n+1)$th and kth ($k=1,2,\ldots,n$) orders we get $(n+2)$ differential equations on the following forms:

$$\dot{h}_0=0 \qquad h'_n=0 \qquad 2\dot{h}_k+h'_{k-1}=0. \tag{30}$$

By iteration we then get that

$$\partial_t^{k+1}h_k(t,x)=0 \qquad \text{and} \qquad \partial_x^{n-k+1}h_k(t,x)=0$$

so that our arbitrary functions become

$$h_k(x,t)=\sum_{p=0}^{n-k}\sum_{m=0}^{k}C_k^{p,m}x^p t^m \tag{31}$$

where $C_k^{p,m}$ are $(k+1)(n-k+1)$ constant coefficients for each $k=0,1,\ldots,n$. These are constrained through

$$2(m+1)C_k^{p,m+1}+(p+1)C_{k-1}^{p+1,m}=0 \qquad k=1,2,\ldots,n \tag{32}$$

due to the relations (30). Consequently the total number of independent parameters is

$$N = \sum_{k=0}^{n} (k+1)(n-k+1) - \sum_{k=1}^{n} k(n-k+1)$$

$$= n+1+\sum_{k=1}^{n} (n-k+1) \equiv \text{equation (29)}.$$

As a last remark connected with a very recent reference [10] (communicated by the referee), we just want to point out that our symmetry operators are conformal ones as discussed by Kalnins *et al.* Indeed, we have solved the invariance condition (3) while our symmetry operators Q do not contain first order time derivatives remembering that they act on solutions of (1) as already mentioned: all the operators $Q_n \equiv (8)$ only contain space derivatives.

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