EXTENDED POINCARÉ PARASUPERALGEBRA WITH CENTRAL CHARGES

A. G. NIKITIN

Institute of Mathematics, Academy of Sciences of the Ukraine,
3 Tereshenkivska str., Kiev–4, Ukraine

Abstract

Irreducible hermitian representations of the extended Poincaré parasuperalgebra with non-trivial central charges are described. These representations include the representations of the usual extended Poincaré superalgebra as a particular case and can serve as a group–theoretical foundation of parasupersymmetric quantum field theory, i.e., as a general viewpoint to reformulate quantum field theory and quantum mechanics of identical particles on the general basis of paraquantization and supersymmetry.

1 Introduction

Poincaré parasuperalgebra (PPSA) is an extension of the Poincaré algebra which is other then Poincaré superalgebra but includes the last as a particular case [1,2]. It appears naturally when the parasupersymmetric quantum mechanics [3] is being relativized and can serve as the group-theoretical base of parasupersymmetric quantum field theory.

There are two approachers in modern physics which in some sense treat bosons and fermions on equal rights. One of them is called *supersymmetry* [4] Indeed all models of supersymmetry quantum field theory admit equivalence transformations which mix fermionic and bosonic states. The other approach is connected with parastatistics and paraquantization [5,6]. Parasupersymmetric quantum field theory [2] is a kind of a syntesis of these two approaches.

In [1,2] the irreducible representations (IRs) of the simplest $N = 1$ Poincaré parasuperalgebra were considered and some representations corresponding to time-like and light-time four-momenta were discussed. A complete description of all nonequivalent IRs for time-like, light-like and space-like four-momenta had been found in [7].

Representations of the extended Poincaré parasuperalgebra $p(1,3;N)$ (i.e., the Poincaré parasuperalgebra with an arbitrary number N of parasupercharges, which includes the external symmetry algebra) were described in [8] and [9]. Moreover, the relations of representations of $p(1,3;N)$ with IRs of the pseudeorthogonal algebras $so(p,q)$ was established [9].

In the following we describe IRs of the extended Poincaré parasuperalgebra with an arbitrary number N of parasupercharges, internal symmetry algebra and n ($n \leq N/2$ for even N and $n \leq (N-1)/2$ for N odd) *central charges.*

2 Extended Poincaré parasuperalgebra.

The Poincaré prasuperalgebra [1, 2, 9] is generated by ten generators P_μ, $J_{\mu\nu}$, $\mu, \nu = 0,1,2,3$ of the Poincaré group, satisfying the commutation relations

$$[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu),$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}), \tag{2.1}$$

and N parasuperchargers Q_A^j, $\left(Q_A^j\right)^\dagger$ ($A = 1,2, j = 1,2,...,N$), which satisfy the following double commutation relations

$$\begin{aligned}
&[Q_A^i, [Q_B^j, Q_C^k]] = 4\varepsilon_{AB}Z^{ij}Q_C^k - 4\varepsilon_{AC}Z^{ik}Q_B^j,\\
&[(Q_A^i)^\dagger, [\left(Q_B^j\right)^\dagger, \left(Q_C^k\right)^\dagger]] = 4\varepsilon^{AB}Z_{ij}^*\left(Q_C^k\right)^\dagger - 4\varepsilon^{AC}Z_{ik}^*\left(Q_B^j\right)^\dagger,\\
&[Q_A^i, [Q_B^j, \left(Q_C^k\right)^\dagger]] = 4\varepsilon_{AB}Z^{ij}\left(Q_C^k\right)^\dagger - 4Q_B^j(\sigma_\mu)_{AC}P^\mu,\\
&[(Q_A^i)^\dagger, [Q_B^j, \left(Q_C^k\right)^\dagger]] = 4\left(Q_C^k\right)^\dagger(\sigma_\mu^*)_{BA}P^\mu - 4\varepsilon_{AB}Z_{ik}^*Q_B^j
\end{aligned} \tag{2.2}$$

where σ_ν are the Pauli matrices, $(.)_{AC}$ relate to matrix elements.

Relations (2.1),(2.2) include operators Z^{ij} which we call the central charges. For the case $Z^{ij}=0$ these relations reduce to the form proposed in [1,2,8].

Like the case of Poincaré superalgebra the central charges are supposed to satisfy the relations $Z^*_{ij}=Z^{ij}$ and $Z^{ij}=-Z^{ji}$ and commute with generators of the PPSA.

The commutation relations between the generators of the Poincaré group and the parasupercharges are:

$$\begin{array}{l} [J_{\mu\nu},Q^j_A]=-\frac{1}{2i}(\sigma_{\mu\nu})_A{}^B Q^j_B,\ [P_\mu,Q^j_A]=0,\\ [J_{\mu\nu},\left(Q^j_A\right)^\dagger]=-\frac{1}{2i}\left(\sigma^*_{\mu\nu}\right)^B{}_A\left(Q^j_B\right)^\dagger,\ [P_\mu,\left(Q^j_A\right)^\dagger]=0 \quad . \end{array} \tag{2.3}$$

We stress that the extended PPSA is a direct (and natural) generalization the Poincaré superalgebra). Indeed, the PSA also includes $10+4N$ elements satisfying (2.1), (2.3), but instead of (2.2) supercharges $Q^j_A,\left(Q^j_A\right)^\dagger$ satisfy the following anticommutation relations

$$[Q^i_A,Q^j_B]_+=Q^i_AQ^j_B+Q^j_BQ^i_A=\varepsilon_{AB}Z^{ij},\quad [Q^i_A,\left(Q^j_B\right)^\dagger]_+=2\delta^{ij}(\sigma_\mu)_{AB}P^\mu. \tag{2.4}$$

Relations (2.2) are mere consequence of (2.4), the converse is not true.

Like the Poincaré superalgebra the PPSA can be extended by adding the generators Σ_α of the internal symmetry group,which satisfy the following relations:

$$[Q^i_A,\Sigma^\alpha]=T^i_{\alpha j}Q^j_A,\quad [\Sigma_\alpha,P_\mu]=[\Sigma_\alpha,J_{\mu\mu}]=0,\quad [\Sigma^\alpha,\Sigma^\sigma]=f^{\alpha\sigma}_\nu\Sigma^{nu} \tag{2.5}$$

were f^k_{lm} are structure constants of the internal symmetry group, the constants T^i_{lj} are specified in the following. Thus, the PSA is a particular case of the more general algebraic structure called PPSA, like the usual Fermi statistics is a particular case of the parastatistics [6]. Moreover, in analogy with the PSA, P_σ and $J_{\mu\nu}$ are called even and $Q^j_A,\left(Q^j_A\right)^\dagger$ are called odd elements of the PPSA.

3 Wigner little parasuperalgebra

The extended Poincaré parasuperalgebra (2.1)-(2.3), (2.5) has two the main Casimir operators [1,2, 8]

$$C_1=P_\mu P^\mu,\qquad C_2=P_\mu P^\mu B_\nu B^\nu-(B_\mu P^\mu)^2 \tag{3.1}$$

where

$$B_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma + \Sigma_{i=1}^N (\sigma_\mu)^{AB} \bar{Q}^i_A Q^i_B.$$

We will use eigenvalues of C_1,C_2 to classify the IRs.

Like the case of the ordinary Poincaré group [10], IRs of the PPSA are qualitatively different for the following cases

I. $P_\mu P^\mu > 0$, II. $P_\mu P^\mu = 0$, III. $P_\mu P^\mu < 0$.

For the cases I and II there exists the additional Casimir operator $C_3 = P_0/|P_0|$ whose eigenvalues are ± 1. Here we consider only such representations which correspond to $C_1 > 0$ and $C_3 > 0$. This class of representations will be denoted as I^+.

As follows from (2.1)-(2.3) four–vector B_μ satisfies the relations

$$[B_\mu, P_\nu] = 0, \qquad [B_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}B_\sigma - g_{\mu\sigma}B_\nu), \tag{3.2}$$

$$[B_\mu, Q^i_A] = \tfrac{1}{2}P_\mu Q^i_A, \;\; [B_\mu, \bar{Q}^i_A] = -\tfrac{1}{2}P_\mu \bar{Q}^i_A, \; [B_\mu, B_\nu] = i\varepsilon_{\mu\nu\rho\sigma}P^\rho B^\sigma. \tag{3.3}$$

Consider these relation in the momentum representation and rest frame of reference $P = (M, 0, 0, 0)$. For this particular choice of P we define the three-vector j_k by the identities

$$B_k = W_k + X_k = -MS_k + X_k \equiv -Mj_k, \qquad k = 1, 2, 3 \tag{3.4}$$

The central charges Z^{ij} have to be equal to the unit matrix multiplied by the numeric coefficients Z^{ij}. We will treat these coefficients as elements of the $N \times N$ antisymmetric matrix Z. Up to the unitary transformation

$$Z \longrightarrow \bar{Z} = UZU^\dagger \tag{3.5}$$

any such matrix can be reduced to the following quasidiagonal form

$$\tilde{Z}^{ij} = U^i{}_k U^{*j}_l Z^{kl}, \tag{3.6}$$

where

$$\tilde{Z}^{ij} = \varepsilon^{ij} \otimes D \; (N \; even); \quad \tilde{Z}^{ij} = \begin{pmatrix} \varepsilon^{ij} \otimes D & 0 \\ 0 & 0 \end{pmatrix} \; (N \; odd), \tag{3.7}$$

where D is a diagonal matrix with the positive real eigenvalues Z_m, $m = 1, 2, ..., \{N/2\}$, $\{N/2\}$ is the integer part of $N/2$, ε^{ij} is the unit antisymmetric tensor.

Relations (2.2) are invariant under the simultaneous transformation

$$Z^{ij} \longrightarrow \bar{Z}^{ij} = U^i{}_k U^{*j}_l Z^{kl}, \quad Q^i_A \longrightarrow \tilde{Q}^i_A = U^j{}_k Q^k_A \tag{3.8}$$

(U^{kL} are elements of the unitary matrix U of (3.6)), where all nonzero Z^{ij} are exhausted by the following ones

$$Z^{2m-1,2m} = -Z^{2m,2m-1} = Z^m. \tag{3.9}$$

Denoting $(\hat{Q}^j_A)\dagger = \hat{\bar{Q}}^j_A$ and choosing a new basis

$$\begin{array}{ll} Q^{2m-1}_1 = \frac{1}{\sqrt{2}}(\hat{Q}^{2m-1}_1 + \hat{Q}^{2m}_1), & Q^{2m-1}_2 = \frac{1}{\sqrt{2}}(\hat{\bar{Q}}^{2m}_2 - \hat{\bar{Q}}^{2m-1}_2), \\ Q^{2m}_1 = \frac{1}{\sqrt{2}}(\hat{Q}^{2m-1}_2 + \hat{Q}^{2m}_2), & Q^{2m}_2 = \frac{1}{\sqrt{2}}(\hat{\bar{Q}}^{2m}_1 - \hat{\bar{Q}}^{2m-1}_1) \end{array} \tag{3.10}$$

we reduce relations (2.2), (2.7), (2.8) in the rest frame $P = (M, 0, 0, 0)$ to the form

$$\begin{array}{l} [j_a, j_b] = i\varepsilon_{abc} j_c, \qquad [j_a, \hat{Q}^j_A] = [j_a, \hat{\bar{Q}}^j_A] = 0, \\ [\hat{Q}^{2k-1}_A, [\hat{\bar{Q}}^{2m-1}_B, \hat{Q}^j_C]] = \delta_{AB}\delta_{km}(2M - Z_k)\hat{Q}^j_C, \\ [\hat{Q}^{2k}_A, [\hat{\bar{Q}}^{2m}_B, \hat{Q}^j_C]] = \delta_{AB}\delta_{km}(2M + Z_m)\hat{Q}^j_C, \\ [\hat{\bar{Q}}^{2k-1}_A, [\hat{Q}^{2m-1}_B, \hat{\bar{Q}}^j_C]] = \delta_{AB}\delta_{km}(2M - Z_m)\hat{\bar{Q}}^j_C, \\ [\hat{\bar{Q}}^{2k}_A, [\hat{Q}^{2m}_B, \hat{\bar{Q}}^j_C]] = \delta_{AB}\delta_{km}(2M + Z_m)\hat{\bar{Q}}^j_C \end{array} \tag{3.11}$$

the remaining double commutators of the parasupercharges are equal to zero.

Let all $Z_m < 2M$ then we find the general solution of relation (3.11) in the form

$$\begin{array}{l} \hat{Q}^{2m-1}_A = (-1)^{A-1}\sqrt{2M - Z_m}(S_{4N+1,8m-11+4A} - iS_{4N+1,8m-10+4A}), \\ \hat{Q}^{2m}_A = (-1)^{A-1}\sqrt{2M + Z_m}(S_{4N+1,8m-9+4A} - iS_{4N+1,8m-8+4A}) \end{array} \tag{3.12}$$

where $S_{\mu\nu}$ are generators of algebra $so(4N+1)$ satisfying the following relations

$$[S_{kl}, S_{mn}] = -i(g_{km}S_{ln} + g_{ln}S_{km} - g_{kn}S_{lm} - g_{lm}S_{km}) \ . \tag{3.13}$$

Here $g_{kl} = -\delta_{kl}$ and δ_{kl} is the Kronecker symbol.

Substituting (3.12) into (3.10) we obtain parasupercharges in the rest frame

$$\begin{array}{l}
\tilde{Q}_A^{2m-1} = \sqrt{M - \frac{Z_m}{2}}((-1)^{A-1}S_{4N+1,8m-11+4A} - iS_{4N+1,8m-10+4A})+ \\
+\sqrt{M + \frac{Z_m}{2}}(S_{4N+1,8m-9+4A} + i(-1)^A S_{4N+1,8m-8+4A}), \\
\tilde{Q}_A^{2m} = \sqrt{M - \frac{Z_m}{2}}(-S_{4N+1,8m-7+4A} + i(-1)^{A-1}S_{4N+1,8m-6+4A})+ \\
+\sqrt{M + \frac{Z_m}{2}}((-1)^A S_{4N+1,8m-5+4A} + iS_{4N+1,8m-4+4A}).
\end{array} \quad (3.14)$$

The related vector of spin S_a has the form

$$\begin{array}{l}
S_1 = (1/2)\sum_{i=1}^N (-1)^{i-1}(S_{2i,2i+3} + S_{2i-1,2i+4}) \oplus j_1, \\
S_2 = (1/2)\sum_{i=1}^N (-1)^{i-1}(S_{2i+3,2i-1} + S_{2i,2i+4}) \oplus j_2, \\
S_3 = (1/2)\sum_{i=1}^N S_{2i-1,2i} \oplus j_3
\end{array} \quad (3.15)$$

where j_3 are generators of the IRs $D(j)$ of algebra $so(3)$, commuting with $S_{\mu\nu}$.

In accordance with the above, the IRs of the class I^+ of the extended Poincaré parasuperalgebra with central charges $Z_m < 2M$ are labelled by the following sets of numbers $(M, j, n_1, n_2, ..., n_{2N}, Z_1, Z_2, ..., Z_{\{n/2\}})$ satisfying the relations $n_1 \geq n_2 \geq ... \geq n_{2N}$, $Z_m < 2M$ (all $n_1, n_2, ...$ are either integer or half integers). The corresponding basis elements P_μ, $J_{\mu\nu}$ and parasupercharges (which can be obtained starting with (4.8) by means of the Lorentz transformation) have the form

$$\begin{array}{l}
P_0 = E, \qquad P_a = p_a, \\
J_{ab} = x_a p_b - x_b p_a + \varepsilon_{abc} S_c, \\
J_{0a} = x_0 p_a - \frac{i}{2}\{\frac{\partial}{\partial p_a}, E\} + -\frac{\varepsilon_{abc} p_b S_c}{E+M} \\
Q_1^j = \frac{1}{\sqrt{2M(E+M)}}[(E + M + p_3)\tilde{Q}_1^j + (p_1 - ip_2)\tilde{Q}_2^j] \\
Q_2^j = \frac{1}{\sqrt{2M(E+M)}}[(p_1 + ip_2)\tilde{Q}_1^j + (E + M - p_3)\tilde{Q}_2^j] \\
j = 1, 2, ..., N
\end{array} \quad (3.16)$$

where $x_a = i\frac{\partial}{\partial p_a}$, $\quad E = \sqrt{M^2 + p^2}$, and $\tilde{Q}_A^j$, ($j = 2m - 1$, or $j = 2m, A = 1, 2$) are matrices given by relations (3.14).

IRs of the PPSA with central charges can be constructed also for the case $Z_m = 2M$. Moreover, in contrast with the PSA, there exist such IRs of the PPSA which correspond to $Z_m > 2M$ for some $m < \{N/2\}$.

Let us consider the most general case when

$$\begin{array}{ll} Z_m < 2M, & m = 1, 2, ..., p, \\ Z_m = 2M, & m = p+1, p+2, ..., s, \\ Z_m > 2M, & m = s+1, s+2, ..., \{\frac{N}{2}\} \end{array} \tag{3.17}$$

for some integers p, s, satisfying $0 \le p \le s \le \{N/2\}$. Using again the basis (3.10), we come to relations (3.11). For $m \le p$ we have the old solutions (3.12), (3.13). For $p < m \le s$ relations (3.11) have only trivial solutions for $\hat{Q}_A^{2m-1}$ (all the double commutators for parasupercharges which are not present in (3.11) should be equal to zero), and formulae (3.12) and (3.14) are replaced by

$$\begin{array}{l} \hat{Q}_A^{2m-1} = 0, \\ \hat{Q}_A^{2m} = (-1)^{A-1} 2\sqrt{M}(S_{4N-2s,4m-5+2A} - iS_{4N-2s,4m-4+2A}) \end{array} \tag{3.18}$$

and

$$\begin{array}{l} \tilde{Q}_1^{2m-1} = 2\sqrt{M}(S_{4N-2s,4m-3} - iS_{4N-2s,4m-2}), \\ \tilde{Q}_2^{2m-1} = 2\sqrt{M}(-S_{4N-2s,4m-1} - iS_{4N-2s,4m}), \\ \tilde{Q}_1^{2m} = 2\sqrt{M}(-S_{4N-2s,4m-1} + iS_{4N-2s,4m}), \\ \tilde{Q}_2^{2m} = 2\sqrt{M}(S_{4N-2s,4m-3} + iS_{4N-2s,4m-2}), \\ m = p+1, p+2, ..., s \end{array} \tag{3.19}$$

For $s < m \le \{N/2\}$ it is convenient to search for solutions of (3.11) in the form

$$\begin{array}{l} \hat{Q}_A^{2m-1} = (-1)^{A-1}\sqrt{Z_m - 2M}(S_{4N-2s,8m+4A-11} - iS_{4N-2s,8m+4A-10}), \\ \hat{Q}_A^{2m} = (-1)^{A-1}\sqrt{Z_m + 2M}(S_{4N-2s,8m+4A-9} - iS_{4N-2s,8m+4A-8}) \end{array}$$

which corresponds to the following parasupercharges in the rest frame

$$\begin{array}{l} \tilde{Q}_A^{2m-1} = \sqrt{\frac{Z_m}{2} - M}((-1)^{A-1}S_{4N-2s+1,8m+4A-11} - iS_{4N-2s+1,8m+4A-10}) + \\ \sqrt{\frac{Z_m}{2} + M}(S_{4N-2s+1,8m+4A-9} + (-1)^A S_{4N-2s+1,8m+4A-8}), \\ \tilde{Q}_A^{2m} = \sqrt{\frac{Z_m}{2} - M}(-S_{4N-2s+1,8m+4A-7} - (-1)^A S_{4N-2s+1,8m+4A-6}) + \\ \sqrt{\frac{Z_m}{2} + M}((-1)^A S_{4N-2s+1,8m+4A-5} + iS_{4N-2s+1,8m+4A-4}), \\ m = s+1, s+2, ..., \{N/2\}. \end{array} \tag{3.20}$$

In order relations (3.11) to be satisfied (and other double commutators for $\tilde{Q}_A^j$ be equal zero), it is necessary and sufficient that $S_{4N-2s+1,\nu}$ belong to the algebra so(2s, 4(N-s)+1), which is defined by relations (3.11) with $g_{\mu\mu} = -1,\ \mu = 2s-1,\ s < m \leq \{N/2\};\quad g_{\mu\mu} = 1,\ \mu = 2m$ or $\mu < 2s+1;\quad g_{\mu\nu} = 0,\ \mu \neq \nu$.

4 Internal symmetries

It was shown in [9] that the IRs of the extended PPSA with N supercharges (but without central charges) can be extended by internal symmetry algebra which is $u(N)$ for $C_1 \leq 0$. If the central charges are nontrivial then the internal symmetry algebra is less extended. Indeed, consider the first of relations (2.2) for A=C=1, B=2:

$$[Q_1^j, [Q_2^j, Q_1^k]] = 4Z^{ij}Q_1^k. \tag{4.1}$$

Calculating commutators of the l.h.s. and r.h.s. of (4.1) with Σ_l and using (2.5) we come to the following condition

$$T_{lj}^I Z^{jk} = T_{lj}^k Z^{ji}. \tag{4.2}$$

In other words, the products of generators of the internal group with the matrix of central charges should be a symmetric matrix.

Let us present the explicit description of the internal symmetry algebra for representations of Class I^+. We consider consequently the following cases:
a) all central charges are nontrivial and $Z_m \neq 0$ for any $m = 1, 2, ..., \{N/2\}$;
b) all central charges are nontrivial, but $Z_m = 0$ for $m = s{+}1, s{+}2, ..., \{N/2\}$;
c) the most general case including all versions (3.17) and also $Z_m = 0$ for some m.

For the case a) and N even the condition (4.2) means that T_l^{ij} belong to the algebra $sp(\frac{N}{2})$. Indeed, the corresponding matrix Z^{ij} is antisymmetric and invertible (see (3.7)) and so matrices T_l^{ij} form a Lie algebra isomorphic to $sp(\frac{N}{2})$.

The $N(N-1)/2$ basis elements of the related internal symmetry algebra can be chosen in the form

$$\begin{array}{l}A^{kk}=Z_k^{-1}(-S_{8k-7,8k-6}-S_{8k-5,8k-4}+S_{8k-38k-2}+S_{8k-1,8k}),\\ B^{kk}=Z_k^{-1}(S_{8k-5,8k}-S_{8k-4,8k-1}+S_{8k-7,8k-2}-S_{8k-6,8k-3})+\\ +i(S_{8k-5,8k-1}+S_{8k-4,8k}+S_{8k-7,8k-3}+S_{8k-6,8k-2}),\\ C_{kk}=(B_{kk})^{\dagger},\\ A^{kn}=(f_{kn}^{-}+f_{nk}^{-})\Sigma_{kn}+(f_{kn}^{+}+f_{nk}^{+})\Sigma_{k+2,n+2},\\ B^{kn}=f_{nk}^{-}\tilde{\Sigma}_{kn}+f_{kn}^{-}\Sigma_{kn}^{\dagger}+f_{nk}^{+}\tilde{\Sigma}_{k+2,n+2}-f_{nk}^{+}\tilde{\Sigma}_{k+2,n+2}^{\dagger}, n>k,\\ C^{kn}=(B^{kn})^{\dagger},\ n<k\end{array}\qquad(4.3)$$

where

$$\begin{array}{l}f_{kn}^{\pm}=\frac{1}{Z_n}\sqrt{\frac{2M\pm Z_k}{2M\pm Z_N}},\quad f_{nk}^{\pm}=\frac{1}{Z_k}\sqrt{\frac{2M\pm Z_n}{2M\pm Z_k}},\\ \Sigma_{kn}=S_{8k-7,8n-6}-S_{8k-6,8n-7}-S_{8k-3,8n-2}+S_{8k-2,8n-3}-\\ -i(S_{8k-7,8n-7}+S_{8k-6,8n-6}+S_{8k-3,8n-3}+S_{8k-2,8n-2}),\\ \tilde{\Sigma}_{kn}=-S_{8k-7,8n-2}+S_{8k-6,8n-3}+S_{8k-3,8n-6}-S_{8k-2,8n-7}-\\ -i(S_{8k-7,8n-3}+S_{8k-6,8n-2}+S_{8k-3,8n-7}+S_{8k-2,8n+6}),\\ n\neq k,\quad k,n=1,2,...,N/2.\end{array}\qquad(4.4)$$

Matrices (4.3) commute with Poincar/'e group generators P_μ, $J_{\mu\nu}$ and satisfy the following relations

$$\begin{array}{l}[A^{kk},Q_A^j]=Z_k^{-1}(\delta_{j,2k-1}-\delta_{j,2k})Q_A^j,\\ [B^{kk},Q_A^j]=2Z_k^{-1}\delta_{j2k-1}Q_{2k}^j,\quad [C^{kk},Q_A^j]=2Z_k^{-1}\delta_{j,2k}Q_A^{2k-1},\\ [A^{kn},Q_A^j]=\delta_{j,2k-1}Z_k^{-1}Q_A^{2k-1}-\delta_{j,2n-1}Z_n^{-1}Q_A^{2k-1}+\\ +\delta_{j,2k}Z_k^{-1}Q_A^{2n}-\delta_{j,2k}Z_n^{-1}Q_A^{2k},\\ [B^{kn},Q_A^j]=\delta_{j,2k-1}Z_k^{-1}Q_A^{2n}+\delta_{j,2n-1}Z_n^{-1}Q_A^{2k},\\ [C^{kn},Q_A^j]=\delta_{j,2k}Z_k^{-1}Q_A^{2n-1}+\delta_{j,2n}Z_n^{-1}Q_A^{2k-1},\end{array}\qquad(4.5)$$

$$\begin{array}{l}[A^{mn},A^{kl}]=Z_k^{-1}\delta^{kn}A^{ml}-Z_m^{-1}\delta^{ml}A^{nk},\\ [A^{mn},B^{kl}]=Z_n^{-1}(\delta^{nk}B^{ml}+\delta^{nl}B^{mk}),\\ [A^{mn},C^{kl}]=[C^{mn},C^{kl}]=0,\\ [B^{mn},C^{kl}]=Z_k^{-1}(\delta^{nk}A^{ml}+\delta^{mk}A^{nl})+Z_k^{-1}(\delta^{nl}A^{mk}+\delta^{ml}A^{nk}).\end{array}\qquad(4.6)$$

Commutation relations (4.5) characterize the Lie algebra which is isomorphic to $sp(n), n=\{N/2\}$.

For the case N odd the corresponding matrix Z^{ij} is equivalent to the direct sum of the invertible and zero matrices and the condition (4.2) defines the direct sum of algebras $Sp(n)\oplus u(1)$, $n=(N-1)/2$. The basis elements of the internal symmetry algebra $sp(n)$ again have the form (4.3) (where $k, n = 1, 2, ..., (N-1)/2$) while the generator of $u(1)$ is

$$\Lambda = S_{4N-3,4N-2} + S_{4N-1,4N}.$$

For the case when $0 < Z_m < 2M, \quad m = 1, 2, ..., s$ and $Z_m = 0, \quad s < m \leq \{N/2\}$ the corresponding internal symmetry algebra reduces to the direct sum $sp(s) \oplus u(N-2s)$. The corresponding generators of algebra $sp(s)$ can again be chosen in the form (4.3) provided we change $\{N/2\}$ by s in the last line of (4.4). The basis elements of the related algebra $u(N-2s)$ can be easily found using results of paper [9].

Finally, for the most complicated case $0 < Z_m < 2M, \quad m = 1, 2, ..., s; \quad Z_m = 2M, \quad m = s+1, ..., p$ and $Z_m = 0, \quad p < m \leq \{N/2\}$ the internal symmetry algebra reduces to $sp(s) \oplus u(n-s-p)$. The explicit expressions for the corresponding basis elements can be easily found using relations (4.3), (4.4) and the results of paper [9].

Thus we describe IRs of the extended Poincaré parasuperalgebra which includes central charges and internal symmetry group. These representations include IRs of the Poincare *superalgebra* as a particular case (which appears when all central charges are less then $2M$ and the Gelfand-Zetlin numbers are equal to $n_1 = n_2 = \cdots = n_{2N} = 1/2$. Essentually new moment in comparison with the Poincaré superalgebra is the existence of IRs with central charges whose value exeeds $2M$.

References

[1] P. D. Jarvis, Aust. J. Phys. **31**, 461 (1978)

[2] J. Beckers and N. Debergh, J. Mod. Phys. **A 8**, 5041 (1993)

[3] V. A. Rubakov and V. P. Spiridonov, Mod. Phys. Lett. **A 3**, 1988, J. Beckers and N. Debergh, Nucl. Phys. **B 340**, 767 (1990)

[4] Yu. A. Gol'fand and E. P. Likhtman, Lett. JETP **13**, 452 (1971); D. V. Volkov and V. P. Akulov, Lett. JETP **16**, 621 (1972)

[5] E. P. Wigner, Phys. Rev. **77**, 711 (1950)

[6] H. E. Green, Phys. Rev. **90**, 270 (1953)

[7] A. G. Nikitin and V. V. Tretynyk, J. Phys. A **28**, 1665 (1995)

[8] A. G. Nikitin, in: Proceedings of the 5th Wigner Symposium, Viena, Austria, 25-29 August 1997, P. Kasperkowitz and D. Grau Ed., World Scientific, 1998, p. 227

[9] J. Niederle and A. G. Nikitin, J. Phys. A **32**, 5141 (1999)

[10] E. P. Wigner, Ann. Math. **40** 149 (1939)